On subalmost contra-*b*-continuous functions

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Abstract. The purpose of this paper is to introduce a new class functions called, subalmost contra-b-continuous functions. Also, we obtain its characterizations and its basic properties.

1 Introduction Generalized open sets play a very important role in General Topology and they are now the research topics of many topologists worldwide. Indeed a significant theme in General Topology and Real analysis concerns the various modified forms of continuity, seperation axioms etc. by utilizing generalized open sets. One of the most well known notions and also an inspiration source is the notion of b-open sets introduced by Andrijević in 1996. Andrijević studied several fundamental and interesting properties of b-open sets. The purpose of this paper is to introduce a new class functions called, subalmost contra-b-continuous functions. Also, we obtain its characterizations and its basic properties.

2 Preliminaries Throughout the paper (X, τ) and (Y, σ) (or simply X and Y) represent topological spaces on which no separation axioms are assumed unless otherwise mentioned. For a subset A of a topological space (X, τ), $Cl(A)$, $Int(A)$ and A^c denote the closure of A, the interior of A and the complement of A in X, respectively. A subset A of X is said to be regular open [14] (resp. semi-open [8], α-open [10], b-open [2]($= \gamma$-open [6])) if $A = Int(Cl(A))$ (resp. $A \subset Cl(Int(A)), A \subset Int(Cl(Int(A)))), A \subset (Int(Cl(A)) \cup Cl(Int(A)))$. The family of all α-open (resp. semi-open, regular open, b-open) subsets of X is denoted by $\alpha(X)$ (resp. $SO(X)$, $RO(X)$, $BO(X)$). The family of all semi-open (resp. regular open, b-closed) subsets of X containing the point x is denoted by $SO(X, x)$ (resp. $RO(X, x)$, $BC(X, x)$). The complement of a semi-open (resp. regular open, b-open) set is called a semiclosed [4] (resp. regular closed, b-closed) set. The intersection of all semi-closed (resp. b-closed) sets containing A is called the semi-closure [3] (resp. b-closure [2]) of A and is denoted by $sCl(A)$ (resp. $bCl(A)$). A subset A is b-closed if and only if A=$bCl(A)$. The θ-semi-closure [7] (resp. the semi-θ-closure [5]) of A, denoted by θ-$sCl(A)$ (resp. $sCl_\theta(A)$), is defined to be the set of all $x \in X$ such that $A \cap Cl(U) \neq \varnothing$ (resp. $A \cap sCl(U) \neq \emptyset$) for every $U \in SO(X, x)$. A subset A is called θ-semi-closed [7] (resp. semi-θ-closed [5]) if and only if $A = \theta$-$sCl(A)$ (resp. $A = sCl_\theta(A)$). The complement of a θ-semi-closed set (resp. semi-θ-closed set) is called a θ-semi-open [7] (resp. semi-θ-open [5]) set. It is well known that θ-$sCl(A) \neq sCl_\theta(A)$ for some subset A of a topological space (X, τ). A function $f : (X, \tau) \to (Y, \sigma)$ is said to be b-continuous [6] (resp. contra-b-continuous [9]) if $f^{-1}(V)$ is b-open (resp. b-closed) set in (X, τ) for each open set V of (Y, σ).

Definition 2.1 [1] A function $f : (X, \tau) \to (Y, \sigma)$ is said to be *almost contra-b-continuous* if $f^{-1}(V) \in BC(X)$ for each $V \in RO(Y)$.

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Lemma 2.2 [11, Lemma 5.3] *If $B \subset A \subset X$ and A is α-open in (X,τ), then $bCl_A(B) = bCl(B) \cap A$.*

Lemma 2.3 [5, Lemma 2.1] *If V is an open set, then $sCl(V) = Int(Cl(V))$.*

Lemma 2.4 [5, Proposition 2.1(a)] *If V is a semi-open set, then $sCl_\theta(V) = sCl(V)$.*

3 Subalmost contra-b-continuous functions

Definition 3.1 A function $f : (X,\tau) \to (Y,\sigma)$ is said to be *subalmost contra-b-continuous* if there exists an open base $\mathcal{B}$ for the topology on Y for which $bCl(f^{-1}(V)) \subset f^{-1}(sCl(V))$ for every $V \in \mathcal{B}$. Sometimes, f is called *subalmost contra-b-continuous with respect to an open base $\mathcal{B}$.*

Theorem 3.2 *For a function $f : (X,\tau) \to (Y,\sigma)$, the following statements are equivalent:*
(1) *f is subalmost contra-b-continuous with respect to an open base $\mathcal{B}$.*
(2) *$bCl(f^{-1}(V)) \subset f^{-1}(sCl_\theta(V))$ for every $V \in \mathcal{B}$.*
(3) *$bCl(f^{-1}(V)) \subset f^{-1}(Int(Cl(V))$ for every $V \in \mathcal{B}$.*

Proof. (1) $\Leftrightarrow$ (2): The proof follows from Lemma 2.4 and a well known property that $\tau \subset SO(X)$.
(1) $\Leftrightarrow$ (3): The proof follows from Lemma 2.3. □

Theorem 3.3 *If $f : (X,\tau) \to (Y,\sigma)$ is subweakly b-continuous [12] and satisfies the additional property that images of b-closed sets are open, then f is subalmost contra-b-continuous.*

Proof. By the definition of subweakly b-continuity [12, Definition 3.1], there exists an open base $\mathcal{B}$ for the topology on Y such that $bCl(f^{-1}(V)) \subset f^{-1}(Cl(V))$ for every $V \in \mathcal{B}$. Since images of b-closed sets are open, $f(bCl(f^{-1}(V))) \subset Int(Cl(V))$ or $bCl(f^{-1}(V)) \subset f^{-1}(Int(Cl(V)))$. Therefore, by Theorem 3.2, f is subalmost contra-b-continuous. □

Recall that for a function $f : (X,\tau) \to (Y,\sigma)$, the subset $\{(x, f(x)); x \in X\} \subset X \times Y$ is called the graph of f and is denoted by $G(f)$.

Definition 3.4 A graph $G(f)$ of a function $f : (X,\tau) \to (Y,\sigma)$ is said to be *regular b-closed* if for each $(x,y) \in (X \times Y) \setminus G(f)$, there exist $U \in BC(X,x)$ and $V \in RO(Y,y)$ such that $(U \times V) \cap G(f) = \emptyset$.

Theorem 3.5 *A graph $G(f)$ of a function $f : (X,\tau) \to (Y,\sigma)$ is regular b-closed if and only if for each $(x,y) \in (X \times Y) \setminus G(f)$, there exist $U \in BC(X,x)$ and $V \in RO(Y,y)$ such that $f(U) \cap V = \emptyset$.* □

Theorem 3.6 *If $f : (X,\tau) \to (Y,\sigma)$ is subalmost contra-b-continuous and (Y,σ) is a Hausdorff space, then the graph of f, $G(f)$ is regular b-closed.*

Proof. Let $(x,y) \in X \times Y \setminus G(f)$. Then $y \neq f(x)$. Let $\mathcal{B}$ be an open base for the topology on Y such that $bCl(f^{-1}(V)) \subset f^{-1}(Int(Cl(V)))$ for every $V \in \mathcal{B}$. Since Y is Hausdorff, there exist disjoint open sets V and W such that $f(x) \in V$, $y \in W$, and $V \in \mathcal{B}$. Then, since $Int(Cl(V)) \cap Int(Cl(W)) = \emptyset$, it follows that $(x,y) \in bCl(f^{-1}(V)) \times Int(Cl(W)) \subset (X \times Y) \setminus G(f)$, which proves that $G(f)$ is regular b-closed. □

Corollary 3.7 *If $f : (X,\tau) \to (Y,\sigma)$ is almost contra-b-continuous and (Y,σ) is a Hausdorff space, then the graph $G(f)$ is regular b-closed.* □

Theorem 3.8 *Let $f : (X,\tau) \to (Y,\sigma)$ be a function and let $\mathcal{B}$ be an open base for σ. Let $\mathcal{C} := \{U \times V : U \in \tau, V \in \mathcal{B}\}$. If the graph function of $f, g : X \to X \times Y$ is subalmost contra-b-continuous with respect to $\mathcal{C}$, then f is subalmost contra-b-continuous with respect to $\mathcal{B}$.*

Proof. If $V \in \mathcal{B}$, then $bCl(f^{-1}(V)) = bCl(g^{-1}(X \times V)) \subset g^{-1}(sCl(X \times V)) = g^{-1}(X \times sCl(V)) = f^{-1}(sCl(V))$. Hence f is subalmost contra-b-continuous with respect to $\mathcal{B}$. □

Recall that a space (X, τ) is said to be *zero-dimensional* provided that (X, τ) has a clopen base (cf. [15]).

Theorem 3.9 *If $f : (X, \tau) \to (Y, \sigma)$ is subalmost contra-b-continuous and X is zero-dimensional, then the graph function of $f, g : X \to X \times Y$ is subalmost contra-b-continuous.*

Proof. Let $\mathcal{B}$ be an open base for the topology on Y such that $bCl(f^{-1}(V)) \subset f^{-1}(Int(Cl(V)))$ for every $V \in \mathcal{B}$. Then $\mathcal{B}_1 = \{U \times V : U \subset X \text{ is clopen and } V \in \mathcal{B}\}$ is a base for the topology on $X \times Y$. For $U \times V \in \mathcal{B}_1$, we have $bCl(g^{-1}(U \times V)) = bCl(U \cap f^{-1}(V)) \subset U \cap bCl(f^{-1}(V)) \subset Int(Cl(U)) \cap f^{-1}(Int(Cl(V))) = g^{-1}(Int(Cl(U)) \times Int(Cl(V))) = g^{-1}(Int(Cl(U \times V)))$. Therefore the graph function g is subalmost contra-b-continuous. □

Definition 3.10 A topological space (X, τ) is said to be weakly Hausdorff [13] if each element of X is an intersection of regular closed sets.

Definition 3.11 A topological space (X, τ) is said to be b-T_1 [11] if for each pair of distinct points x and y of X, there exist b-open sets U and V containing x and y, respectively such that $y \notin U$ and $x \notin V$.

Theorem 3.12 *If $f : (X, \tau) \to (Y, \sigma)$ is a subalmost contra-b-continuous injection and (Y, σ) is weakly Hausdorff, then (X, τ) is b-T_1.*

Proof. Let x_1 and x_2 be distinct points in X. Then $f(x_1) \neq f(x_2)$, and since Y is weakly Hausdorff, there exists a regular closed subset F of Y such that $f(x_1) \in F$ and $f(x_2) \notin F$. Then $f(x_2) \in X \setminus F$, which is regular open. Let $\mathcal{B}$ be an open base for the topology on Y such that $bCl(f^{-1}(V)) \subset f^{-1}(sCl(V))$ for every $V \in \mathcal{B}$. Then let $V \in \mathcal{B}$ such that $f(x_2) \in V \subset Y \setminus F$. Then $x_2 \notin X \setminus bCl(f^{-1}(V))$, which is b-open. Also $f(x_1) \in F$, which is regular closed and therefore also semi-open. Since $F \cap V = \emptyset$, it follows that $f(x_1) \notin sCl(V)$, and hence $x_1 \notin f^{-1}(sCl(V))$. Then $x_1 \in X \setminus f^{-1}(sCl(V)) \subset X \setminus bCl(f^{-1}(V))$. Hence $X \setminus bCl(f^{-1}(V))$ is a b-open set containing x_1 but not x_2, which proves that X is b-T_1. □

Theorem 3.13 *If $f : (X, \tau) \to (Y, \sigma)$ is subalmost contra-b-continuous with respect to the open base $\mathcal{B}$ for the topology on Y and A is an α-open subset of X, then $f_A : (A, \tau_A) \to (Y, \sigma)$ is subalmost contra-b-continuous with respect to $\mathcal{B}$, where τ_A is the relative topology for A and f_A is the restriction of f to A.*

Proof. Let $V \in \mathcal{B}$. Then $bCl_A(f_A^{-1}(V)) \subset A \cap bCl(f_A^{-1}(V)) = A \cap bCl(f^{-1}(V) \cap A) \subset A \cap bCl(f^{-1}(V)) \cap bCl(A) = A \cap bCl(f^{-1}(V)) \subset A \cap f^{-1}(sCl(V)) = f_A^{-1}(sCl(V))$. Hence, $f_A : A \to Y$ is subalmost contra-b-continuous with respect to $\mathcal{B}$. □

If we take $\mathcal{B}$ to be the topology on Y in the above theorem, we obtain the following result.

Corollary 3.14 *If $f : (X, \tau) \to (Y, \sigma)$ is almost contra-b-continuous and A is an α-open subset of X, then $f_A : (A, \tau_A) \to (Y, \sigma)$ is subalmost contra-b-continuous .* □

Theorem 3.15 *If $f : (X, \tau) \to (Y, \sigma)$ is subalmost contra-b-continuous and A is an open subset of (Y, σ) with $f(X) \subset A$, then $f : (X, \tau) \to (A, \sigma_A)$ is subalmost contra-b-continuous.*

Proof. Let $\mathcal{B}$ be an open base for the topology on Y such that $bCl(f^{-1}(V)) \subset f^{-1}(sCl(V))$ for every $V \in \mathcal{B}$. Then $\mathcal{B}_A := \{V \cap A : V \in \mathcal{B}\}$ is an open base for the relative topology σ_A on A. For $V \cap A \in \mathcal{B}_A$, where $V \in \mathcal{B}$, we have $bCl(f^{-1}(V \cap A)) = bCl(f^{-1}(V)) \subset f^{-1}(Int(Cl(V))) = f^{-1}(Int(Cl(V)) \cap A) \subset f^{-1}(Int_A(Cl_A(V \cap A)))$, which proves that $f : (X, \tau) \to (A, \sigma_A)$ is subalmost contra-b-continuous with respect to the base $\mathcal{B}_A$. □

Definition 3.16 The θ-closure [16] of A, denoted by $Cl_\theta(A)$, is defined to be the set of all $x \in X$ such that $Cl(U) \cap A \neq \emptyset$ for every open set U containing x. A subset A is called θ-closed [16] if and only if $A = Cl_\theta(A)$. The complement of a θ-closed set is called a θ-open set [16].

Theorem 3.17 *If $f : (X, \tau) \to (Y, \sigma)$ is subalmost contra-b-continuous, then for every θ-open* (resp. *θ-closed*) *subset W of Y, $f^{-1}(W)$ is a union of b-closed sets* (resp. *an intersection of b-open sets*).

Proof. Let $\mathcal{B}$ be an open base for the topology on Y such that $bCl(f^{-1}(V)) \subset f^{-1}(sCl(V))$ for every $V \in \mathcal{B}$. Let W be a θ-open set of Y and let $x \in f^{-1}(W)$. Let $V \in \mathcal{B}$ such that $f(x) \in V \subset Cl(V) \subset W$. Then $x \in bCl(f^{-1}(V)) \subset f^{-1}(sCl(V)) \subset f^{-1}(Cl(V)) \subset f^{-1}(W)$. Since $bCl(f^{-1}(V))$ is b-closed, it follows that $f^{-1}(W)$ is a union of b-closed sets. An argument using complements will prove the remaining part of the theorem. □

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