# INVARIANT MEASURE OF ISOMETRIC ACTIONS ON METRIC SPACE

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Abstract. The late Prof. Y. Mibu proved that there exists an invariant measure under a condition that the action is transitive. Our aim in this note is to show that when space is a metric space, we do not need this condition.

**1 Introduction.** In [2] Prof. Matsumoto approched this problem from an angle of ergodic theory. Hence it is hard for us to understand its proof. Our proof is done by reducing the result of Mibu. Hence the proof can understand easily.

**2 Notation and preparation.** In this section we explain notation used in this note and do preparation for the proof of main theorem.

**Definition 2.1.** A metric space $X$ is called proper if any closed ball $B_r(x) = \{y \in X \mid d(x,y) \leq r\}$ is compact for any $x \in X$ and $r > 0$. (The word comes from the properness of the distance function from $x$.)

Let $X$ be a proper metric space and let $G = \{f \mid f\colon X \to X \text{ surjective, isometric}\}$ be the set of all the surjective isometries. We denote by $\mathcal{K}(X)$ (resp. $\mathcal{O}(X)$) the set of all compact (resp. open) subsets of $X$.

We introduce in $G$ the compact-open topology [1]. Put

$$(*) \qquad W(K,U) = \{f \in G \mid f(K) \subset U\}$$

for any $K$ (resp. $U$) belonging $\mathcal{K}(X)$ (resp. $\mathcal{O}(X)$).

**Definition 2.2.** For any finite subset $\{W(K_i,U_i) \mid 1 \leq i \leq n\}$ of the form (*), all of their meet $\bigcap_{i=1}^{n} W(K_i,U_i)$ form an open base. One says this topology the compact-open topology in $G$.

For any compact sets $\{K_1, K_2\}$ and open sets $\{U_1, U_2\}$ such that $K_1 \subset K_2$ and $U_2 \subset U_1$, we have $W(K_2,U_2) \subset W(K_1,U_1)$.

**Proposition 2.3.** *(i) $G$ is separable and metrizable with respect to the compact-open topology.*

*(ii) For any $x \in X$ and a bounded closed ball $B_r(x_0)$ of $X$, $W(\{x\}, B_r(x_0))$ is a compact set with respect to the compact-open topology.*

*Proof.* (i) Since $X$ is a proper metric space, $X$ is separable metric space. Let $D = \{d_k\}$ be a countable dense subset of $X$. Since $X$ is a proper metric space, the closed ball $B_r(x_0)$ is a compact set in $X$.

Let $V$ be a bounded open set in $X$. Let $B_r(x)$ be a closed ball. Since $X$ is a proper metric space, for any $r > 0$ and $x \in X$, $B_r(x)$ is compact. For any $K \in \mathcal{K}(X)$ there exists

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an $r \in Q^+$ such that $K \subset B_r(x)$, where $Q^+$ is the set of all positive rational numbers. Since $D$ is dense in $X$, there exists a $d_k$ in $D$ with $d_k \in V$. Hence there exists an open set $U_t(d_k)$ such that $d_k \in U_t(d_k) \subset \overline{U_t}(d_k) \subset V$ and $t \in Q^+$. We have $W(B_r(x), U_t(d_k)) \subset W(K, V)$. $\{W(B_r(d_l), U_t(d_k)) \,|\, r, t \in Q^+, d_k, d_l \in D\}$ is a countable open base and satisfies the regular condition. Hence $G$ is metrizable.

(ii) $W(B_r(x_0), \overline{U_t}(d_k))$ is closed with respect to the compact-open topology, for $\overline{U_t}(d_k)$ is compact. Let $x \in \underline{B_r(x_0)}$. By Ascoli's theorem $W(\{x\}, \overline{U_t}(d_k))$ is compact. Since $W(B_r(x), U_t(d_k)) \subset W(\{x\}, \overline{U_t}(d_k))$, $\overline{W}(B_r(x_0), U_t(d_k))$ is compact. Hence $G$ is locally compact. ☐

**3 Main theorem.** Let $Y$ be a locally compact metric space with a distance function $d(x,y)$. Assume a group $\Gamma$ acts on $Y$ isometrically. For $x$ in $Y$, put $\Gamma_x = \{\gamma x \,|\, \gamma \in \Gamma\}$, which is the orbit of $y$. We denote by $X$ the closure $\overline{\Gamma_x}$ of the orbit.

**Definition 3.1.** When all the orbits are dense, one says that the action is minimal.

In this section we prove the main theorem.

**Theorem 3.2** (main theorem)**.** *Let $Y$ be a locally compact metric space. Assume a group $\Gamma$ acts on $Y$ isometrically. Then there is a $\Gamma$-invariant Radon measure on $Y$.*

The next proposition is a key to prove the main theorem.

**Proposition 3.3.** *$\Gamma$ acts minimally on $X$. This means that any orbit contained in $X$ is dense in $X$.*

*Proof.* We shall show that for any $y, z \in X$ and $\epsilon > 0$, there is $\gamma_y \in \Gamma$ such that $d({\gamma_y}^{-1}y, z) < \epsilon$. The point $x$ used in the definition of $X$ has the property that for any $\epsilon > 0$ and $y \in X$, there is $\gamma_y$ such that $d(y, \gamma_y x) < \epsilon/2$, since $X$ is the closure of the orbit of $x$. Hence we have

$$d(\gamma_z {\gamma_y}^{-1} y, z) \le d(\gamma_z {\gamma_y}^{-1} y, \gamma_z x) + d(\gamma_z x, z) = d(y, \gamma_y x) + d(\gamma_z x, z) < \epsilon/2 + \epsilon/2 = \epsilon$$

and the proof is complete. ☐

This means that any orbit contained in $X$ is dense in $X$. Since a closed subspace of a locally compact metric space is again locally compact, we have only to show the following.

**Theorem 3.4.** *Let $X$ be a locally compact metric space. Assume a group $\Gamma$ acts on $X$ isometrically and minimally. Then there is a $\overline{\Gamma}$-invariant Radon measure on $X$.*

*Proof.* This is a special case of the main theorem of [2]. However if the space $X$ is proper, we have a conceptually easier proof, which we shall discuss below.

Let $G$ be the set of all the surjective isometries from $X$ to $X$. By proposition 2.3, if $X$ is a proper metric space, then standard argument shows that the group $G$ of all the surjective isometries forms a locally compact metrizable group with respect to the compact-open topology (see proposition 2.3).

The closure $\overline{\Gamma}$ in $G$ is also locally compact metrizable, and acts transitively on $X$, that is, for any $x, y \in X$ there exists an element $g$ in $\overline{\Gamma}$ such that $gx = y$. Indeed, by proposition 3.3, for any positive integer $n$, there exists a $\gamma_n \in \overline{\Gamma} \subset G$ such that $d(\gamma_n x, y) < 1/n$. For any compact neighbourhood $\overline{U}(y)$ of $y$, $W(\{x\}, \overline{U}(y))$ is compact. Hence there exists a positive integer $N$ such that $\gamma_n x \in \overline{U}(y)$ for any $n > N$. Accordingly, for any positive integer $n > N$, we have $\gamma_n \in W(\{x\}, \overline{U}(y))$. Hence there exists a convergent subsequence $\{\gamma_{n_j}\}$ of $\{\gamma_n\}$. Its limit $g$ is contained in the closure of $\Gamma \subset G$. Hence $gx = y$ holds, that is, the closure of $\Gamma$ acts transitively on $X$. Therefore, Theorem 3.4 follows from a result of Mibu. ☐

**The proof of main Theorem 3.2.** Let $Y$ be a locally compact metric space with a distance function $d(x, y)$. Put

$$G = C(Y, Y) = \{f \mid f : Y \to Y \text{ surjective, isometric}\}.$$

Let $\Gamma_x = \{\gamma x \mid \gamma \in \Gamma\}$. $\Gamma$ acts isometrically on $Y$, that is, for any $x, y \in Y$ and $\gamma \in \Gamma$, $d(x, y) = d(\gamma x, \gamma y)$ holds. $\Gamma$ acts minimally, that is, any orbit $\Gamma_x$ is dense in $Y$. Assume the $Y$ is proper. A group $G$ is dense in $Y$. A group $G$ is locally compact and metrizable with the compact-open topology and acts transitively on $Y$, that is, for any $x, y$ in $Y$ there exists $g$ in $\overline{\Gamma}$ such that $gx = y$. By the result of Mibu we have $\Gamma$-invariant measure.

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## References

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