JI-DISTRIBUTIVE, DUALLY QUASI-DE MORGAN SEMI-HEYTING AND HEYTING ALGEBRAS

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Dedicated to Professor P.N. Shivakumar
A Great Humanitarian who changed the course of my life

ABSTRACT. The variety **DQD** of dually quasi-De Morgan semi-Heyting algebras and several of its subvarieties were investigated in the series [26] - [31]. In this paper we define and investigate a new subvariety **JID** of **DQD**, called "JI-distributive, dually quasi-De Morgan semi-Heyting algebras", defined by the identity: $x' \vee (y \to z) \approx (x' \vee y) \to (x' \vee z)$, as well as the (closely related) variety **DSt** of dually Stone semi-Heyting algebras. Firstly, we prove that **DSt** and **JID** are discriminator varieties of level 1 and level 2 respectively. Secondly, we give a characterization of subdirectly irreducible algebras of the subvariety $\mathbf{JID_1}$ of **JID** of level 1. As applications, we derive that the variety $\mathbf{JID_1}$ is the join of the variety **DSt** and the variety of De Morgan Boolean semi-Heyting algebras, give a concrete description of the subdirectly irreducible algebras in the subvariety $\mathbf{JIDL_1}$ of $\mathbf{JID_1}$ defined by the linear identity: $(x \to y) \vee (y \to x) \approx 1$, and deduce that the variety $\mathbf{JIDL_1}$ is the join of the variety **DStHC** generated by the dually Stone Heyting chains and the variety generated by the 4-element De Morgan Boolean Heyting algebra. Furthermore, we present an explicit description of the lattice of subvarieties of $\mathbf{JIDL_1}$ and equational bases for all subvarieties of $\mathbf{JIDL_1}$. Finally, we prove that the amalgamation property holds for all subvarieties of **DStHC**.

1. Introduction

The De Morgan (strong) negation and the pseudocomplement are two of the fairly well known negations that generalize the classical negation. A common generalization of these two negations led to a

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new variety of algebras, called "semi-De Morgan algebras", which was investigated in [24]. Several subvarieties of this variety, including a subvariety called "(upper) quasi-De Morgan algebras" were also studied in [24].

In a different vein, semi-Heyting algebras were introduced in [25] as an abstraction of Heyting algebras. Using the dual version of quasi-De Morgan negation, an expansion of semi-Heyting algebras, called "dually quasi-De Morgan semi-Heyting algebras (**DQD**, for short)" was defined and investigated in [26], as a common generalization of De Morgan (or symmetric) Heyting algebras [23] (see also [19]) and dually pseudocomplemented Heyting algebras [22]. It may also be mentioned here that [8] has proposed recently a propositional logic, called "Dually quasi-De Morgan semi-Heyting logic", which has dually quasi-De Morgan semi-Heyting algebras as an equivalent algebraic semantics.

Several new subvarieties of **DQD** were studied in [26]- [31], including the variety **DStHC** generated by the dually Stone Heyting chains (i.e., the expansion of the Gödel variety by the dual Stone operation), the variety **DMB** of De Morgan Boolean semi-Heyting algebras and the variety **DMBH** generated by the 4-element De Morgan Boolean Heyting algebra. These investigations led us naturally to the problem of equational axiomatization for the join of the variety **DStHC** and the variety **DMBH**. Our investigations into this problem led us to the results of the present paper that include a solution to the just mentioned problem.

In this paper we define and investigate a new subvariety of **DQD**, called "JI-distributive, dually quasi-De Morgan semi-Heyting algebras (**JID**, for short)", defined by the identity: $x' \vee (y \to z) \approx (x' \vee y) \to (x' \vee z)$, as well as the (closely related) variety **DSt** of dually Stone semi-Heyting algebras. We first prove that **DSt** and **JID** are discriminator varieties of level 1 and level 2 respectively (see Section 2 for definitions). Secondly, we prove that the lattice of subvarieties of **DStHC** is an $\omega + 1$-chain. Thirdly, we give a characterization of subdirectly irreducible algebras of the subvariety $\mathbf{JID_1}$ of level 1. As a first application of it, we derive that the variety $\mathbf{JID_1}$ is the join of the variety **DSt** and the variety **DMB**. As a second application, we give a concrete description of the subdirectly irreducible algebras in the subvariety $\mathbf{JIDL_1}$ of $\mathbf{JID_1}$ defined by the linear identity: $(x \to y) \vee (y \to x) \approx 1$, and deduce that the variety $\mathbf{JIDL_1}$ is the join of the variety **DStHC** generated by the dually Stone Heyting chains and the variety **DMBH**. Other applications include a description of the lattice of subvarieties of $\mathbf{JIDL_1}$, equational bases of all subvarieties of $\mathbf{JIDL_1}$, and the fact that the amalgamation property holds in all subvarieties of **DStHC**.

More explicitly, the paper is organized as follows: In Section 2 we recall definitions, notations and results from [26], [27] and [28] and also prove some new results needed in the rest of the paper. In Section 3, we define the variety **JID** of JI-distributive, dually quasi-De Morgan semi-Heyting algebras and give some arithmetical properties of **JID**. In particular, we show that **JID** satisfies the $\vee$-De Morgan law and the level 2 identity: $(x \wedge x'^{*})'^{*} \approx (x \wedge x'^{*})'^{*'*}$. These two propertes allow us to apply [26, Corollary 8.2(a)] to deduce that **JID** is a discriminator variety. These properties also play a crucial role in the rest of the paper. Section 4 will prove that the variety **DSt** is a discriminator variety of level 1. It will also present some properties of **DSt**, which, besides being of interest in their own right, will also be useful in the later sections. It is also proved that the lattice of subvarieties of **DStHC** is an $\omega + 1$-chain. In Section 5, we give a characterization of subdirectly irreducible (= simple) algebras in the variety $\mathbf{JID_1}$ of level 1 and deduce that $\mathbf{JID_1}$ is the join of **DSt** and the variety **DMB** of De Morgan Boolean semi-Heyting algebras. Several applications of this characterization are given in Section 6 and Section 7. We investigate, in Section 6, the variety $\mathbf{JIDL_1}$ of JI-distributive, dually quasi-De Morgan, linear semi-Heyting algebras of level 1. An explicit description of subdirectly irreducible algebras in $\mathbf{JIDL_1}$ is given, and from this description it is deduced that $\mathbf{JIDL_1} = \mathbf{DStHC} \vee \mathbf{DMBH}$, which solves the aforementioned problem of axiomatizing the join of **DStHC** and **DMBH**. In Section 7, some applications of the just-mentioned result are given. It is shown that the lattice of subvarieties of $\mathbf{JIDL_1}$ is isomorphic to $\mathbf{1} \oplus [(\omega + \mathbf{1}) \times \mathbf{2}]$, where $\mathbf{1}$ and $\mathbf{2}$ are the 1-element and the 2-element lattices, respectively. Also, (small) equational bases for all subvarieties of $\mathbf{JIDL_1}$ are given. Finally, it is shown that all subvarieties of **DStHC** have the amalgamation property.

2. **Preliminaries**

In this section we recall some notions and known results needed to make this paper as self-contained as possible. However, for other information used but not mentioned here, we refer the reader to [5], [7] and [20].

An algebra $\mathbf{L} = \langle L, \vee, \wedge, \to, 0, 1\rangle$ is a *semi-Heyting algebra* ([25]) if $\langle L, \vee, \wedge, 0, 1\rangle$ is a bounded lattice and $\mathbf{L}$ satisfies:

(SH1) $x \wedge (x \to y) \approx x \wedge y$,
(SH2) $x \wedge (y \to z) \approx x \wedge [(x \wedge y) \to (x \wedge z)]$,

(SH3) $x \to x \approx 1$.

Semi-Heyting algebras are distributive and pseudocomplemented, with $a^* := a \to 0$ as the pseudocomplement of an element a.
Let **L** be a semi-Heyting algebra. **L** is a *Heyting algebra* if **L** satisfies:

(H) $(x \wedge y) \to y \approx 1$.

L is a *Boolean semi-Heyting algebra* if **L** satisfies:

(Bo) $x \vee x^* \approx 1$.

L is a *Boolean Heyting algebra* if **L** is a Heyting algebra and satisfies (Bo).

The following definition, taken from [26], is central to this paper.

DEFINITION 2.1. *An algebra* $\mathbf{L} = \langle L, \vee, \wedge, \to, ', 0, 1\rangle$ *is a semi-Heyting algebra with a dual quasi-De Morgan operation or dually quasi-De Morgan semi-Heyting algebra* (**DQD**-algebra, for short) if $\langle L, \vee, \wedge, \to, 0, 1\rangle$ is a semi-Heyting algebra, and **L** satisfies:

(a) $0' \approx 1$ and $1' \approx 0$,
(b) $(x \wedge y)' \approx x' \vee y'$,
(c) $(x \vee y)'' \approx x'' \vee y''$,
(d) $x'' \leq x$.

Let **L** be a **DQD**-algebra. **L** is a *dually pseudocomplemented semi-Heyting algebra* (**DPC**-algebra) (see [24]) if **L** satisfies:

(e) $x \vee x' \approx 1$.

L is a *dually Stone semi-Heyting algebra* (**DSt**-algebra) if **L** satisfies the dual Stone identity:

(DSt) $x' \wedge x'' \approx 0$.

It should be noted that if (DSt) holds in a **DQD**-algebra **L**, then (e) holds in **L** as well, and hence $'$ is indeed the dual pseudocomplement satisfying the dual Stone identity, and so **L** has, indeed, a dual Stone algebra as a reduct. **L** is a *De Morgan semi-Heyting algebra* (**DM**-algebra) if **L** satisfies:

(DM) $x'' \approx x$.

The varieties of **DQD**-algebras, **DPC**-algebras, **DSt**-algebras, **DM**-algebras are denoted, respectively, by **DQD**, **DPC**, **DSt**, and **DM**. If the underlying semi-Heyting algebra of a **DQD**-algebra is a Heyting algebra, then we add "**H**" at the end of the names of the varieties that will be considered in the sequel. Thus, for example, **DStH** denotes the variety of dually Stone Heyting algebras.

The following lemmas are basic to this paper. The proof of the first lemma is straightforward and is left to the reader.

LEMMA 2.2. *Let* $\mathbf{L} \in \mathbf{DQD}$ *and let* $x, y, z \in L$. *Then*

(i) $1'^{*} = 1$, *and* $1 \to x = x$,
(ii) $x \leq y$ *implies* $x' \geq y'$,
(iii) $(x \wedge y)'^{*} = x'^{*} \wedge y'^{*}$,
(iv) $x''' = x'$,
(v) $(x \vee y)' = (x'' \vee y)'$,
(vi) $x \wedge [y \vee (x \to z)] = x \wedge (y \vee z)$,
(vii) $x \wedge (x \to y)'' \leq y$.

LEMMA 2.3. *Let* $L \in \mathbf{DQD}$ *and* $x, y \in L$. *Then*

(1) $(x \vee y)' \leq x' \to (x \vee y)'$,
(2) $[x \vee (y \vee z)']' = (x \vee y')' \vee (x \vee z')'$,
(3) $x \wedge [(x \to y) \vee z] = x \wedge (y \vee z)$,
(4) $y \wedge [x \to (y \wedge z)] = y \wedge (x \to z)$,
(5) $x \to (y \wedge z) \;\geq\; y \wedge (x \to z)$,
(6) $x \;\leq\; y \to (x \wedge y)$,
(7) $(x \vee y)' = x' \wedge [(x \vee y)' \vee \{x' \to (x \vee y)'\}'']$,
(8) $x \leq (x \to y) \to y$.

Proof.
(1) is straightforward to verify since $(x \vee y)' \leq x'$.

(2):
$$\begin{aligned} [(x \vee (y \vee z)']' &= [x'' \vee (y \vee z)']' && \text{by Lemma 2.2 (v)} \\ &= [x' \wedge (y \vee z)]'' \\ &= [(x' \wedge y) \vee (x' \wedge z)]'' \\ &= (x' \wedge y)'' \vee (x' \wedge z)'' \\ &= (x'' \vee y')' \vee (x'' \vee z')' \\ &= (x \vee y')' \vee (x \vee z')' && \text{by Lemma 2.2 (v)} \end{aligned}$$
(3) and (4) are easy to verify.

(5):
$$\begin{aligned} [x \to (y \wedge z)] \wedge y \wedge (x \to z) &= y \wedge [x \to (y \wedge z)] \wedge (x \to z) \\ &= y \wedge (x \to z) \text{ by (4).} \end{aligned}$$
(6): $x = x \wedge (y \to y) \leq y \to (x \wedge y)$ by (5).

(7): $(x \vee y)' = x' \wedge (x \vee y)'$.

$= x' \wedge [x' \to (x \vee y)']$

$= x' \wedge [\{x' \to (x \vee y)'\} \vee \{x' \to (x \vee y)'\}'']$

$= [x' \wedge (x \vee y)'] \vee [x' \wedge \{x' \to (x \vee y)'\}'']$

$= x' \wedge [(x \vee y)' \vee \{x' \to (x \vee y)'\}''],$

(8): $x \wedge [(x \to y) \to y] = x \wedge [\{x \wedge (x \to y)\} \to (x \wedge y)] = x \wedge [(x \wedge y) \to (x \wedge y)] = x \wedge 1 = x$, completing the proof. □

The following three 4-element algebras, called $\mathbf{D_1}$, $\mathbf{D_2}$, and $\mathbf{D_3}$ (following the notation of [26]), in **DQD**, play an important role in the sequel. All three of them have the Boolean lattice reduct with the universe $\{0, a, b, 1\}$, where b is the Boolean complement of a, and the operation $'$ is defined as follows: $a' = a$, $b' = b$, $0' = 1$, $1' = 0$, while the operation $\to$ is defined in Figure 1.

$\mathbf{D_1}$:

$\to$	0	1	a	b
0	1	0	b	a
1	0	1	a	b
a	b	a	1	0
b	a	b	0	1

$\mathbf{D_2}$:

$\to$	0	1	a	b
0	1	1	1	1
1	0	1	a	b
a	b	1	1	b
b	a	1	a	1

$\mathbf{D_3}$:

$\to$	0	1	a	b
0	1	a	1	a
1	0	1	a	b
a	b	a	1	0
b	a	1	a	1

Figure 1

Let **DQB** and **DMB** denote respectively the subvarieties of **DQD** and **DM** defined by (Bo). Also, by an earlier convention, **DQBH** and **DMBH** denote, respectively, the subvarieties of **DQB** and **DMB** defined by (H). $\mathbf{V}(\mathbf{K})$ denotes the variety generated by the class K of algebras in **DQD**. The following proposition is proved in ([26]) and is needed later in this paper.

PROPOSITION 2.4.

(a) $\mathbf{DQB} = \mathbf{DMB} = \mathbf{V}(\{\mathbf{D_1}, \mathbf{D_2}, \mathbf{D_3}\})$,

(b) $\mathbf{DQBH} = \mathbf{DMBH} = \mathbf{V}(\mathbf{D_2})$.

The following definition is from [26].

DEFINITION 2.5. *Let* $\mathbf{L} \in \mathbf{DQD}$ *and* $x \in \mathbf{L}$. *For* $n \in \omega$, *we define* $t_n(x)$ *recursively as follows:*

$$x^{0('*)} := x;$$
$$x^{(n+1)('*)} := (x^{n('*)})'^*, \text{ for } n \geq 0;$$
$$t_0(x) := x,$$
$$t_{n+1}(x) := t_n(x) \wedge x^{(n+1)('*)}, \text{ for } n \geq 0.$$

Let $n \in \omega$. *The subvariety* $\mathbf{DQD_n}$ *of level* n *of* $\mathbf{DQD}$ *is defined by the identity:*

(lev n) $$t_n(x) \approx t_{(n+1)}(x);$$

For a subvariety $\mathbf{V}$ *of* $\mathbf{DQD}$, *we let* $\mathbf{V_n} := \mathbf{V} \cap \mathbf{DQD_n}$.

Recall from [26] (or [27]) that **BDQDSH** is the subvariety of **DQD** (= **DQDSH**) defined by the identity:

(BL) $(x \vee y^*)' \approx x' \wedge y^{*\prime}$.

We will abbreviate **BDQDSH** by **BDQD**.

The following "simplicity condition", (SC), is crucial in the rest of the paper.

(SC) For every $x \in L$, if $x \neq 1$, then $x \wedge x'^* = 0$.

The following theorem, which was proved in [27, Corollary 4.1] (which is, in turn, a consequence of Corollaries 7.6 and 7.7 of [26]), will play a fundamental role in this paper.

THEOREM 2.6. [27, Corollary 4.1] *Let* $\mathbf{L} \in \mathbf{BDQD_1}$ *with* $|L| \geq 2$. *Then the following are equivalent:*

(1) **L** *is simple,*
(2) **L** *is subdirectly irreducible,*
(3) **L** *satisfies* (SC).

3. JI-distributive, dually quasi-De Morgan semi-Heyting algebras

The identity, $x \vee (y \to z) \approx (x \vee y) \to (x \vee z)$, was shown in [28, Corollary 3.55] to be an equational base for the variety generated by $\mathbf{D_2}$, relative to **DQD**. Let us refer to this identity as "strong

JI-distributive identity". We now introduce a slightly weaker identity, called "JI-distributive identity" (by restricting the first variable to "primed" elements). The subvariety **JID** of **DQD** defined by this identity and some of its subvarieties are the subject of our investigation in the rest of this paper.

DEFINITION 3.1. *The subvariety* **JID** *of* **DQD** *is defined by:*

(JID) $x' \vee (y \to z) \approx (x' \vee y) \to (x' \vee z)$ (restricted **D**istribution of **J**oin over **I**mplication).

Members of the variety **JID** are called "JI-distributive, dually quasi-De Morgan semi-Heyting algebras" and will be referred to as **JID**-algebras. Examples of **JID**-algebras come from a surprising source to which we shall now turn. But, first we need some notation.

A **DQD**-algebra is a **DQD**-chain if its lattice reduct is a chain. Let **DQDC** [**DPCC**] denote the variety generated by the **DQD**-chains [**DPC**-chains]. The following lemma provides an important class of examples of **JID**, which is partly the motivation for our interest in **JID**.

LEMMA 3.2. DPCC $\subseteq$ JID.

Proof. It suffice to show that **DPCC** $\models$ (JID). Let **A** be a **DPC**-chain and let $a \in A \setminus \{1\}$. Since **A** is a chain, we have $a' \leq a$ or $a \leq a'$, from which we get that $a \vee a' \leq a$ or $a \vee a' \leq a'$. Since **A** is dually pseudocomplemented, we have $a \vee a' = 1$, implying $a' = 1$, as $a \neq 1$. Now, it is routine to verify (JID) holds in **A**. □

For **L** a **DPC**-chain, it was observed in the proof of the preceding lemma that the dual pseudocomplement $'$ satisfies: $a' = 1$, if $a \neq 1$, and hence **L** $\models$ (DSt). Thus, we have the following corollary, where **DStC** denotes the variety generated by the dually Stone semi-Heyting chains.

COROLLARY 3.3. DPCC = DStC.

From now on, we use **DPCC** and **DStC** interchangeably. We note that $\mathbf{D_1}$, $\mathbf{D_2}$, and $\mathbf{D_3}$ are also examples of **JID**-algebras.

In the rest of this section we present several useful arithmetical properties of **JID**. Following our convention made earlier, **JIDH** denotes the subvariety of **JID** defined by the identity (H).

Throughout this section, we assume that L $\in$ JID.

LEMMA 3.4. *Let* $x, y, z \in$ **L**. *Then*

(1) $x' \to (x' \vee y) = x' \vee (x' \to y)$,
(2) $x' \to (x' \vee y) = x' \vee (0 \to y)$,

(3) $x' \vee (x' \to y) = x' \vee (0 \to y)$; *in particular,* $x' \vee x'^{*} = 1$,
(4) $(x' \vee y) \to x' = x' \vee y^{*}$,
(5) $(x' \vee y) \to x' = x' \vee (y \to x')$,
(6) $x' \vee (y \to x') = x' \vee y^{*}$,
(7) $x' \to (x \vee y)' = x'^{*} \vee (x \vee y)'$.

Proof. Observe that $x' \to (x' \vee y) = (x' \vee x') \to (x' \vee y) = x' \vee (x' \to y)$ by (JID), which proves (1). To prove (2), again using (JID), we get $x' \vee (0 \to y) = (x' \vee 0) \to (x' \vee y) = x' \to (x' \vee y)$. (3) is immediate from (1) and (2). For (4), $(x' \vee y) \to x' = (x' \vee y) \to (x' \vee 0) = x' \vee (y \to 0) = x' \vee y^{*}$, in view of (JID). Next, $(x' \vee y) \to x' = (x' \vee y) \to (x' \vee x') = x' \vee (y \to x')$, proving (5), and (6) is immediate from (4) and (5). For (7), we have

$$\begin{aligned} x' \to (x \vee y)' &= (x \vee y)' \vee [x' \to (x \vee y)'] \text{ by Lemma 2.3 (1)} \\ &= (x \vee y)' \vee x'^{*} \text{ by (6).} \end{aligned}$$

□

We now prove an important property of the variety **JID**, namely the $\vee$-De Morgan law. We denote by **Dms** the subvariety of **DQD** (called "dually ms semi-Heyting algebras") defined by

$$(x \vee y)' \approx x' \wedge y' \qquad (\vee\text{-De Morgan Law}).$$

THEOREM 3.5. JID $\subseteq$ Dms.

Proof. Let $x, y \in \mathbf{L}$. As $x' \wedge x'^{*\prime\prime} \leq x' \wedge x'^{*} = 0$, we get $x' \wedge y' = (x' \wedge x'^{*\prime\prime}) \vee (x' \wedge y')$. Hence,

$$\begin{aligned} x' \wedge y' &= x' \wedge (x'^{*\prime\prime} \vee y') \\ &= x' \wedge [(x \vee y)' \vee x'^{*\prime\prime} \vee y'] && \text{since } (x \vee y)' \leq y' \\ &= x' \wedge [(x \vee y)' \vee x'^{*\prime\prime} \vee y'''] \\ &= x' \wedge [(x \vee y)' \vee (x'^{*} \vee y')''] \\ &= x' \wedge [(x \vee y)' \vee \{(x'^{*} \vee x')' \vee (x'^{*} \vee y')'\}'] && \text{by Lemma 3.4 (3.4)} \\ &= x' \wedge [(x \vee y)' \vee \{x'^{*} \vee (x \vee y)'\}''] && \text{by Lemma 2.3 (2)} \\ &= x' \wedge [(x \vee y)' \vee \{x' \to (x \vee y)'\}''] && \text{by Lemma 3.4 (7)} \\ &= (x \vee y)' && \text{by Lemma 2.3 (7).} \end{aligned}$$

Hence, **JID** $\subseteq$ **Dms**. □

The following lemma is useful in this and later sections.

LEMMA 3.6. *Let* $x, y, z \in \mathbf{L}$. *Then*

(1) $x'^{*\prime\prime} = x'^{*}$,
(2) $x''^{*} = x'^{*\prime}$,
(3) $x \to (x \wedge y') = x^{*} \vee y'$,
(4) $(x \wedge y'^{*})^{*} = y' \vee x^{*}$,

(5) $(x' \vee y''^{*})^{*'} = (x'' \wedge y'^{*})^{*}$.

Proof. (1): From Lemma 3.4 (3) we have $x' \vee x'^{*} = 1$, which yields $x''' \vee x'^{*''} = 1$, implying $x' \vee x'^{*''} = 1$, leading to $x'^{*} \leq x'^{*''}$; thus, $x'^{*} = x'^{*''}$.

(2): From $x' \vee x'^{*} = 1$ and Theorem 3.5 we get $x'' \wedge x'^{*'} = 0$, implying $x'^{*'} \leq x''^{*}$. To prove the reverse inequality, from $x' \wedge x'^{*} = 0$, we get $x'' \vee x'^{*'} = 1$, from which it follows that $x''^{*} \leq x'^{*'}$.

(3):
$$\begin{aligned} x^{*} \vee y' &= (y' \vee x) \to y' && \text{by (JID)} \\ &= (y' \vee x) \to [y' \vee (x \wedge y')] \\ &= y' \vee [x \to (x \wedge y')] && \text{by (JID)} \\ &= x \to (x \wedge y') && \text{by Lemma 2.3 (6).} \end{aligned}$$

(4):
$$\begin{aligned} (x \wedge y'^{*})^{*} &= (x \wedge y'''^{*})^{*} && \text{by Lemma 2.2 (v)} \\ &= (x \wedge y'^{*''})^{*} && \text{by (2) (twice)} \\ &= (x \wedge y'^{*''}) \to (y' \wedge x \wedge y'^{*''}) && \text{as } y' \wedge y'^{*''} = 0 \\ &= y' \vee (x \wedge y'^{*''})^{*} && \text{by (3)} \\ &= y' \vee (x \wedge y'''^{*})^{*} && \text{by (2) (twice)} \\ &= y' \vee (x \wedge y'^{*})^{*} \\ &= y' \vee [(x \wedge y'^{*}) \to 0] \\ &= [y' \vee (x \wedge y'^{*})] \to y' && \text{by (JID)} \\ &= [(y' \vee x) \wedge (y' \vee y'^{*})] \to y' \\ &= [(y' \vee x) \wedge 1] \to y' && \text{by Lemma 3.4 (3)} \\ &= (y' \vee x) \to y' \\ &= y' \vee (x \to 0) && \text{by (JID)} \\ &= y' \vee x^{*}. \end{aligned}$$

(5): $(x' \vee y''^{*})^{*\prime} = (x' \vee y'^{*\prime})^{*\prime}$ by (2)

$= (x \wedge y'^{*})'^{*\prime}$

$= (x \wedge y'^{*})''^{*}$ by (2)

$= (x' \vee y'^{*\prime})'^{*}$

$= (x'' \wedge y'^{*\prime\prime})^{*}$ by Theorem 3.5

$= (x'' \wedge y'''^{*})^{*}$ by (2) (twice)

$= (x'' \wedge y'^{*})^{*}$.

This completes the proof. □

3.1. An Alternate Definition of "level n", for $n \geq 1$.

The following lemmas enable us to give an alternate definition of "Level n", for $n \geq 1$.

LEMMA 3.7. *Let $x \in \mathbf{L}$. Then $x'^{**} = x'$.*

Proof. Since $x' \vee x'^{*} = 1$ by Lemma 3.4, and $x' \wedge x'^{*} = 0$, we get $x'^{**} = x'$. □

LEMMA 3.8. *Let $x \in \mathbf{L}$. Then $x \wedge x'^{*} \wedge x'^{*\prime *} = (x \wedge x'^{*})'^{*}$.*

Proof. $x \wedge x'^{*} \wedge x'^{*\prime *} = x \wedge x'^{*} \wedge x''^{**}$ by Lemma 3.6 (2)

$= x \wedge x'^{*} \wedge x''$ by Lemma 3.7

$= x'^{*} \wedge x''$

$= x'^{*} \wedge x''^{**}$ by Lemma 3.7

$= x'^{*} \wedge x'^{*\prime *}$ by Lemma 3.6 (2)

$= (x \wedge x'^{*})'^{*}$.

□

Since $\mathbf{JID_n} = \mathbf{JID} \cap \mathbf{DQD_n}$, the above lemma allows us to make the following alternate (but equivalent) definition for $\mathbf{JID_n}$, for $n \in \omega$ such that $n \geq 1$.

DEFINITION 3.9. *Let n be an integer ≥ 1. The variety $\mathbf{JID_n}$ is the subvariety of $\mathbf{JID}$ defined by*

(Lev n) $\quad (x \wedge x'^{*})^{(n-1)('*)} \approx (x \wedge x'^{*})^{n('*)}$.

Thus, in particular, $\mathbf{JID_1}$ and $\mathbf{JID_2}$ are, respectively, defined, relative to $\mathbf{JID}$, by

(Lev 1) $\quad x \wedge x'^{*} \approx (x \wedge x'^{*})'^{*}$, (Lev 2) $(x \wedge x'^*)'^* \approx (x \wedge x'^*)'^{*'*}$.

In the rest of the paper we will use these definitions for the levels of $\mathbf{JID_1}$ and $\mathbf{JID_2}$.

3.2. **The Level of JID.**

Next, we wish to prove that **JID** is at Level 2.

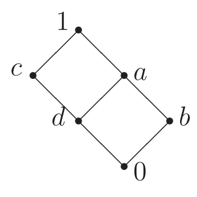
THEOREM 3.10. *We have*

(1) $\mathbf{JID_1} \subset \mathbf{JID}$,
(2) $\mathbf{JID} = \mathbf{JID_2}$.

Proof. First, we prove (2). That is, we need to prove that the "level 2" identity holds in **JID**. Let $x \in \mathbf{L}$.

$$\begin{aligned}
(x \wedge x'^*)'^{*'*} &= (x' \vee x'^{*'})^{*'*} \\
&= (x' \vee x''^*)^{*'*} && \text{by Lemma 3.6 (2)} \\
&= (x'' \wedge x'^*)^{**} && \text{by Lemma 3.6 (5)} \\
&= (x' \vee x''^*)^{*} && \text{by Lemma 3.6 (4)} \\
&= (x' \vee x'^{*'})^{*} && \text{by Lemma 3.6 (2)} \\
&= (x \wedge x'^*)'^*.
\end{aligned}$$

Hence (2) is proved. For (1), we consider the following algebra **SIX** with its lattice reduct, $\to$ and $'$ as given in Figure 2. We note that $\mathbf{SIX} \in \mathbf{JID}$; but it is not of level 1 (at a). ☐



$\to$:	0	1	a	b	c	d
0	1	1	1	1	1	1
1	0	1	a	b	c	d
a	0	1	1	b	c	c
b	c	1	1	1	c	c
c	b	1	a	b	1	a
d	b	1	1	b	1	1

$'$:	0	1	a	b	c	d
	1	0	b	b	c	1

Figure 2

The following corollary is immediate from the above theorem and [26, Corollary 8.2(a)].

COROLLARY 3.11. **JID** *is a discriminator variety of* level 2.

4. Dually Stone Semi-Heyting algebras

The study of dually Stone Heyting algebras goes back to [22], while the investigations into the variety **DSt** of dually Stone semi-Heyting algebras were initiated in [26]. In this section we will prove that the variety **DSt** is a discriminator variety of level 1 and also present some of its properties that, besides being of interest in their own right, will be needed in the later sections. We will also consider the subvariety **DStHC** of **DStH** generated by dually Stone Heyting chains and prove that the lattice of subvarieties of **DStHC** is an $\omega + 1$-chain–a result which was implicit in [26, Section 13].

It is well-known that the identity $(x \wedge y)^* \approx x^* \vee y^*$ holds in Stone algebras. The following lemma is just its dual.

LEMMA 4.1. *Let* $\mathbf{L} \in \mathbf{DSt}$*. Then* $\mathbf{L}$ *satisfies:* $(x \vee y)' \approx x' \wedge y'$*.*

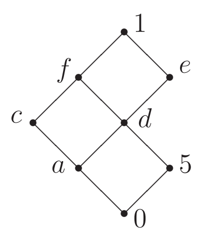
See Section 2 for the definition of the condition (SC). The following theorem will be useful in the sequel.

THEOREM 4.2. *Let* $\mathbf{L} \in \mathbf{DSt}$*. Then*

(a) $\mathbf{L} \models x'' \approx x'^*$*;*
(b) $\mathbf{L} \models$ (Lev 1)*;*
(c) *If* $\mathbf{L} \models$ (SC)*, then* $\mathbf{L} \in \mathbf{JID_1}$*.*

Proof. We note that (a) is the dual of a well known property of Stone algebras. From (a) we have $(x \wedge x'^*)'^* = (x \wedge x'')'^* = x'''^* = x'^* = x'' = x \wedge x'^*$, implying that (b) holds. Finally, let $\mathbf{L} \in \mathbf{DSt}$ and satisfy (SC), and let $a \in L \setminus \{1\}$. Then, by (SC) and (a), we have $a'' = a \wedge a'' = a \wedge a'^* = 0$, implying $a' = 1$. Then it is straightforward to verify that $\mathbf{L} \models$ (JID). Hence, (c) holds, in view of (b). □

REMARK 4.3. In contrast to **DSt**, **DPC** is not, however, at level 1. For example, the algebra **EIGHT** with its lattice reduct, $\to$ *and* $'$ *as given below, is, in fact, in the subvariety of* **DPC***, defined by:* $(x \vee y)' \wedge (x' \vee y)' \wedge (x \vee y')' = 0$*; but it fails to satisfy* (Lev 1) *identity.*



$\to$:	0	1	e	c	a	b	f	d
0	1	0	0	b	b	c	0	0
1	0	1	e	c	a	b	f	d
e	0	1	1	c	c	b	f	f
c	b	c	a	1	e	0	c	a
a	b	c	c	1	1	0	c	c
b	c	b	b	0	0	1	b	b
f	0	1	e	c	a	b	1	e
d	0	1	1	c	c	b	1	1

$'$:	0	1	e	c	a	b	f	d
	1	0	c	e	1	1	e	1

The following corollary is immediate from Lemma 4.1, Theorem 4.2 and [26, Corollary 8.2(a)].

COROLLARY 4.4. **DSt** *is a discriminator variety of level* 1.

Observe that Lemma 4.1 impies that **DSt** satisfies (BL). The following corollary is, therefore, immediate from Theorem 4.2(b) and Theorem 2.6.

COROLLARY 4.5. *Let* $\mathbf{L} \in \mathbf{DSt}$ *with* $|L| \geq 2$. *Then the following are equivalent:*

(1) **L** *is simple,*
(2) **L** *is subdirectly irreducible,*
(3) **L** *satisfies* (SC).

4.1. **The variety DStHC.**

Recall that **DStHC** is the variety generated by dually Stone Heyting chains. We now give an application of Corollary 4.5.

DEFINITION 4.6. *For* $n \in \mathbb{N}$, *let* $\mathbf{C_n^{dp}}$ *denote the* n*-element* **DStH**-*chain such that*
$C_n^{dp} = \{0, a_1, a_2, \ldots, a_{n-2}, 1\}$, *where* $0 < a_1 < a_2 < \cdots < a_{n-2} < 1$.

We denote by $\mathbf{V}(\mathbf{C_n^{dp}})$ the variety generated by $\mathbf{C_n^{dp}}$. (Note that $\mathbf{C_3^{dp}}$ is the same as $\mathbf{L_1^{dp}}$ given in [26].)

It follows from Corollary 3.3 that $\mathbf{DPCHC} = \mathbf{DStHC}$. The following theorem was implicit in [26, Section 13].

THEOREM 4.7. *The lattice of subvarieties of* **DStHC** *is the following* $\omega + 1$*-chain:*

$$\mathbf{V}(\mathbf{C_1^{dp}}) < \mathbf{V}(\mathbf{C_2^{dp}}) < \cdots < \mathbf{V}(\mathbf{C_n^{dp}}) < \cdots < \mathbf{DStHC}.$$

Proof. We claim that subdirectly irreducible algebras in **DStHC** are precisely the **DStH**-chains. For, let $\mathbf{C^{dp}}$ be a **DStHC**-chain and let $x \in \mathbf{C^{dp}}$. Since $x \leq x'$ or $x' \leq x$, it follows that $x = 1$ or $x' = 1$, for every $x \in \mathbf{C^{dp}}$, which implies that $\mathbf{C^{dp}}$ satisfies (SC). On the other hand, let $\mathbf{A} \in \mathbf{DStHC}$ satisfy (SC). Let $a \in A \setminus \{1\}$. By Theorem 4.2 (a) we have $a'^* \leq a$; hence by (SC), we get $a'^* = 0$, implying $a' = 1$, again by Theorem 4.2 (a). Since each **DStHC**-chain satisfies the identity: (L) $(x \to y) \vee (y \to x) \approx 1$, it follows that **DStHC** satisfies it too, implying that $\mathbf{A} \models$ (L). Hence, any two elements of $\mathbf{A}$ are comparable in $\mathbf{A}$, so $\mathbf{A}$ is a **DStH**-chain. Thus, $\mathbf{A} \in \mathbf{DStHC}$ is subdirectly irreducible iff $\mathbf{A}$ is a **DStH**-chain. Now it is not hard to observe that if an identity fails in an infinite **DStHC**-chain, then it fails in a finite **DStHC**-chain. Thus **DStHC** is generated by finite **DStH**-chains. Hence, the conclusion of the theorem follows. □

Note, however, that if we consider **DStC**-chains with semi-Heyting reducts that are not Heyting algebras, the situation gets more complicated, since the structure of the lattice of subvarieties of **DStC** is quite complex, as shown by the following class of examples: Let A be a semi-Heyting algebra. Let A^e be the expansion of A by adding a unary operation $'$ as follows:

$$x' = 0, \text{ if } x = 1, \text{ and } x' = 1, \text{ otherwise.}$$

Then it is clear that A^e is a **DSt**-algebra and is simple. In particular, if A is a semi-Heyting-chain, then $A^e \in \mathbf{DStC}$ and is simple. Furthermore, the number of semi-Heyting chains even for a small size is large; for example, there are 160 semi-Heyting chains of size 4 and, therefore, there are 160 **DStC**-chains of size 4. If we denote the 2-element, non-Boolean, dually Stone semi-Heyting algebra by $\mathbf{\bar{2}^e}$, then it is interesting to observe that $\mathbf{\bar{2}^e} \in \mathbf{DStC} \setminus \mathbf{DStHC}$, and **DStHC** is only a "small" subvariety of **DStC**. These observations naturally suggest that the following open problem is of interest:

Problem: Investigate the structure of the lattice of subvarieties of DStC.

5. Subdirectly Irreducible Algebras in $\mathbf{JID_1}$

Recall that the variety $\mathbf{JID_1}$ is the subvriety of $\mathbf{JID}$ defined by

$$\text{(Lev 1)} \qquad x \wedge x'^{*} \approx (x \wedge x'^{*})'^{*}.$$

In this section we give a somewhat concrete characterization of subdirectly irreducible (=simple) algebras in the variety $\mathbf{JID_1}$.

The following theorem follows immediately from Theorem 3.10 and Theorem 2.6.

THEOREM 5.1. *Let* $\mathbf{L} \in \mathbf{JID_1}$ *with* $|L| \geq 2$. *Then the following are equivalent:*

(1) $\mathbf{L}$ *is simple,*
(2) $\mathbf{L}$ *is subdirectly irreducible,*
(3) $\mathbf{L}$ *satisfies* (SC).

We now wish to refine further the above characterization of the subdirectly irreducible algebras in $\mathbf{JID_1}$. In view of the above theorem, it suffices to characterize the algebras in $\mathbf{JID_1}$ satisfying the condition (SC).

Unless otherwise stated, in the rest of this section we assume that $\mathbf{L} \in \mathbf{JID_1}$ with $|L| \geq 2$ and satisfies the simplicity condition (SC).

LEMMA 5.2. *Let* $a, b \in L$ *such that* $a' = a$. *Then*

$$a \vee b \vee b^{*} = 1.$$

Proof. From Lemma 3.4 (4) and $a' = a$, we have

$$(a \vee b) \to a = a \vee b^{*}. \tag{1}$$

Now,

$$\begin{aligned} a \vee (a \vee b)'^{*} &= a' \vee [(a \vee b)' \to 0] \\ &= [a' \vee (a \vee b)'] \to (a' \vee 0), \text{ by (JID)} \\ &= a' \to a' \qquad\qquad \text{as } a' \geq (a \vee b)' \\ &= 1. \end{aligned}$$

Thus, we have

$$a \vee (a \vee b)'^{*} = 1. \tag{2}$$

If $a \vee b = 1$, then clearly the lemma is true. So, we assume that $a \vee b \neq 1$. Then $(a \vee b) \wedge (a \vee b)'^* = 0$ by (SC), and hence, we have

$$\begin{aligned}
a &= a \vee (b \wedge b^*) \\
&= (a \vee b) \wedge (a \vee b^*) \\
&= [(a \vee b) \wedge (a \vee b)'^*] \vee [(a \vee b) \wedge (a \vee b^*)] && \text{by (SC)} \\
&= (a \vee b) \wedge [(a \vee b)'^* \vee (a \vee b^*)] \\
&= (a \vee b) \wedge [(a \vee b)'^* \vee \{(a \vee b) \to a\}] && \text{by (1)} \\
&= (a \vee b) \wedge [\{(a \vee b) \to a\} \vee (a \vee b)'^*] \\
&= (a \vee b) \wedge [a \vee (a \vee b)'^*] && \text{by Lemma 2.3(3)} \\
&= a \vee b && \text{by (2).}
\end{aligned}$$

Hence, $a \vee b = a$, which implies, by (1), that $a \vee b^* = 1$. The conclusion of the lemma is now immediate. □

LEMMA 5.3. *Let $x \in L \setminus \{1\}$. Then $x \leq x'$.*

Proof. Since $x \neq 1$, we have $x \wedge x'^* = 0$ by (SC), from which we get $(x' \vee x) \wedge (x' \vee x'^*) = x'$, whence $x' \vee x = x'$, as $x' \vee x'^* = 1$ by Lemma 3.4 (3), proving the lemma. □

LEMMA 5.4. *Let $|L| > 2$ and let $a \in L$ such that $a' = a$. Then the height of L is at most 2.*

Proof. Suppose there are $b, c \in L$ such that $0 < b < c < 1$. We wish to arrive at a contradiction.
From Lemma 5.3 we have $c \leq c'$, from which it follows that

$$b \leq c'. \tag{3}$$

Claim 1: $b' = 1$.

Suppose $b' \neq 1$. Then, by Lemma 5.3, we get $b' \leq b'' \leq b \leq c$; thus $b' \leq c$. Next, $b \leq c$ implies $c' \leq b'$; and also $c \leq c'$ from Lemma 5.3, whence $c \leq b'$. Thus we conclude that $b' = c$, whence $c' = b'' \leq b$, implying $c' = b$, by (3). Then, in view of Lemma 5.3. we have $c \leq c' = b$; thus $c \leq b$, which is a contradiction, proving the claim.

From Lemma 5.2 we have $a \vee b \vee b^* = 1$. Hence, $a' \wedge b' \wedge b^{*\prime} = 0$ by Theorem 3.5, implying $a \wedge b^{*\prime} = 0$ by **Claim 1** and the hypothesis. Thus

$$a \wedge b^{*\prime} = 0. \tag{4}$$

Therefore, $a \vee b^* \geq a \vee b^{*\prime\prime} = 1$ as $a' = a$, yielding $b \leq a$. Hence, again from (4), we obtain

$$b \wedge b^{*\prime} = 0. \tag{5}$$

Claim 2: $b \vee b^* = 1$.

Suppose the claim is false. Then $b \leq b \vee b^* \leq (b \vee b^*)'$ by Lemma 5.3, whence $b \leq b' \wedge b^{*\prime}$, which implies $b = b \wedge b' \wedge b^{*\prime} = 0$ by the equation (5), contrary to $b > 0$, proving the claim.

From **Claim 2** and Theorem 3.5 we have $b' \wedge b^{*\prime} = 0$, Since $b' = 1$ by **Claim 1**, it follows that $b^{*\prime} = 0$, whence $b^* \geq b^{*\prime\prime} = 1$; so $b \leq b^{**} = 0$, contradicting $b > 0$, proving the lemma. □

LEMMA 5.5. *For every $x \in \mathbf{L}$, $x = 1$ or $x' = 1$ or $x = x'$.*

Proof. Suppose $x \in L$ such that $x \neq 1$ and $x' \neq 1$. Then by Lemma 5.3, we have $x \leq x'$. Also, since $x' \neq 1$, we have $x' \leq x'' \leq x$, again by Lemma 5.3. So, $x = x'$, proving the lemma. □

LEMMA 5.6. *Let $a \in L$ such that $a' = a$. Then $a^{*\prime} = a^*$.*

Proof. First, observe that $a \neq 0$ and $a \neq 1$, since $a = a'$. Suppose $a^{*\prime} \neq a^*$. The following claims will lead to a contradiction.

Claim 1: $a^* = a^{*\prime\prime}$.

$a \vee a^{*\prime\prime} = a'' \vee a^{*\prime\prime} = (a \vee a^*)'' = [a' \vee (a' \to 0)]'' = 1$ by Lemma 3.4(3). Hence, $a \vee a^{*\prime\prime} = 1$, implying $a^* \leq a^{*\prime\prime}$, and so $a^* = a^{*\prime\prime}$, proving the claim.

Claim 2: $a^* = 0$.

We have, by Lemma 5.5, that $a^{*\prime} = 1$ or $a^{*\prime\prime} = 1$ or $a^{*\prime} = a^{*\prime\prime}$. So, by **Claim 1**, we get $a^* = a^{*\prime\prime} = 0$ or $a^* = 1$ (as $a^* \geq a^{*\prime\prime}$) or $a^{*\prime} = a^*$. But, we know, by our assumption, that $a^* \neq a^{*\prime}$. Hence, $a^* = 0$ or $a^* = 1$, which clearly implies $a^* = 0$ or $a = 0$. Since we know that $a \neq 0$, the claim is proved.

Now, in view of (JID) and **Claim 2**, we have $a = a \vee 0 = a \vee a^* = a' \vee (a \to 0) = (a' \vee a) \to (a' \vee 0) = a \to a = 1$, implying $a = 1$, which is a contradiction, proving the lemma. □

PROPOSITION 5.7. *Let $|L| > 2$. Suppose there is an $a \in L$ such that $a' = a$. Then $\mathbf{L} \in \{\mathbf{D_1}, \mathbf{D_2}, \mathbf{D_3}\}$, up to isomorphism.*

Proof. In view of Lemma 5.4 and $|L| > 2$, the height of L is exactly 2. Since the lattice reduct of L is distributive, L is either a 3-element chain or a 4-element Boolean lattice. We know from Lemma 5.6 that $a^{*\prime} = a^*$. Thus a and a^* are complementary, implying that the lattice reduct of $\mathbf{L}$ is a 4-element Boolean lattice; so $\mathbf{L} \models (Bo)$, and hence $\mathbf{L} \in \mathbf{DQB}$. Then, from Proposition 2.4 (a) it follows that $\mathbf{L} \in \{\mathbf{D_1}, \mathbf{D_2}, \mathbf{D_3}\}$, up to isomorphism. □

PROPOSITION 5.8. *Suppose $x' \neq x$, for every $x \in L$. Then* $\mathbf{L} \in \mathbf{DSt}$.

Proof. Let $x \in L$. Without loss of generalty, we can assume that $x \neq 1$. Then we claim that $x' = 1$. For, assume that $x' \neq 1$. Then, by Lemma 5.3 we get $x \leq x'$ and $x' \leq x''$, which implies $x = x'$, as $x'' \leq x$, contradicting the hypothesis. So, we have $x' = 1$, which implies $x' \wedge x'' = 0$, Hence $\mathbf{L}$ is a dually Stone semi-Heyting algebra, completing the proof. □

We are now ready to prove our main theorem of this section.

THEOREM 5.9. *Let* $\mathbf{L} \in \mathbf{DQD}$ *with* $|L| \geq 2$. *Then the following are equivalent:*

(a) $\mathbf{L}$ *is a subdirectly irreducible algebra in* $\mathbf{JID_1}$*,*
(b) $\mathbf{L}$ *is a simple algebra in* $\mathbf{JID_1}$*,*
(c) $\mathbf{L} \in \mathbf{JID_1}$ *such that* (SC) *holds in* $\mathbf{L}$*,*
(d) $\mathbf{L} \in \{\mathbf{D_1}, \mathbf{D_2}, \mathbf{D_3}\}$*, up to isomorphism, or* $\mathbf{L} \in \mathbf{DSt}$ *such that* $\mathbf{L}$ *satisfies* (SC).

Proof. In view of Theorem 5.1, we only need to prove (c) $\Leftrightarrow$ (d). Now, suppose (d) holds. First, let us suppose $\mathbf{L} \in \{\mathbf{D_1}, \mathbf{D_2}, \mathbf{D_3}\}$, up to isomorphism. Then it is routine to verify that $\{\mathbf{D_1}, \mathbf{D_2}, \mathbf{D_3}\} \subseteq \mathbf{JID_1}$ and $\{\mathbf{D_1}, \mathbf{D_2}, \mathbf{D_3}\}$ satisfies (SC), implying (c). Next, suppose $\mathbf{L} \in \mathbf{DSt}$ such that $\mathbf{L}$ satisfies (SC). Then $\mathbf{L} \in \mathbf{JID_1}$, in view of Theorem 4.2(c), implying that (c) holds. Thus (d) $\Rightarrow$ (c). To prove the converse, suppose (c) holds. We consider two cases. First, suppose there is an $a \in L$ such that $a' = a$. Then, by Proposition 5.7, $\mathbf{L} \in \{\mathbf{D_1}, \mathbf{D_2}, \mathbf{D_3}\}$, up to isomorphism, implying (d).

Next, suppose there is no element $a \in L$ such that $a' = a$. Hence, $\mathbf{L}$ satisfies:

(6) $$\text{For every } x \in L,\ x' \neq x.$$

Then, using Proposition 5.8, we obtain that $\mathbf{L}$ is dually Stone, which, together with the hypothesis, leads us to conclude (c) $\Rightarrow$ (d). □

We have the following important consequence of Theorem 5.9.

COROLLARY 5.10. $\mathbf{JID_1} = \mathbf{DSt} \ \vee \ \mathbf{V}(\mathbf{D_1}, \mathbf{D_2}, \mathbf{D_3})$.

Recall that $\mathbf{JIDH}$ is the subvariety of $\mathbf{JID}$ defined by the identity: $(x \wedge y) \to x \approx 1$, and $\mathbf{DStH}$ is the variety of dually Stone Heyting algebras. Now, we focus on the subvariety $\mathbf{JIDH_1}$ of $\mathbf{JIDH}$. Note that the variety of Boolean algebras is the only atom in the lattice of subvarieties of $\mathbf{JIDH_1}$. For $\mathbf{V}$ a subvariety of $\mathbf{JIDH_1}$, let $\mathcal{L}(\mathbf{V})$ and $\mathcal{L}^+(\mathbf{V})$ denote, respectively, the lattice of subvarieties of $\mathbf{V}$ and the lattice of nontrivial subvarieties of $\mathbf{V}$. Let $\mathbf{1} \oplus \mathbf{L}$ denote the ordinal sum of the trivial lattice $\mathbf{1}$ and a lattice $\mathbf{L}$.

Restricting the semi-Heyting reduct in the above corollary to Heyting algebras, we obtain the following interesting corollary, where $\mathbf{2}$ denotes a 2-element lattice.

COROLLARY 5.11. *We have*
(1) $\mathbf{JIDH_1} = \mathbf{DStH} \vee \mathbf{V}(\mathbf{D_2})$,
(2) $\mathcal{L}(\mathbf{JIDH_1}) \cong \mathbf{1} \oplus (\mathcal{L}^{+}(\mathbf{DStH}) \times \mathbf{2})$.

The preceding corollary leads to the following open problem.

PROBLEM: Investigate the structure of $\mathcal{L}^{+}(\mathbf{DStH})$.

6. JI-distributive, dually quasi-De Morgan, linear Semi-Heyting Algebras

In this section we focus on the linear identity:

(L) $\qquad (x \to y) \vee (y \to x) \approx 1.$

Let **DQDL** [**JIDL**] denote the subvariety of **DQD** [**JID**] defined by (L), and let **JIDLH** denote the subvariety of **JIDL** consisting of JI-distributive, dually quasi-De Morgan, linear Heyting algebras.

The following result is needed later in this section. Part (a) of it is proved in [26, Lemma 12.1(f)], and (b) follows immediately from (a).

PROPOSITION 6.1. [26, Lemma 12.1(f)] *Let* $\mathbf{L}$ *be a linear semi-Heyting algebra* (i.e., $\mathbf{L} \models$ (L)). Then
(a) $\mathbf{L} \models$ (H),
(b) $\mathbf{JIDL} = \mathbf{JIDLH}$.

LEMMA 6.2. *Let* $\mathbf{L} \in \mathbf{DQDL}$ *and let* $x, y \in L$. *Then*
(a) $(x \to y) \vee (y \to x)'' = 1$,
(b) $x \ \leq \ y \vee (y \to x)''$.

Proof. $(x \to y) \vee (y \to x)'' \geq (x \to y)'' \vee (y \to x)'' = [(x \to y) \vee (y \to x)]'' = 1$ by (L), proving (a). Using (a), we get $x \wedge [y \vee (y \to x)''] = (x \wedge y) \vee [x \wedge (y \to x)''] = [x \wedge (x \to y) \vee [x \wedge (y \to x)''] = x \wedge [(x \to y) \vee (y \to x)''] = x \wedge 1 = x$, implying (b). □

Note that the algebra **SIX** described earlier in Section 3 is actually an algebra in **JIDL**. Hence **JIDL** does not satisfy (Lev 1); but **JIDL** is at level 2, in view of Theorem 3.10.

In this section, our goal is to present, as an application of Theorem 5.9, an explicit description of subdirectly irreducible (= simple) algebras in the variety $\mathbf{JIDL_1}$ of JI-distributive, dually quasi-De Morgan, linear semi-Heyting algebras of level 1.

Recall that $\mathbf{DPCC} = \mathbf{DStC}$ and $\mathbf{DPCHC} = \mathbf{DStHC}$. So, we use these names interchangeably.

LEMMA 6.3. $\mathbf{DPCC} \models$ (Lev1).

Proof. Let $\mathbf{L}$ be a $\mathbf{DPC}$-chain and let $x \in L$. Since x, x' are comparable, we have $x \vee x' = x$ or $x \vee x' = x'$, implying $x = 1$ *or* $x' = 1$, as $x \vee x' = 1$. Then it is easy to see that (Lev 1) holds in $\mathbf{L}$, and hence in $\mathbf{DPCC}$. □

PROPOSITION 6.4. $\mathbf{DPCHC} \vee \mathbf{V}(\mathbf{D_2}) \subseteq \mathbf{JIDL_1}$.

Proof. It follows from Lemma 3.2, and Lemma 6.3 that $\mathbf{DPCHC}$ satisfies (JID) and (Lev 1), and it is easy to see that $\mathbf{DPCHC} \models$ (L). Also, it is routine to verify that (JID), (L) and (Lev 1) hold in $\mathbf{D_2}$. □

Our goal in this section is to prove that, in fact, the equality holds in the statement of the above Proposition.

Unless otherwise stated, in the rest of this section we assume that $\mathbf{L} \in \mathbf{JIDL_1}$ with $|L| > 2$ and satisfies (SC).

LEMMA 6.5. *Let $x, y \in L$ such that $x \vee y \neq 1$. Then, $x \leq y'$.*

Proof. Let $x \vee y \neq 1$. Since $y' \vee (x \vee y)'^* \geq y' \vee y'^* = 1$ by Lemma 3.4 (3), we get, using (SC), that $x = x \wedge (x \vee y) \wedge [y' \vee (x \vee y)'^*] = x \wedge [\{(x \vee y) \wedge y'\} \vee \{(x \vee y) \wedge (x \vee y)'^*\}] = x \wedge (x \vee y) \wedge y' = x \wedge y'$, whence $x \leq y'$. □

LEMMA 6.6. *Let $a, b \in L$ such that $a' \neq a$, $a \neq 1$, and $a \not\leq b$. Then $(a \to b)'' = 0$.*

Proof. First, we claim that $a \not\leq (a \to b)''$. For, suppose $a \leq (a \to b)''$; then $a = a \wedge (a \to b)'' \leq b$ by Lemma 2.2 (vii), implying $a \leq b$, which is a contradiction to the hypothesis $a \not\leq b$. Hence $a \not\leq (a \to b)''$. Then $a \vee (a \to b)' = 1$ by (the contrapositive of) Lemma 6.5, whence $a'' \vee (a \to b)''' = 1$. Since $a \neq 1$ and $a' \neq a$ by hypothesis, we get $a' = 1$ by Lemma 5.5, whence $a'' = 0$. Then we conclude that $(a \to b)'' = 0$, proving the lemma. □

We are now ready to give an explicit description of subdirectly irreducible (=simple) algebras in $\mathbf{JIDL_1}$.

THEOREM 6.7. *Let* $\mathbf{L} \in \mathbf{DQD_1}$ *with* $|L| > 2$*. Then the following are equivalent:*

(1) $\mathbf{L}$ *is a subdirectly irreducible algebra in* $\mathbf{JIDL_1}$*,*
(2) $\mathbf{L}$ *is a simple algebra in* $\mathbf{JIDL_1}$*,*
(3) $\mathbf{L} \in \mathbf{JIDL_1}$ *such that* (SC) *holds in* $\mathbf{L}$*,*
(4) $\mathbf{L} \cong \mathbf{D_2}$*, or* $\mathbf{L}$ *is a* $\mathbf{DStH}$*-chain.*

Proof. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) follow from Theorem 5.9. So we need to prove (3) $\Rightarrow$ (4) $\Rightarrow$ (3). Suppose (3) holds. Then, by Theorem 5.9, either $\mathbf{L} \in \{\mathbf{D_1}, \mathbf{D_2}, \mathbf{D_3}\}$, or $\mathbf{L} \in \mathbf{DSt}$ and satisfies (SC). In the former case, since $\mathbf{L} \models$ (L), it follows from Proposition 6.1 that $\mathbf{L} \models$ (H). Hence $\mathbf{L} \cong \mathbf{D_2}$. Next, we assume the latter case. So, $\mathbf{L} \in \mathbf{DSt}$ and satisfies (SC). Since $\mathbf{L} \models$ (L) by hypothesis, we get, by Proposition 6.1, that $\mathbf{L} \in \mathbf{DStH}$. So, we need only prove that $\mathbf{L}$ is a chain. Let $a, b \in L \setminus \{1\}$ such that $a \not\leq b$. Then, from Lemma 6.2(b), we have that $b \leq a \vee (a \to b)''$. Also, since $\mathbf{L} \models$ (DSt), it is clear that $a' \neq a$. Hence, by Lemma 6.6, we get $(a \to b)'' = 0$, implying $b \leq a$. Thus, the lattice reduct of $\mathbf{L}$ is a chain, and so, (3) $\Rightarrow$ (4). Finally, assume (4) holds. First, if $\mathbf{L} \cong \mathbf{D_2}$, then it is routine to verify that (3) holds. Next, suppose $\mathbf{L}$ is a $\mathbf{DStH}$-chain and let $x \in L$. Then, $x' \leq x''$ or $x'' \leq x'$, implying $x' \wedge x'' = x'$ or $x' \wedge x'' = x''$. Hence, by (DSt), we get $x' = 0$ or $x' = 1$, from which it is easy to see that $\mathbf{L}$ satisfies (SC). So, from Theorem 4.2 (c), we conclude that $\mathbf{L} \in \mathbf{JID_1}$. Also, it is well known that Heyting chains satisfy (L). Thus, $\mathbf{L} \in \mathbf{JIDL_1}$ and $\mathbf{L}$ satisfies (SC), implying (3). □

The following corollary is immediate from Theorem 6.7.

COROLLARY 6.8. $\mathbf{JIDL_1} = \mathbf{DStHC} \ \vee \ \mathbf{V}(\mathbf{D_2})$.

We would like to mention here that the attempt to solve the problem of axiomatization of $\mathbf{DStHC} \vee \mathbf{V}(\mathbf{D_2})$ led to the results of this paper, with Corollary 6.8 yielding a solution to that problem.

We conclude this section with an axiomatization of $\mathbf{DStHC}$.

THEOREM 6.9. $\mathbf{DStHC} = \mathbf{DStL}$.

Proof. We know from the proof of Theorem 4.7 that the subdirectly irreducible algebras in $\mathbf{DStHC}$ are precisely the $\mathbf{DStH}$-chains. So, to complete the proof, it suffices to prove that the subdirectly irreducible algebras in $\mathbf{DStL}$ are precisely the $\mathbf{DStH}$-chains. For this, first note that from Proposition 6.1 we have that a linear semi-Heyting algebra satisfies (L) and hence $\mathbf{DStL} \models$ (H), implying $\mathbf{DStL} \subseteq \mathbf{DStH}$. Now, let $\mathbf{L}$ be a subdirectly irreducible (= simple) algebra in $\mathbf{DStL}$. We wish to show that $\mathbf{L}$ is a Heyting chain. Let $a, b \in L$ be arbitrary. By

Corollary 4.5, **L** satisfies (SC); and $\mathbf{L} \models$ (H), as $\mathbf{DStL} \models$ (H). Hence, by Lemma 6.2 (a), we have

$$\mathbf{L} \models (x \to y) \vee (y \to x)'' = 1. \tag{7}$$

Suppose that $a \neq 1$. Then from (SC) we have $a \wedge a'^* = 0$, whence $a \wedge a'' = 0$ by Lemma 4.2 (a), implying $a' = 1$. Thus we have proved

$$\text{For every } x \in L,\ x = 1 \text{ or } x' = 1. \tag{8}$$

Hence, by (8), we get $(a \to b)' = 1$ or $(a \to b)'' = 1$, implying $(a \to b)'' = 0$ or $(a \to b) = 1$. So,, by (7), we have $b \to a = 1$ or $a \to b = 1$, implying $b \leq a$ or $a \leq b$, as $\mathbf{L} \models$ (H). Thus **L** is a **DStH**-chain, completing the proof. □

7. More Consequences of Theorem 6.7

In this section we present some more consequences of Theorem 6.7.

As mentioned earlier, the axiomatizations of the variety **DPCHC** (= **DStHC**) and all of its subvarieties were given in [26].

The following corollary is immediate from Corollary 6.8 and Theorem 4.7.

COROLLARY 7.1.

(1) $\mathcal{L}(\mathbf{JIDL_1}) \cong \mathbf{1} \oplus [(\omega + \mathbf{1}) \times \mathbf{2}]$.
(2) $\mathbf{JIDL_1}$ *and* **DStHC** *are the only two elements of infinite height in the lattice* $\mathcal{L}(\mathbf{JIDL_1})$.
(3) $\mathbf{V} \in \mathcal{L}^+(\mathbf{JIDL_1})$ *is of finite height iff* **V** *is either* $\mathbf{V}(\mathbf{D_2})$, *or* $\mathbf{V}(\mathbf{C_n^{dp}})$ *for some* $n \in \mathbb{N} \setminus \{1\}$, *or* $\mathbf{V}(\mathbf{C_m^{dp}}) \vee \mathbf{V}(\mathbf{D_2})$ *for some* $m \in \mathbb{N} \setminus \{1\}$.

In Corollaries 7.2-7.5, we give equational bases to all subvarieties of $\mathbf{JIDL_1}$.

COROLLARY 7.2. *The variety* **DStHC** *is defined, modulo* $\mathbf{JIDL_1}$, *by*

$$x \vee x' \approx 1.$$

Proof. Observe that $\mathbf{DStHC} \models x \vee x' \approx 1$, but $\mathbf{V}(\mathbf{D_2}) \not\models x \vee x' \approx 1$, and then apply Theorem 6.7. □

The variety $\mathbf{V}(\mathbf{D_2})$ was axiomatized in [26]. Here is a new one.

COROLLARY 7.3. *The variety* $\mathbf{V}(\mathbf{D_2})$ *is defined, modulo* $\mathbf{JIDL_1}$, *by*

$x'' \approx x$.

Proof. Observe that $\mathbf{DStHC} \not\models x'' \approx x$, but $\mathbf{V(D_2)} \models x'' \approx x$, and then use Theorem 6.7. □

COROLLARY 7.4. *Let $n \geq 2$. The variety $\mathbf{V(C_n^{dp})} \vee \mathbf{V(D_2)}$ is defined, modulo $\mathbf{JIDL_1}$, by*

$$(\mathrm{C}_n)\ x_1 \vee x_2 \vee \cdots \vee x_n \vee (x_1 \to x_2) \vee (x_2 \to x_3) \vee \cdots \vee (x_{n-1} \to x_n) = 1.$$

Proof. We now prove that $\mathbf{C_n^{dp}} \models (\mathrm{C_n})$. Let $\langle c_1, c_2, \ldots, c_n \rangle \in C_n^{dp}$ be an arbitrary assignment in C_n^{dp} for the variables x_i such that c_i is the value of x_i, for $i = 1, \cdots, n$. If $c_i \leq c_{i+1}$ for some i, then $c_i \to c_{i+1} = 1$, as $\mathbf{C_n^{dp}}$ has a Heyting algebra reduct, and hence, the identity holds in $\mathbf{C_n^{dp}}$. So, we assume that $c_i > c_{i+1}$, for $i = 1, 2, \cdots, n$. Then, $c_1 = 1$ since $|C_n^{dp}| = n$, implying that (C_n) holds in $\mathbf{C_n^{dp}}$. Also, it is routine to check that that $\mathbf{D}_2 \models (\mathrm{C}_2)$ and (C_i) implies (C_{i+1}), for $i = 2, \cdots, n-1$. So, $\mathbf{D}_2 \models (\mathrm{C}_n)$, implying that $\mathbf{V(C_n^{dp})} \vee \mathbf{V(D_2)} \models (\mathrm{C_n})$.

Next, suppose that $\mathbf{V}$ is the subvariety of $\mathbf{JIDL_1}$ satisfying (C_n). Then, by Corollary 3.11, $\mathbf{V}$ is a discriminator variety. Let $\mathbf{L}$ be a simple (= subdirectly irreducible) algebra in $\mathbf{V}$. Then, it follows from Corollary 6.8 (or Theorem 6.7) that $\mathbf{L}$ is a $\mathbf{DStH}$-chain or $\mathbf{L} \cong \mathbf{D_2}$. Suppose that $\mathbf{L}$ is a $\mathbf{DStH}$-chain. Assume, if possible, $|L| > n$. Then, there exist $b_1, b_2 \cdots, b_{n-1} \in L$ such that $0 < b_1 < \cdots, < b_{n-1} < 1$. Since $\mathbf{L} \models (\mathrm{C}_n)$, we can assign $\langle b_{n-1}, b_{n-2}, \cdots, b_1, 0 \rangle$ for $\langle x_1, x_2, \cdots, x_{n-1}, x_n \rangle$. Then, $b_{n-1} \vee (b_{n-1} \to b_{n-2}) \vee \cdots, \vee (b_1 \to 0) = 1$, yielding $b_{n-1} \vee b_{n-2} \vee \cdots \vee b_1 \vee 0 = 1$, implying that $b_{n-1} = 1$, which is a contradiction. Thus we have $|L| \leq n$, from which it follows that $\mathbf{V} \subseteq \mathbf{V(C_n^{dp})} \vee \mathbf{V(D_2)}$, completing the proof. □

COROLLARY 7.5. *The variety $\mathbf{V(C_n^{dp})}$ is defined, modulo $\mathbf{JIDL_1}$, by*

(1) $x \vee x' \approx 1$,

(2) $x_1 \vee x_2 \vee \cdots \vee x_n \vee (x_1 \to x_2) \vee (x_2 \to x_3) \vee \cdots \vee (x_{n-1} \to x_n) = 1$.

For a different base for $\mathbf{V(C_n^{dp})}$, see [26]. Regularity was studied in [26], [27], [28] and [29]. Here is another use of it.

COROLLARY 7.6. *The variety $\mathbf{V(C_3^{dp})} \vee \mathbf{V(D_2)}$ is defined, modulo $\mathbf{JIDL_1}$, by*

$x \wedge x^+ \leq y \vee y^*$ (Regularity), where $x^+ := x'^{*'}$.

It is also defined, modulo $\mathbf{JIDL_1}$, by

$x \wedge x' \leq y \vee y^*$.

The variety $\mathbf{V(C_3^{dp})}$ is axiomatized in [26]. Here is another axiomatization for it.

COROLLARY 7.7. *The variety* $\mathbf{V}(\mathbf{C_3^{dp}})$ *is defined, modulo* $\mathbf{JIDL_1}$, *by*

(1) $x \wedge x^+ \leq y \vee y^*$ (Regularity),
(2) $x' = x^+$.

7.1. **Amalgamation Property.** We now examine the Amalgamation Property for subvarieties of the variety **DStHC**. For this purpose, we need the following lemma whose proof is straightforward.

We use "$\leq$" to abbreviate "is a subalgebra of" in the next lemma.

Recall from Theorem 4.7 (see also [26]) that the proper, nontrivial subvarieties of **DStHC** are precisely the subvarieties of the form $\mathbf{V}(\mathbf{C_n^{dp}})$, for $n \in \mathbb{N}$.

LEMMA 7.8. *Let* $m, n \in \mathbb{N}$. *Then*

$\mathbf{C_m^{dp}} \leq \mathbf{C_n^{dp}}$, *for* $m \leq n$.

COROLLARY 7.9. *Every subvariety of* **DStHC** *has Amalgamation Property.*

Proof. It follows from Corollary 4.4 that **DStHC** is a discriminator variety; and hence has CEP. Also, from Theorem 6.7 we obtain that every subalgebra of each subdirectly irreducible (= simple) algebra in **DStHC** is subdirectly irreducible. Let **V** be a subvariety of **DStHC**. Then, using a result from [11] that we need only consider an amalgam (**A**: **B**, **C**), where **A**, **B**, **C** are simple in **V** and **A** a subalgebra of **B** and **C**. First, suppose $\mathbf{V} = \mathbf{V}(\mathbf{C_n^{dp}})$ for some n. Then **B** and **C** are **DStHC**-chains. Then, in view of the preceding lemma, it is clear that the amalgam (**A**: **B**, **C**) can be amalgamated in **V**. Next, suppose **V** = **DStHC**, then it is clear that the amalgamation can be achieved as in the previous case. □

We conclude this section with the following remark: Since every subvariety **V** of **DStHC** has Congruence Extension Property and Amalgamation Property, it follows from Banachewski [6] that all subvarieties of **DStHC** have enough injectives (see [6] for the definition of this notion).

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