THE n-TH OPERATOR VALUED DIVERGENCES $\Delta_{i,x}^{[n]}(A|B)$

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Abstract. Let A and B be strictly positive linear operators on a Hilbert space $\mathcal{H}$. As a generalization of the relative operator entropy $S(A|B) \equiv A^{\frac{1}{2}}(\log A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}}$ and the Tsallis relative operator entropy $T_x(A|B) \equiv A^{\frac{1}{2}}\dfrac{(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^x - I}{x}A^{\frac{1}{2}}$, we have introduced the n-th relative operator entropy $S^{[n]}(A|B)$ and the n-th Tsallis relative operator entropy $T_x^{[n]}(A|B)$ for $n \in \mathbb{N}$ and $x \in \mathbb{R}$. In this paper, we define the n-th generalized Petz-Bregman divergence $\mathscr{D}_x^{[n]}(A|B) \equiv T_x^{[n]}(A|B) - S^{[n]}(A|B)$ $(x \in R)$ corresponding to the operator valued divergence $\Delta_{1,\alpha}(A|B) \equiv T_\alpha(A|B) - S(A|B)$ $(\alpha \in [0,1])$ which is a generalization of Petz-Bregman divergence $D_{FK}(A|B) \equiv B - A - S(A|B)$. Similatly, by using $\mathscr{D}_x^{[n]}(A|B)$, we introduce the n-th operator valued divergences $\Delta_{2,x}^{[n]}(A|B)$, $\Delta_{3,x}^{[n]}(A|B)$ and $\Delta_{4,x}^{[n]}(A|B)$ corresponding to $\Delta_{2,\alpha}(A|B) \equiv S_\alpha(A|B) - T_\alpha(A|B)$, $\Delta_{3,\alpha}(A|B) \equiv -T_{1-\alpha}(B|A) - S_\alpha(A|B)$ and $\Delta_{4,\alpha}(A|B) \equiv S_1(A|B) + T_{1-\alpha}(B|A)$, respectively, and show their properties and relations among them.

1 Introduction. A bounded linear operator T on a Hilbert space $\mathcal{H}$ is positive (denoted by $T \geq 0$) if $(T\xi, \xi) \geq 0$ for all $\xi \in \mathcal{H}$, and T is said to be strictly positive (denoted by $T > 0$) if T is invertible and positive. Throughout this paper, A and B denote strictly positive operators.

Based on the concept of the α-divergence introduced by Amari [1], Fujii [2] defined the operator valued α-divergence:

$$D_\alpha(A|B) \equiv \frac{A \nabla_\alpha B - A \sharp_\alpha B}{\alpha(1-\alpha)} \quad (\alpha \in (0,1)),$$

where $A \nabla_\alpha B \equiv (1-\alpha)A + \alpha B$ is the weighted arithmetic operator mean and $A \sharp_\alpha B \equiv A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^\alpha A^{\frac{1}{2}}$ is the weighted geometric operator mean [18]. We use the representation $A \natural_\alpha B$ instead of $A \sharp_\alpha B$ below if $\alpha \in \mathbb{R}$ ([17]).

Aside from this, Petz [19] introduced the Bregman divergence for an operator valued smooth function $\psi : C \to B(\mathcal{H})$ as

$$\psi(x) - \psi(y) - \lim_{t \to +0} \frac{\psi(y + t(x-y)) - \psi(y)}{t},$$

where C is a convex set in a Banach space. As an analogy of this kind of divergence, we had given an operator valued divergence

$$\psi(1) - \psi(0) - \frac{d}{dt}\,\psi(t)\Big|_{t=0} = B - A - S(A|B)$$

for $\psi(t) \equiv A \natural_t B$. We call it the Petz-Bregman divergence and denote it by

$$D_{FK}(A|B) = B - A - S(A|B),$$

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where

$$S(A|B) \equiv A^{\frac{1}{2}} \left(\log A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right) A^{\frac{1}{2}}$$

is the relative operator entropy introduced by Fujii and Kamei [3, 9, 11]. Fujii, et. al. [5, 6] showed that the operator valued α-divergence coincides with the Petz-Bregman divergence at the end points for interval $(0,1)$. That is,

$$D_0(A|B) \equiv \lim_{\alpha \to +0} D_\alpha(A|B) = B - A - S(A|B) = D_{FK}(A|B).$$

In addition, since $D_1(A|B) \equiv \lim_{\alpha \to 1-0} D_\alpha(A|B) = D_{FK}(B|A)$ holds, $D_\alpha(A|B)$ combines $D_{FK}(A|B)$ with $D_{FK}(B|A)$. This is a symmetric property for $D_\alpha(A|B)$ in the sense of [4].

In [10], we had given the following relations among relative operator entropies:

$$S(A|B) \le T_\alpha(A|B) \le S_\alpha(A|B) \le -T_{1-\alpha}(B|A) \le S_1(A|B) \ \mathit{for}\ \alpha \in (0,1), \tag{1}$$

where $S_x(A|B) \equiv A^{\frac{1}{2}} \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right)^x \left(\log A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right) A^{\frac{1}{2}}$ $(x \in \mathbb{R})$ is the generalized relative operator entropy defined by Furuta [8] and $T_\alpha(A|B) \equiv \dfrac{A \sharp_\alpha B - A}{\alpha}$ $(\alpha \in (0,1])$ is the Tsallis relative operator entropy defined by Yanagi, Kuriyama and Furuichi [20]. The Tsallis relative operator entropy $T_x(A|B)$ can be defined for all $x \in \mathbb{R}$ and the inequalities (1) hold also at $\alpha = 0$ and 1.

In [12], we obtained the following representations of the operator valued α-divergence and the Petz-Bregman divergence:

$$\begin{aligned} D_\alpha(A|B) &= -T_{1-\alpha}(B|A) - T_\alpha(A|B) \quad (\alpha \in (0,1)), \\ D_{FK}(A|B) &= -T_1(B|A) - T_0(A|B) = T_1(A|B) - S(A|B). \end{aligned}$$

Since these are differences between the terms in (1), we also regarded other differences as operator divergences [14]: For $\alpha \in (0,1)$,

$$\begin{aligned} \Delta_{1,\alpha}(A|B) &\equiv T_\alpha(A|B) - S(A|B), & \Delta_{2,\alpha}(A|B) &\equiv S_\alpha(A|B) - T_\alpha(A|B), \\ \Delta_{3,\alpha}(A|B) &\equiv -T_{1-\alpha}(B|A) - S_\alpha(A|B), & \Delta_{4,\alpha}(A|B) &\equiv S_1(A|B) + T_{1-\alpha}(B|A) \end{aligned}$$

and so on.

Since the relative operator entropy $S(A|B)$ is given as the derivative of the path $A \natural_t B$ at $t = 0$, Fujii et. al. [7] gave the viewpoint that $S(A|B)$ is the velocity on the path $A \natural_t B$ at $t = 0$. Similarly, we regarded $S_\alpha(A|B)$ as the velocity on $A \natural_t B$ at $t = \alpha$ and based on this viewpoint, we tried to introduce a notion of the acceleration on the path $A \natural_t B$ at $t = \alpha$ which was given as the second derivative of the path at $t = \alpha$ in [15]. As an extension of such perspective, we regarded the Tsallis relative operator entropy $T_x(A|B)$ as the average rate of change of the path $A \natural_t B$ over the interval $[0, x]$ and $S(A|B) = \lim_{x \to 0} T_x(A|B)$ as the rate of change of the path at $t = 0$ in [16].

The n-th Tsallis relative operator entropy $T_x^{[n]}(A|B)$ is constructed inductively as follows:

$$T_x^{[1]}(A|B) \equiv T_x(A|B)$$

and for $n \ge 2$,

$$T_x^{[n]}(A|B) \equiv \frac{T_x^{[n-1]}(A|B) - S^{[n-1]}(A|B)}{x} \quad (x \in \mathbb{R} \backslash \{0\}),$$

where $S^{[n]}(A|B)$ is defined by

$$S^{[n]}(A|B) \equiv \frac{1}{n!}A^{\frac{1}{2}}\left(\log A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^n A^{\frac{1}{2}} = \frac{1}{n!}A(A^{-1}S(A|B))^n$$

and we call it the n-th relative operator entropy. Since $T_x^{[n]}(A|B)$ is represented specifically as

$$T_x^{[n]}(A|B) = \frac{1}{x^n}\left(A \natural_x B - A - \sum_{k=1}^{n-1} x^k S^{[k]}(A|B)\right) \quad (x \in \mathbb{R}\backslash\{0\}),$$

the corresponding functions to $T_x^{[n]}(A|B)$ and $S^{[n]}(A|B)$ are

$$\frac{1}{x^n}\left(\lambda^x - 1 - \sum_{k=1}^{n-1}\frac{x^k}{k!}(\log\lambda)^k\right) \quad \text{and} \quad \frac{1}{n!}(\log\lambda)^n \quad (\lambda > 0),$$

respectively. Since $\lim_{x\to 0}\frac{1}{x^n}\left(\lambda^x - 1 - \sum_{k=1}^{n-1}\frac{x^k}{k!}(\log\lambda)^k\right) = \frac{1}{n!}(\log\lambda)^n$, we obtain $\lim_{x\to 0} T_x^{[n]}(A|B) = S^{[n]}(A|B)$ for all $n \in \mathbb{N}$. Therefore, we defined $T_0^{[n]}(A|B)$ by

$$T_0^{[n]}(A|B) \equiv S^{[n]}(A|B).$$

For $n \geq 2$, the n-th Tsallis relative operator entropy $T_x^{[n]}(A|B)$ is regarded as the average rate of change of $T_x^{[n-1]}(A|B)$ over the interval $[0, x]$.

In addition, we defined $S_y^{[n]}(A|B)$ by

$$S_y^{[n]}(A|B) \equiv \frac{1}{n!}\frac{d^n}{dx^n}\ A \natural_x B\ \Bigg|_{x=y} = (A \sharp_y B)A^{-1}S^{[n]}(A|B) \quad (y \in \mathbb{R})$$

and call it the n-th generalized relative operator entropy. We remark that $S_0^{[n]}(A|B)$ coincides with $S^{[n]}(A|B)$ and $(A \natural_x B)A^{-1}S_y^{[n]}(A|B) = S_{x+y}^{[n]}(A|B)$ holds for x, $y \in \mathbb{R}$.

In [16], we defined the n-th Petz-Bregman divergence $D_{FK}^{[n]}(A|B)$ and the n-th operator valued divergence $\mathscr{D}_\alpha^{[n]}(A|B)$ by

$$D_{FK}^{[n]}(A|B) \equiv T_1^{[n]}(A|B) - S^{[n]}(A|B) = B - A - \sum_{k=1}^{n} S^{[k]}(A|B),$$

$$\mathscr{D}_\alpha^{[n]}(A|B) \equiv T_\alpha^{[n]}(A|B) - S^{[n]}(A|B) = \frac{1}{\alpha^n}\left(A \natural_\alpha B - A - \sum_{k=1}^{n}\alpha^k S^{[k]}(A|B)\right) \quad (\alpha \in [0,1]),$$

and showed their properties. We remark $\mathscr{D}_1^{[1]}(A|B) = D_{FK}^{[1]}(A|B) = D_{FK}(A|B)$ and $\mathscr{D}_\alpha^{[1]}(A|B) = \Delta_{1,\alpha}(A|B)$. So we think $\mathscr{D}_\alpha^{[n]}(A|B)$ is a generalization of the Petz-Bregman divergence. In addition, it is natural that $\mathscr{D}_\alpha^{[n]}(A|B)$ is regarded as the n-the operator valued divergence corresponding to $\Delta_{1,\alpha}(A|B)$. In this paper, we propose the definitions of the n-th operator valued divergences corresponding to $\Delta_{2,\alpha}(A|B)$, $\Delta_{3,\alpha}(A|B)$ and $\Delta_{4,\alpha}(A|B)$ and to show properties of them. For this purpose, we need to extend $\mathscr{D}_\alpha^{[n]}(A|B)$ $(\alpha \in (0,1))$ to $\mathscr{D}_x^{[n]}(A|B)$ $(x \in \mathbb{R})$. We call $\mathscr{D}_x^{[n]}(A|B)$ the n-th generalized Petz-Bregman divergence and show some properties of it in section 2. In section 3, we define the n-th operator valued divergences $\Delta_{2,x}^{[n]}(A|B)$, $\Delta_{3,x}^{[n]}(A|B)$ and $\Delta_{4,x}^{[n]}(A|B)$ with $x \in \mathbb{R}$ which correspond to $\Delta_{2,\alpha}(A|B)$, $\Delta_{3,\alpha}(A|B)$ and $\Delta_{4,\alpha}(A|B)$ by using the n-th generalized Petz-Bregman divergence $\mathscr{D}_x^{[n]}(A|B)$ and show some properties for them.

2 The n-th Generalized Petz-Bregman Divergence. Our idea of defining the n-th operator valued divergences corresponding to $\Delta_{2,\alpha}(A|B)$, $\Delta_{3,\alpha}(A|B)$ and $\Delta_{4,\alpha}(A|B)$ is to use $\mathscr{D}_\alpha^{[n]}(A|B)$ defined in [16]. In order to achieve such purpose, we need to broaden the range of α for $\mathscr{D}_\alpha^{[n]}(A|B)$ from $[0,1]$ to $\mathbb{R}$. For strictly positive operators A and B, $n \in \mathbb{N}$ and $x \in \mathbb{R}$, we define $\mathscr{D}_x^{[n]}(A|B)$ as follows:

$$\mathscr{D}_x^{[n]}(A|B) \equiv T_x^{[n]}(A|B) - S^{[n]}(A|B).$$

We call it the n-th generalized Petz-Bregman divergence.

By Proposition 4.5 in [16], the following proposition holds for the n-th generalized Petz-Bregman divergence.

Proposition 2.1. *Let n be a fixed natural number and x be a fixed real number in $\mathbb{R}\backslash\{0\}$. Then the following holds for any strictly positive operators A and B:*

$$\mathscr{D}_x^{[n]}(A|B) = O \quad \textit{if and only if} \quad A = B.$$

Remark 1. Since $\mathscr{D}_1^{[n]}(A|B) = D_{FK}^{[n]}(A|B)$,

$$D_{FK}^{[n]}(A|B) = O \quad \textit{if and only if} \quad A = B$$

holds for any fixed natural number n.

The following are fundamental properties for $\mathscr{D}_x^{[n]}(A|B)$.

Theorem 2.2. *Let A and B be strictly positive operators and $x \in \mathbb{R}$. Then the following hold for $n \in \mathbb{N}$:*

(a) $\mathscr{D}_0^{[n]}(A|B) = O$.

(b) *If $x > 0$ then*

$$\mathscr{D}_x^{[n]}(A|B) \begin{cases} \geq O, & \textit{if } n \textit{ is odd or } A \leq B, \\ \leq O, & \textit{if } n \textit{ is even and } A \geq B. \end{cases}$$

(c) *If $x < 0$ then*

$$\mathscr{D}_x^{[n]}(A|B) \begin{cases} \leq O, & \textit{if } n \textit{ is odd or } A \leq B, \\ \geq O, & \textit{if } n \textit{ is even and } A \geq B. \end{cases}$$

Proof. Since it is obvious that $\mathscr{D}_0^{[n]}(A|B) = O$ holds, we suppose $x \in \mathbb{R}\backslash\{0\}$. Let $\lambda > 0$. Since $\mathscr{D}_x^{[n]}(A|B) = \frac{1}{x^n}\left(A \natural_x B - A - \sum_{k=1}^{n} x^k S^{[k]}(A|B)\right)$, the corresponding function $f^{[n]}(\lambda, x)$ for $\mathscr{D}_x^{[n]}(A|B)$ is given as follows:

$$f^{[n]}(\lambda, x) = \frac{1}{x^n}\left(\lambda^x - 1 - \sum_{k=1}^{n} \frac{x^k}{k!}(\log\lambda)^k\right).$$

On the other hand, λ^x can be represented by using some $\theta \in (0,1)$ as

$$\lambda^x = 1 + \sum_{k=1}^{n} \frac{x^k}{k!}(\log\lambda)^k + \frac{x^{n+1}}{(n+1)!}\lambda^{\theta x}(\log\lambda)^{n+1}.$$

Hence, by using this θ, we get

$$f^{[n]}(\lambda, x) = \frac{x}{(n+1)!}\lambda^{\theta x}(\log \lambda)^{n+1}.$$

Let $x > 0$. Then $f^{[n]}(\lambda, x) \geq 0$ if n is odd or $\lambda \geq 1$, and $f^{[n]}(\lambda, x) \leq 0$ if n is even and $0 < \lambda \leq 1$. Therefore, (b) holds. Let $x < 0$. We obtain (c) since $f^{[n]}(\lambda, x) \leq 0$ if n is odd or $\lambda \geq 1$, and $f^{[n]}(\lambda, x) \geq 0$ if n is even and $0 < \lambda \leq 1$. □

In [16], we have obtained the following properties for the n-th relative operator entropies.

Lemma 2.3. (Theorem 2.4 and Theorem 3.4 in [16]) *Let A and B be strictly positive operators, r, $s \in \mathbb{R}$ and $x \in \mathbb{R}\backslash\{0\}$. Then*

(a) $T_x^{[n]}(A \natural_r B|A \natural_s B) = (s-r)^n(A \natural_r B)A^{-1}T_{(s-r)x}^{[n]}(A|B)$,

(b) $S_x^{[n]}(A \natural_r B|A \natural_s B) = (s-r)^n(A \natural_r B)A^{-1}S_{(s-r)x}^{[n]}(A|B) = (s-r)^n S_{(1-x)r+xs}^{[n]}(A|B)$

hold for all $n \in \mathbb{N}$. In particular, $S^{[n]}(A \natural_r B|A \natural_s B) = (s-r)^n(A \natural_r B)A^{-1}S^{[n]}(A|B)$.

By using Lemma 2.3, the n-th generalized Petz-Bregman divergence has also similar properties.

Proposition 2.4. (cf. Theorem 4.8 in [16]) *Let A and B be strictly positive operators and r, s, $x \in \mathbb{R}$. Then*

$$\mathscr{D}_x^{[n]}(A \natural_r B|A \natural_s B) = (s-r)^n(A \natural_r B)A^{-1}\mathscr{D}_{(s-r)x}^{[n]}(A|B)$$

holds for $n \in \mathbb{N}$.

Corollary 2.5. *Let A and B be strictly positive operators and r, x, $y \in \mathbb{R}$. Then the following holds for $n \in \mathbb{N}$:*

(a) $\mathscr{D}_x^{[n]}(A \natural_r B|A) = (-r)^n(A \natural_r B)A^{-1}\mathscr{D}_{-rx}^{[n]}(A|B)$,

(b) $\mathscr{D}_x^{[n]}(B|A) = (-1)^n BA^{-1}\mathscr{D}_{-x}^{[n]}(A|B)$,

(c) $(A \natural_y B)A^{-1}\mathscr{D}_x^{[n]}(A|B) = (-1)^n(B \natural_{1-y} A)B^{-1}\mathscr{D}_{-x}^{[n]}(B|A)$.

Since $A \natural_y B = B \natural_{1-y} A$ holds for $y \in \mathbb{R}$, we obtain (c) by (b) in Corollary 2.5.

Remark 2. By putting $x = 1$ in (a) in Lemma 2.3, we have

$$D_{FK}^{[n]}(A \natural_r B|A \natural_s B) = (s-r)^{n+1}(A \natural_r B)A^{-1}T_{s-r}^{[n+1]}(A|B).$$

The following relation between $\mathscr{D}_x^{[n]}(A|B)$ and $D_{FK}^{[n]}(A|B)$ holds, which is an extension of Corollary 4.9 in [16].

Proposition 2.6. *Let A and B be strictly positive operators and $x \in \mathbb{R}\backslash\{0\}$. Then the following holds for all $n \in \mathbb{N}$:*

$$\mathscr{D}_x^{[n]}(A|B) = \frac{1}{x^n}D_{FK}^{[n]}(A|A \natural_x B).$$

3 The n-th Operator Valued Divergences corresponding to $\Delta_{i,\alpha}(A|B)$. The n-th generalized Petz-Bregman divergence $\mathscr{D}_x^{[n]}(A|B)$ defined in section 2 coincides with $\Delta_{1,\alpha}(A|B)$ when $n = 1$, $0 < x < 1$ and $x = \alpha$. So it is natural to regard $\mathscr{D}_x^{[n]}(A|B)$ as the n-the operator valued divergence corresponding to $\Delta_{1,\alpha}(A|B)$ and we can write it as $\Delta_{1,x}^{[n]}(A|B)$. In this section, we define the n-th operator valued divergences corresponding to $\Delta_{2,\alpha}(A|B)$, $\Delta_{3,\alpha}(A|B)$ and $\Delta_{4,\alpha}(A|B)$ by using $\mathscr{D}_x^{[n]}(A|B)$ and show some properties of them.

For $r, s \in \mathbb{R}$, $(A \natural_r B)A^{-1}(A \natural_s B) = A \natural_{r+s} B$ holds (cf. [13]). Then the Tsallis relative operator entropy $T_x(A|B)$ can be rewritten as

$$\begin{aligned} T_x(A|B) &= \frac{A \natural_x B - A}{x} = (A \natural_x B)A^{-1}\frac{A \natural_{-x} B - A}{-x} \\ &= (A \natural_x B)A^{-1}T_{-x}(A|B) \quad (x \in \mathbb{R}\backslash\{0\}). \end{aligned}$$

In addition, since $S_x(A|B) = (A \natural_x B)A^{-1}S(A|B)$ holds for $x \in \mathbb{R}$, we can rewrite $\Delta_{2,\alpha}(A|B)$ as follows:

$$\begin{aligned} \Delta_{2,\alpha}(A|B) &= S_\alpha(A|B) - T_\alpha(A|B) = -(A \natural_\alpha B)A^{-1}(T_{-\alpha}(A|B) - S(A|B)) \\ &= -(A \natural_\alpha B)A^{-1}\mathscr{D}_{-\alpha}^{[1]}(A|B) \quad (\alpha \in (0,1)). \end{aligned}$$

Similarly, $\Delta_{3,\alpha}(A|B)$ and $\Delta_{4,\alpha}(A|B)$ can be rewritten as follows ($\alpha \in (0,1)$):

$$\begin{aligned} \Delta_{3,\alpha}(A|B) &= -T_{1-\alpha}(B|A) - S_\alpha(A|B) = (A \natural_\alpha B)A^{-1}(T_{1-\alpha}(A|B) - S(A|B)) \\ &= (A \natural_\alpha B)A^{-1}\mathscr{D}_{1-\alpha}^{[1]}(A|B), \\ \Delta_{4,\alpha}(A|B) &= S_1(A|B) + T_{1-\alpha}(B|A) = -(A \natural_1 B)A^{-1}(T_{\alpha-1}(A|B) - S(A|B)) \\ &= -(A \natural_1 B)A^{-1}\mathscr{D}_{\alpha-1}^{[1]}(A|B). \end{aligned}$$

Based on such representations, we define the n-th operator valued divergences corresponding to $\Delta_{2,\alpha}(A|B)$, $\Delta_{3,\alpha}(A|B)$ and $\Delta_{4,\alpha}(A|B)$.

Definition 1. Let A and B be strictly positive operators, $n \in \mathbb{N}$ and $x \in \mathbb{R}$. We define the n-th operator valued divergence $\Delta_{2,x}^{[n]}(A|B)$, $\Delta_{3,x}^{[n]}(A|B)$ and $\Delta_{4,x}^{[n]}(A|B)$ as follows:

$$\Delta_{2,x}^{[n]}(A|B) \equiv -(A \natural_x B)A^{-1}\mathscr{D}_{-x}^{[n]}(A|B), \qquad \Delta_{3,x}^{[n]}(A|B) \equiv (A \natural_x B)A^{-1}\mathscr{D}_{1-x}^{[n]}(A|B),$$

$$\Delta_{4,x}^{[n]}(A|B) \equiv -(A \natural_1 B)A^{-1}\mathscr{D}_{x-1}^{[n]}(A|B).$$

We remark that $\Delta_{2,x}^{[n]}(A|B)$, $\Delta_{3,x}^{[n]}(A|B)$ and $\Delta_{4,x}^{[n]}(A|B)$ are defined for all $x \in \mathbb{R}$ as $\Delta_{1,x}^{[n]}(A|B) = \mathscr{D}_x^{[n]}(A|B)$ was. They are also written as follows by (c) in Corollary 2.5.

Proposition 3.1. *Let A and B be strictly positive operators, $n \in \mathbb{N}$ and $x \in \mathbb{R}$. Then the followings hold:*

(a) $\Delta_{1,x}^{[n]}(A|B) = (-1)^n(B \natural_1 A)B^{-1}\mathscr{D}_{-x}^{[n]}(B|A)$,

(b) $\Delta_{2,x}^{[n]}(A|B) = (-1)^{n+1}(B \natural_{1-x} A)B^{-1}\mathscr{D}_x^{[n]}(B|A)$,

(c) $\Delta_{3,x}^{[n]}(A|B) = (-1)^n(B \natural_{1-x} A)B^{-1}\mathscr{D}_{x-1}^{[n]}(B|A)$,

(d) $\Delta_{4,x}^{[n]}(A|B) \ = \ (-1)^{n+1}(B \natural_0 A)B^{-1}\mathscr{D}_{1-x}^{[n]}(B|A).$

Properties shown in Theorem 3.2 and Theorem 3.3 are fundamental where the n-th operator valued divergences have in common.

Theorem 3.2. *Let A and B be strictly positive operators, $n \in \mathbb{N}$ and $x \in \mathbb{R}$. Then the followings hold for $i = 1, 2$:*

(a) $\Delta_{i,0}^{[n]}(A|B) = O$.

(b) *If $x \neq 0$ then*

$$\Delta_{i,x}^{[n]}(A|B) = O \ \textit{ if and only if } \ A = B.$$

(c) *If $x > 0$ then*

$$\Delta_{i,x}^{[n]}(A|B) \begin{cases} \geq O, & \textit{if } n \textit{ is odd or } A \leq B, \\ \leq O, & \textit{if } n \textit{ is even and } A \geq B. \end{cases}$$

(d) *If $x < 0$ then*

$$\Delta_{i,x}^{[n]}(A|B) \begin{cases} \leq O, & \textit{if } n \textit{ is odd or } A \leq B, \\ \geq O, & \textit{if } n \textit{ is even and } A \geq B. \end{cases}$$

Proof. By Proposition 2.1 and Theorem 2.2, $\Delta_{1,x}^{[n]}(A|B)$ satisfies (a), (b), (c) and (b).

Let $\lambda > 0$. As with the proof of Theorem 2.2, the corresponding function $f_2^{[n]}(\lambda, x)$ for $\Delta_{2,x}^{[n]}(A|B)$ is represented by using $\theta_2 \in (0, 1)$ as follows:

$$f_2^{[n]}(\lambda, x) = \frac{x}{(n+1)!}\lambda^{(1-\theta_2)x}(\log\lambda)^{n+1}.$$

We obtain (a) since $f_2^{[n]}(\lambda, 0) = 0$ holds. Let $x \neq 0$. Then we have (b) since $f_2^{[n]}(\lambda, x) = 0$ if and only if $\lambda = 1$ holds. Assume that $x > 0$. Since $f_2^{[n]}(\lambda, x) \geq 0$ holds if n is odd or $\lambda \geq 1$, and $f_i^{[n]}(\lambda, x) \leq 0$ holds if n is even and $0 < \lambda \leq 1$. Hence, we have (c). We can get (d) in the same way as (c). □

Theorem 3.3. *Let A and B be strictly positive operators, $n \in \mathbb{N}$ and $x \in \mathbb{R}$. Then the followings hold for $i = 3, 4$:*

(a) $\Delta_{i,1}^{[n]}(A|B) = O$.

(b) *If $x \neq 1$ then*

$$\Delta_{i,x}^{[n]}(A|B) = O \ \textit{ if and only if } \ A = B.$$

(c) *If $x < 1$ then*

$$\Delta_{i,x}^{[n]}(A|B) \begin{cases} \geq O, & \textit{if } n \textit{ is odd or } A \leq B, \\ \leq O, & \textit{if } n \textit{ is even and } A \geq B. \end{cases}$$

(d) *If $x > 1$ then*

$$\Delta_{i,x}^{[n]}(A|B) \begin{cases} \leq O, & \textit{if } n \textit{ is odd or } A \leq B, \\ \geq O, & \textit{if } n \textit{ is even and } A \geq B. \end{cases}$$

Proof. The corresponding functions $f_3^{[n]}(\lambda, x)$ and $f_4^{[n]}(\lambda, x)$ for $\Delta_{3,x}^{[n]}(A|B)$ and $\Delta_{4,x}^{[n]}(A|B)$ are represented by using θ_2, $\theta_3 \in (0,1)$ as follows, respectively ($\lambda > 0$):

$$f_3^{[n]}(\lambda, x) = \frac{1-x}{(n+1)!}\lambda^{(1-\theta_3)x+\theta_3}(\log\lambda)^{n+1},$$

$$f_4^{[n]}(\lambda, x) = \frac{1-x}{(n+1)!}\lambda^{\theta_4 x+(1-\theta_4)}(\log\lambda)^{n+1}.$$

We obtain the assertions in the same way as Theorem 3.2. □

Corollary 3.4. (Proposition 2.1 and Proposition 4.2 in [16]) *Let A and B be strictly positive operators and $n \in \mathbb{N}$. Then the following holds:*

(a) $D_{FK}^{[n]}(A|B) = O \iff A = B,$

(b) $D_{FK}^{[n]}(A|B) = O \begin{cases} \geq O, & \text{if } n \text{ is odd or } A \leq B, \\ \leq O, & \text{if } n \text{ is even and } A \geq B. \end{cases}$

By Proposition 2.4, $\Delta_{1,x}^{[n]}(A \natural_r B|A \natural_s B) = (s-r)^n(A \natural_r B)A^{-1}\mathscr{D}_{(s-r)x}^{[n]}(A|B)$ holds for $n \in \mathbb{N}$ and r, s, $x \in \mathbb{R}$. We can also obtain similar results for remaining.

Theorem 3.5. *Let A and B be strictly positive operators, $n \in \mathbb{N}$ and r, s, $x \in \mathbb{R}$. Then the followings hold:*

(a) $\Delta_{2,x}^{[n]}(A \natural_r B|A \natural_s B) = -(s-r)^n(A \natural_{(1-x)r+xs} B)A^{-1}\mathscr{D}_{-(s-r)x}^{[n]}(A|B),$

(b) $\Delta_{3,x}^{[n]}(A \natural_r B|A \natural_s B) = (s-r)^n(A \natural_{(1-x)r+xs} B)A^{-1}\mathscr{D}_{(s-r)(1-x)}^{[n]}(A|B),$

(c) $\Delta_{4,x}^{[n]}(A \natural_r B|A \natural_s B) = -(s-r)^n(A \natural_s B)A^{-1}\mathscr{D}_{(s-r)(x-1)}^{[n]}(A|B).$

Proof. For r, s, $x \in \mathbb{R}$, $(A \natural_r B) \natural_x (A \natural_s B) = A \natural_{(1-x)r+xs} B$ holds (cf. (1) in Lemma 2.2 in [13]). By using Proposition 2.4, these are shown as follows:

(a)
$$\begin{aligned}&\Delta_{2,x}^{[n]}(A \natural_r B|A \natural_s B)\\ &= -((A \natural_r B) \natural_x (A \natural_s B))(A \natural_r B)^{-1}(s-r)^n(A \natural_r B)A^{-1}\mathscr{D}_{-(s-r)x}^{[n]}(A|B)\\ &= -(s-r)^n(A \natural_{(1-x)r+xs} B)A^{-1}\mathscr{D}_{-(s-r)x}^{[n]}(A|B).\end{aligned}$$

(b)
$$\begin{aligned}&\Delta_{3,x}^{[n]}(A \natural_r B|A \natural_s B)\\ &= ((A \natural_r B) \natural_x (A \natural_s B))(A \natural_r B)^{-1}(s-r)^n(A \natural_r B)A^{-1}\mathscr{D}_{(s-r)(1-x)}^{[n]}(A|B)\\ &= (s-r)^n(A \natural_{(1-x)r+xs} B)A^{-1}\mathscr{D}_{(s-r)(1-x)}^{[n]}(A|B).\end{aligned}$$

(c)
$$\begin{aligned}&\Delta_{4,x}^{[n]}(A \natural_r B|A \natural_s B)\\ &= -(A \natural_s B)(A \natural_r B)^{-1}(s-r)^n(A \natural_r B)A^{-1}\mathscr{D}_{(s-r)(x-1)}^{[n]}(A|B)\\ &= -(s-r)^n(A \natural_s B)A^{-1}\mathscr{D}_{(s-r)(x-1)}^{[n]}(A|B).\end{aligned}$$

□

In the following sense, $\Delta_{1,\alpha}(A|B)$ and $\Delta_{4,\alpha}(A|B)$ are symmetric as well as $\Delta_{2,\alpha}(A|B)$ and $\Delta_{3,\alpha}(A|B)$ are:

$$\Delta_{1,1-\alpha}(B|A) = T_{1-\alpha}(B|A) - S(B|A) = T_{1-\alpha}(B|A) + S_1(A|B) = \Delta_{4,\alpha}(A|B),$$

$$\Delta_{2,1-\alpha}(B|A) = S_{1-\alpha}(B|A) - T_{1-\alpha}(B|A) = -T_{1-\alpha}(B|A) - S_\alpha(A|B) = \Delta_{3,\alpha}(A|B).$$

These properties are some kind of duality. By Proposition 3.1, similar properties hold between $\Delta_{1,x}^{[n]}(A|B)$ and $\Delta_{4,x}^{[n]}(A|B)$ and between $\Delta_{2,x}^{[n]}(A|B)$ and $\Delta_{3,x}^{[n]}(A|B)$.

Theorem 3.6. *Let A and B be strictly positive operators, $n \in \mathbb{N}$ and $x \in \mathbb{R}$. Then the followings hold:*

(a) $\Delta_{1,1-x}^{[n]}(B|A) = (-1)^{n+1}\Delta_{4,x}^{[n]}(A|B)$,

(b) $\Delta_{2,1-x}^{[n]}(B|A) = (-1)^{n+1}\Delta_{3,x}^{[n]}(A|B)$.

In [14], we have shown the following relations between $\Delta_{i,\alpha}(A|B)$ and the Petz-Bregman divergence:

$$\Delta_{1,\alpha}(A|B) = \frac{1}{\alpha}D_{FK}(A|A \natural_\alpha B), \qquad \Delta_{2,\alpha}(A|B) \equiv \frac{1}{\alpha}D_{FK}(A \natural_\alpha B|A),$$

$$\Delta_{3,\alpha}(A|B) \equiv \frac{1}{1-\alpha}D_{FK}(A \natural_\alpha B|B), \qquad \Delta_{4,\alpha}(A|B) \equiv \frac{1}{1-\alpha}D_{FK}(B|A \natural_\alpha B).$$

By Proposition 2.6, the corresponding relation between the n-th operator valued divergence $\Delta_{1,x}^{[n]}(A|B)$ and the n-th Petz-Bregman divergence holds:

$$\Delta_{1,x}^{[n]}(A|B) = \frac{1}{x^n}D_{FK}^{[n]}(A|A \natural_x B).$$

We show the corresponding relations between remaining $\Delta_{i,x}^{[n]}(A|B)$ $(2 \le i \le 4)$ and the n-th Petz-Bregman divergence. The next theorem comes from Corollary 2.5, Proposition 2.6 and Theorem 3.6.

Theorem 3.7. *Let A and B be strictly positive operators, $n \in \mathbb{N}$ and $x \in \mathbb{R}$. Then the followings hold:*

(a) $\Delta_{2,x}^{[n]}(A|B) = (-1)^{n+1}\dfrac{1}{x^n}D_{FK}^{[n]}(A \natural_x B|A)$,

(b) $\Delta_{3,x}^{[n]}(A|B) = \dfrac{1}{(1-x)^n}D_{FK}^{[n]}(A \natural_x B|B)$,

(c) $\Delta_{4,x}^{[n]}(A|B) = (-1)^{n+1}\dfrac{1}{(1-x)^n}D_{FK}^{[n]}(B|A \natural_x B)$.

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