FURUTA TYPE INEQUALITIES RELATED TO ANDO-HIAI INEQUALITY WITH NEGATIVE POWERS

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Abstract. Furuta inequality and Ando-Hiai inequality have been actively investigated since they were established about thirty years ago. Recently, Kian and Seo obtained the Ando-Hiai type inequality with negative powers as follows: For $A, B > 0$, $A \natural_{-\alpha} B \le I$ for $\alpha \in [0,1]$ implies $A^r \natural_{-\alpha} B^r \le I$ for $r \ge 1$. Related to this result, Fujii and Nakamoto obtained Furuta type inequality with negative powers. Moreover, they discussed these generalizations. In this paper, we improve their results based on properties of Furuta inequality and Ando-Hiai inequality.

1 Introduction

Throughout this paper, an operator means a bounded linear operator on a complex Hilbert space. For convenience, we denote $A \ge 0$ (resp. $A > 0$) if A is a positive (resp. strictly positive) operator.

First of all, we state Furuta inequality [10] established in 1987 (cf. [2, 11, 15, 19, 23]): If $A \ge B \ge 0$, then for each $r \ge 0$,

$$\text{(i) } (B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\frac{1}{q}} \ge B^{\frac{p+r}{q}} \quad \text{and} \quad \text{(ii) } A^{\frac{p+r}{q}} \ge (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{1}{q}}$$

hold for $p \ge 0$ and $q \ge 1$ with $(1+r)q \ge p+r$. We remark that Furuta inequality is a generalization of Loewner-Heinz theorem *"$A \ge B \ge 0$ ensures $A^\alpha \ge B^\alpha$ for any $\alpha \in [0,1]$,"* and also it is known that (i) is equivalent to (ii) under the assumption $A \ge B \ge 0$. As stated in [19] (cf. [11]), Furuta inequality can be arranged in terms of the weighted geometric mean $\sharp_\alpha$ defined by $A \sharp_\alpha B = A^{\frac{1}{2}}(A^{\frac{-1}{2}} B A^{\frac{-1}{2}})^\alpha A^{\frac{1}{2}}$ for $A, B > 0$ and $\alpha \in [0,1]$:

$$\text{(F)} \quad A \ge B > 0 \quad \text{implies} \quad A^{-r} \sharp_{\frac{1+r}{p+r}} B^p \le B \le A \le B^{-r} \sharp_{\frac{1+r}{p+r}} A^p \quad \text{for } p \ge 1 \text{ and } r \ge 0.$$

(F) is sometimes called the satellite theorem for Furuta inequality.

On the other hand, in 1994, Ando and Hiai [1] obtained the following inequality called Ando-Hiai inequality as follows: For $A, B > 0$,

$$\text{(AH)} \qquad A \sharp_\alpha B \le I \text{ for } \alpha \in (0,1) \quad \text{implies} \quad A^r \sharp_\alpha B^r \le I \text{ for } r \ge 1.$$

We remark that they obtained the log majorization theorem by using (AH).

As a generalization of Furuta and Ando-Hiai inequalities, Furuta established grand Furuta inequality in [13] (cf. [7, 14, 15, 16, 25]): If $A \ge B \ge 0$ with $A > 0$, then for each $t \in [0,1]$ and $p \ge 1$,

$$A^{1-t+r} \ge \{A^{\frac{r}{2}}(A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^s A^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}}$$

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holds for $r \geq t$ and $s \geq 1$. Similarly to (F), it is known in [13] that grand Furuta inequality can be arranged in terms of the weighted geometric mean, that is, we can get the satellite theorem for grand Furuta inequality:

$$\text{(SGF)} \quad A \geq B > 0 \quad \text{implies} \quad A^{-r+t} \sharp_{\frac{1-t+r}{(p-t)s+r}} (A^{-t} \natural_s B^p) \leq A^{-r+t} \sharp_{\frac{1-t+r}{p-t+r}} B^p \leq B \leq A$$

for $t \in [0, 1]$, $p \geq 1$, $r \geq t$ and $s \geq 1$, where $A \natural_\alpha B = A^{\frac{1}{2}}(A^{\frac{-1}{2}} B A^{\frac{-1}{2}})^\alpha A^{\frac{1}{2}}$ for $\alpha \in \mathbb{R}$. The notation $\natural_\alpha$ is the same as $\sharp_\alpha$ if $\alpha \in [0, 1]$. We remark that (SGF) leads (F) by putting $t = 0$ and $s = 1$, and also (SGF) leads the equivalent inequality to (AH) by putting $t = 1$ and $s = r$. On Ando-Hiai inequality, its generalization was shown in [8], and also related topics were discussed in [5, 18].

Recently, Kian and Seo [22] obtained the Ando-Hiai type inequality with negative powers as follows:

Theorem 1.A ([22])**.** *For $A, B > 0$,*

$$\text{(KS)} \qquad A \natural_{-\alpha} B \leq I \quad \textit{for } \alpha \in [0, 1] \quad \textit{implies} \quad A^r \natural_{-\alpha} B^r \leq I \quad \textit{for } 0 \leq r \leq 1.$$

Fujii and Nakamoto [9] discussed generalizations of Theorem 1.A, and also they obtained the Furuta type inequality with negative powers as follows:

Theorem 1.B ([9, Theorem 3.1])**.** *If $A \geq B > 0$, then $A^{-r} \natural_{\frac{1+r}{p+r}} B^p \leq A$ holds for $p \leq -1$ and $r \in [-1, 0]$.*

By replacing p, r by $-p, -r$ respectively, we can rewrite Theorem 1.B as follows:

$$\text{(FN)} \qquad A \geq B > 0 \quad \text{implies} \quad A^{-r} \sharp_{\frac{1-r}{p+r}} B^p \leq A^{1-2r} \quad \text{for } p \geq 1 \text{ and } 0 \leq r \leq 1.$$

We remark that the equivalence between two inequalities

$$A^r \natural_{\frac{1-r}{-p-r}} B^{-p} \leq A \quad \text{and} \quad A^{-r} \sharp_{\frac{1-r}{p+r}} B^p \leq A^{1-2r}$$

can be shown by using the relation

$$(*) \qquad A \natural_{-r} B = A(A^{-1} \natural_r B^{-1})A.$$

Fujii and Nakamoto [9] also discussed the grand Furuta type inequalities. We state them later.

In this paper, from the viewpoint of the satellite theorem for Furuta inequality, we improve some results in [9], and also we discuss relations among Theorem 1.A, Theorem 1.B and our results.

2 Furuta type inequalities and their grand Furuta type generalizations

Firstly, we show an improvement of (FN).

Theorem 2.1. *Let $A \geq B > 0$ and $r > 0$. Then for $p \geq 1$, the following inequalities hold.*

$$A^{-r} \sharp_{\frac{1-r}{p+r}} B^p \begin{cases} \leq B^{1-2r} \leq A^{1-2r} & \text{if } 0 \leq r \leq \frac{1}{2}, \\ \leq A^{1-2r} \leq B^{1-2r} & \text{if } \frac{1}{2} \leq r \leq 1, \end{cases} \tag{2.1}$$

$$A^{-r} \natural_{\frac{1-r}{p+r}} B^p \geq A^{1-2r} \ \text{if } r > 1. \tag{2.2}$$

Proof. Firstly, we show (2.1). If $0 \leq r \leq 1$, then we have

$$A^{-r} \sharp_{\frac{1-r}{p+r}} B^p \leq B^{-r} \sharp_{\frac{1-r}{p+r}} B^p = B^{1-2r}$$

and

$$\begin{aligned} A^{-r} \sharp_{\frac{1-r}{p+r}} B^p &= A^{-r} \sharp_{\frac{1-r}{1+r}} (A^{-r} \sharp_{\frac{1+r}{p+r}} B^p) \\ &\leq A^{-r} \sharp_{\frac{1-r}{1+r}} (B^{-r} \sharp_{\frac{1+r}{p+r}} B^p) = A^{-r} \sharp_{\frac{1-r}{1+r}} B \leq A^{-r} \sharp_{\frac{1-r}{1+r}} A = A^{1-2r}. \end{aligned}$$

Therefore we obtain (2.1) since $B^{1-2r} \leq A^{1-2r}$ holds if $0 \leq r \leq \frac{1}{2}$ and $A^{1-2r} \leq B^{1-2r}$ holds if $\frac{1}{2} \leq r \leq 1$.

If $r > 1$, then we have (2.2) since

$$\begin{aligned} A^{-r} \natural_{\frac{1-r}{p+r}} B^p &= A^{-r}(A^r \sharp_{\frac{r-1}{p+r}} B^{-p})A^{-r} \\ &= A^{-r}(B^{-p} \sharp_{\frac{1+p}{r+p}} A^r)A^{-r} \geq A^{-r}AA^{-r} = A^{1-2r} \end{aligned}$$

holds by $(*)$ and (F). □

Next, we discuss grand Furuta type generalizations of Theorem 2.1. As a generalization of Theorem 1.B, Fujii and Nakamoto [9] showed the following result related to grand Furuta inequality.

Theorem 2.A ([9, Theorem 3.4])**.** *If $A \geq B > 0$ and $t \in [0, 1]$, then*

$$A^{-r+t} \natural_{\frac{1-t+r}{r+(p-t)s}} (A^t \sharp_s B^p) \leq A$$

holds for $p \leq -1$, $r \in [0, t]$ and $s \in \left[\max\left\{\frac{-t}{p-t}, \frac{-2r-(1-t)}{p-t}\right\}, 1\right]$.

Replacing p by $-p$ and using $(*)$, Theorem 2.A can be rewritten as follows: If $A \geq B > 0$ and $t \in [0, 1]$, then

$$A^{-r+t} \natural_{\frac{1-t+r}{r-(p+t)s}} (A^t \sharp_s B^{-p}) \leq A, \quad \text{that is,} \quad A^{r-t} \natural_{\frac{1-t+r}{(p+t)s-r}} (A^{-t} \sharp_s B^p) \leq A^{1-2(t-r)}$$

holds for $p \geq 1$, $r \in [0, t]$ and $s \in \left[\max\left\{\frac{t}{p+t}, \frac{1-t+2r}{p+t}\right\}, 1\right]$.

Here, we show an improvement of Theorem 2.A.

Theorem 2.2. *Let $A \geq B > 0$ and $0 \leq r \leq t \leq 1$. Then*

$$A^{r-t} \natural_{\frac{1-t+r}{(p+t)s-r}} (A^{-t} \natural_s B^p) \begin{cases} \leq B^{1-2(t-r)} \leq A^{1-2(t-r)} & \text{if } 0 \leq t-r \leq \frac{1}{2}, \\ \leq A^{1-2(t-r)} \leq B^{1-2(t-r)} & \text{if } \frac{1}{2} \leq t-r \leq 1 \end{cases}$$

holds for $p \geq 1$ and $\frac{1-t+2r}{p+t} \leq s \leq 1$.

Proof. Noting that $0 \leq \frac{1-t+r}{(p+t)s-r} \leq 1$ and $0 \leq t-r \leq 1$ hold, we have

$$A^{r-t} \natural_{\frac{1-t+r}{(p+t)s-r}} (A^{-t} \natural_s B^p) \leq B^{r-t} \natural_{\frac{1-t+r}{(p+t)s-r}} (B^{-t} \natural_s B^p) = B^{1-2(t-r)}.$$

Next we show $A^{r-t} \natural_{\frac{1-t+r}{(p+t)s-r}} (A^{-t} \natural_s B^p) \leq A^{1-2(t-r)}$ by dividing into three cases. If $(p+t)s-t \geq 1$ holds, then

$$\begin{aligned} A^{r-t} \natural_{\frac{1-t+r}{(p+t)s-r}} (A^{-t} \natural_s B^p) &\leq A^{r-t} \natural_{\frac{1-t+r}{(p+t)s-r}} (B^{-t} \natural_s B^p) = A^{r-t} \natural_{\frac{1-t+r}{(p+t)s-r}} B^{(p+t)s-t} \\ &= A^{r-t} \natural_{\frac{1-t+r}{1+t-r}} (A^{r-t} \natural_{\frac{1+(t-r)}{(p+t)s-t+(t-r)}} B^{(p+t)s-t}) \\ &\leq A^{r-t} \natural_{\frac{1-t+r}{1+t-r}} (B^{r-t} \natural_{\frac{1+(t-r)}{(p+t)s-t+(t-r)}} B^{(p+t)s-t}) \\ &= A^{r-t} \natural_{\frac{1-t+r}{1+t-r}} B \\ &\leq A^{r-t} \natural_{\frac{1-t+r}{1+t-r}} A = A^{1-2(t-r)}. \end{aligned}$$

If $0 \leq (p+t)s-t \leq 1$ holds, then

$$\begin{aligned} A^{r-t} \natural_{\frac{1-t+r}{(p+t)s-r}} (A^{-t} \natural_s B^p) &\leq A^{r-t} \natural_{\frac{1-t+r}{(p+t)s-r}} (B^{-t} \natural_s B^p) = A^{r-t} \natural_{\frac{1-t+r}{(p+t)s-r}} B^{(p+t)s-t} \\ &\leq A^{r-t} \natural_{\frac{1-t+r}{(p+t)s-r}} A^{(p+t)s-t} = A^{1-2(t-r)}. \end{aligned}$$

If $(p+t)s-t \leq 0$ holds, then

$$\begin{aligned} A^{-t} \natural_s B^p &= A^{-t} \natural_{\frac{(p+t)s}{t}} (A^{-t} \natural_{\frac{t}{p+t}} B^p) \\ &\leq A^{-t} \natural_{\frac{(p+t)s}{t}} (B^{-t} \natural_{\frac{t}{p+t}} B^p) = A^{-t} \natural_{\frac{(p+t)s}{t}} I = A^{(p+t)s-t}, \end{aligned}$$

so that we have

$$A^{r-t} \natural_{\frac{1-t+r}{(p+t)s-r}} (A^{-t} \natural_s B^p) \leq A^{r-t} \natural_{\frac{1-t+r}{(p+t)s-r}} A^{(p+t)s-t} = A^{1-2(t-r)}.$$

Therefore we obtain the desired result since $B^{1-2(t-r)} \leq A^{1-2(t-r)}$ holds if $0 \leq t-r \leq \frac{1}{2}$ and $A^{1-2(t-r)} \leq B^{1-2(t-r)}$ holds if $\frac{1}{2} \leq t-r \leq 1$. □

We remark that Theorem 2.2 (grand Furuta type inequality) interpolates Theorem 2.1 (Furuta type inequality) and Theorem 1.A (Ando-Hiai type inequality) as follows: By putting $s = 1$ and $r = 0$ (and replacing t by r) in Theorem 2.2, we have (2.1) in Theorem 2.1.

On the other hand, by putting $t = 1$, Theorem 2.2 implies the following Theorem 2.3, which is an improvement of [9, Theorem 3.2].

Theorem 2.3. *Let $A \geq B > 0$ and $0 \leq r \leq 1$. Then*

$$A^{r-1} \sharp_{\frac{r}{(p+1)s-r}} (A^{-1} \sharp_s B^p) \begin{cases} \leq B^{2r-1} \leq A^{2r-1} & \text{if } \frac{1}{2} \leq r \leq 1, \\ \leq A^{2r-1} \leq B^{2r-1} & \text{if } 0 \leq r < \frac{1}{2} \end{cases}$$

holds for $p \geq 1$ and $\frac{2r}{p+1} \leq s \leq 1$.

Theorem 2.3 implies the following result by putting $s = r$.

Corollary 2.4. *Let $A \geq B > 0$ and $0 \leq r \leq 1$. Then*

$$A^{r-1} \sharp_{\frac{1}{p}} (A^{-1} \sharp_r B^p) \leq A^{2r-1}$$

holds for $p \geq 1$.

We understand that Theorem 1.A is equivalent to Corollary 2.4 by the replacements $S = A^{-1}$, $T = (A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^{-\alpha}$ and $p = \frac{1}{\alpha}$ as follows: For $\alpha \in [0, 1]$,

$$A \natural_{-\alpha} B \leq I \iff (A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^{-\alpha} \leq A^{-1} \iff S \geq T.$$

Since $T = (A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^{-\alpha}$ is equivalent to $B = (S^{\frac{1}{2}} T^{\frac{1}{\alpha}} S^{\frac{1}{2}})^{-1}$, for $r \in [0, 1]$,

$$\begin{aligned} A^r \natural_{-\alpha} B^r \leq I &\iff S^{-r} \natural_{-\alpha} (S^{\frac{1}{2}} T^{\frac{1}{\alpha}} S^{\frac{1}{2}})^{-r} \leq I \\ &\iff S^{-r} \left\{ S^r \sharp_\alpha (S^{\frac{1}{2}} T^{\frac{1}{\alpha}} S^{\frac{1}{2}})^r \right\} S^{-r} \leq I \quad \text{by } (*) \\ &\iff S^{\frac{1}{2}-r} \left\{ S^{r-1} \sharp_\alpha (S^{-1} \sharp_r T^{\frac{1}{\alpha}}) \right\} S^{\frac{1}{2}-r} \leq I \\ &\iff S^{r-1} \sharp_{\frac{1}{p}} (S^{-1} \sharp_r T^p) \leq S^{2r-1}. \end{aligned}$$

3 Inequalities for chaotic order

In this section, we show a generalization of Theorems 2.1 and 2.2.

Theorem 3.1. *Let $\log A \geq \log B$ for $A, B > 0$ and $0 \leq r \leq t$.*

(i) *For $p > 0$ and $\frac{1-t+2r}{p+t} \leq s \leq 1$,*

$$A^{r-t} \sharp_{\frac{1-t+r}{(p+t)s-r}} (A^{-t} \sharp_s B^p) \begin{cases} \leq B^{1-2(t-r)} & \text{if } 0 \leq t-r \leq \frac{1}{2}, \\ \leq A^{1-2(t-r)} & \text{if } \frac{1}{2} \leq t-r \leq 1. \end{cases}$$

(ii) *For $p > 0$ and $\frac{t-1}{p+t} \leq s \leq 1$,*

$$A^{r-t} \natural_{\frac{1-t+r}{(p+t)s-r}} (A^{-t} \sharp_s B^p) \geq A^{1-2(t-r)} \text{ if } t - r > 1.$$

We remark that inequalities in Theorem 3.1 hold for chaotic order $\log A \geq \log B$, which is weaker assumption than usual order $A \geq B$, and Theorem 3.1 holds for some looser conditions of parameters than Theorem 2.2. Moreover, Theorem 3.1 gives a generalization of Theorem 2.1 by putting $s = 1$ and $r = 0$ (and replacing t by r).

In order to prove Theorem 3.1, we use the following theorem in [16] (see also [3, 6, 12, 21, 26]).

Theorem 3.A ([16])**.** *Let $A, B > 0$. Then the following assertions are mutually equivalent.*

(i) $\log A \geq \log B$.

(ii) *For any fixed $q \geq 0$, $F_q(p,r) = A^{-r} \natural_{\frac{q+r}{p+r}} B^p$ is decreasing for $p \geq q$ and $r \geq 0$.*

(iii) *For any fixed $q \leq 0$, $F_q(p,r) = A^{-r} \natural_{\frac{q+r}{p+r}} B^p$ is decreasing for $p \geq 0$ and $r \geq -q$.*

Since $\log A \geq \log B$ is equivalent to $\log B^{-1} \geq \log A^{-1}$, Theorem 3.A ensures that $\log A \geq \log B$ implies the following two statements.

(i) For any fixed $q \geq 0$, $\widehat{F}_q(p,r) = B^{-r} \natural_{\frac{q+r}{p+r}} A^p$ is increasing for $p \geq q$ and $r \geq 0$,

(ii) For any fixed $q \leq 0$, $\widehat{F}_q(p,r) = B^{-r} \natural_{\frac{q+r}{p+r}} A^p$ is increasing for $p \geq 0$ and $r \geq -q$.

We remark that $\log A \geq \log B$ implies that

$$\begin{aligned} &F_q(p,r) \leq F_q(p,0) = B^q \quad \text{for } p \geq q \text{ and } r \geq 0 \text{ if } q \geq 0, \\ &F_q(p,r) \leq F_q(p,-q) = A^q \quad \text{for } p \geq 0 \text{ and } r \geq -q \text{ if } q \leq 0 \end{aligned} \tag{3.1}$$

by Theorem 3.A, and also the similar inequalities hold for $\widehat{F}_q(p,r)$.

Proof of Theorem 3.1. By (3.1), $\log A \geq \log B$ implies

$$A^{-t} \natural_s B^p = A^{-t} \natural_{\frac{(p+t)s-t+t}{p+t}} B^p \begin{cases} \leq B^{(p+t)s-t} & \text{if } (p+t)s-t \geq 0, \\ \leq A^{(p+t)s-t} & \text{if } (p+t)s-t \leq 0 \end{cases} \tag{3.2}$$

for $p > 0$, $t \geq 0$ and $0 \leq s \leq 1$.

Firstly, we show (i). We may assume $t-r < 1$. We note that $1-t+r > 0$, $(p+t)s-r > 0$ and $0 < \frac{1-t+r}{(p+t)s-r} \leq 1$ hold. If $(p+t)s-t \geq 0$, then

$$A^{r-t} \natural_{\frac{1-t+r}{(p+t)s-r}} (A^{-t} \natural_s B^p) \leq A^{r-t} \natural_{\frac{1-t+r}{(p+t)s-r}} B^{(p+t)s-t} \begin{cases} \leq B^{1-2(t-r)} & \text{if } 1-2(t-r) \geq 0, \\ \leq A^{1-2(t-r)} & \text{if } 1-2(t-r) \leq 0 \end{cases}$$

holds for $t - r \geq 0$, where the first inequality holds by (3.2) and the second ones hold by (3.1) since $\frac{1-t+r}{(p+t)s-r} = \frac{1-2(t-r)+(t-r)}{(p+t)s-t+(t-r)}$. If $(p+t)s-t \leq 0$, then

$$A^{r-t} \natural_{\frac{1-t+r}{(p+t)s-r}} (A^{-t} \natural_s B^p) \leq A^{r-t} \natural_{\frac{1-t+r}{(p+t)s-r}} A^{(p+t)s-t} = A^{1-2(t-r)}$$

holds by (3.2). In this case, $\frac{1-t+2r}{p+t} \leq s \leq \frac{t}{p+t}$ holds, so that $1-2(t-r) \leq 0$ holds. Therefore the proof of (i) is complete.

Next we show (ii). We note that $1-t+r<0$, $(p+t)s-r>0$ and $-1 \le \frac{1-t+r}{(p+t)s-r} < 0$ hold. If $(p+t)s-t \ge 0$, then

$$\begin{aligned}
A^{r-t} \natural_{\frac{1-t+r}{(p+t)s-r}} (A^{-t} \sharp_s B^p) &= A^{r-t} \left\{ A^{t-r} \sharp_{\frac{-1+t-r}{(p+t)s-r}} (A^{-t} \sharp_s B^p)^{-1} \right\} A^{r-t} \\
&\ge A^{r-t} \left\{ A^{t-r} \sharp_{\frac{-1+t-r}{(p+t)s-r}} B^{-((p+t)s-t)} \right\} A^{r-t} \\
&= A^{r-t} \left\{ B^{-((p+t)s-t)} \sharp_{\frac{1+(p+t)s-t}{t-r+(p+t)s-t}} A^{t-r} \right\} A^{r-t} \\
&\ge A^{r-t} A A^{r-t} = A^{1-2(t-r)}
\end{aligned}$$

holds for $t-r>1$ by (3.2) and Theorem 3.A. If $(p+t)s-t \le 0$, then

$$A^{r-t} \natural_{\frac{1-t+r}{(p+t)s-r}} (A^{-t} \sharp_s B^p) \ge A^{r-t} \natural_{\frac{1-t+r}{(p+t)s-r}} A^{(p+t)s-t} = A^{1-2(t-r)}$$

holds by (3.2). Therefore the proof of (ii) is complete. □

4 Inequalities for $s < \frac{1-t+2r}{p+t}$

In [9], Fujii and Nakamoto also considered the case $s = \frac{t}{p+t} < \frac{1-t+2r}{p+t}$. As a generalization of [9, Theorems 3.6 and 3.8], we obtain the following results.

Theorem 4.1. *Let $A \ge B > 0$ and $0 \le r \le t$. Then*

$$A^{r-t} \natural_{\frac{1-t+r}{(p+t)s-r}} (A^{-t} \sharp_s B^p) \le A^{-t} \sharp_{\frac{1-t+2r}{p+t}} B^p \le A^{1-2(t-r)}$$

holds for $p \ge 1$ and $\max\left\{\frac{1-t+2r}{2(p+t)}, \frac{r}{p+t}\right\} \le s \le \frac{1-t+2r}{p+t}$ *with $(p+t)s-r \neq 0$.*

Theorem 4.2. *Let $\log A \ge \log B$ for $A, B > 0$ and $0 \le r \le t$ with $t-r \ge \frac{1}{2}$. Then*

$$A^{r-t} \natural_{\frac{1-t+r}{(p+t)s-r}} (A^{-t} \sharp_s B^p) \le A^{-t} \natural_{\frac{1-t+2r}{p+t}} B^p \le A^{1-2(t-r)}$$

holds for $p > 0$ and $\max\left\{\frac{1-t+2r}{2(p+t)}, \frac{r}{p+t}\right\} \le s \le \frac{1-t+2r}{p+t}$ *with $(p+t)s-r \neq 0$.*

In order to prove Theorems 4.1 and 4.2, we use the following inequalities in [20] known as the Furuta type inequalities with negative powers (cf. [4, 17, 24, 27]).

Theorem 4.A ([20]). *If $A \ge B \ge 0$ with $A > 0$, then the following inequalities hold.*

(i) $A^t \natural_{\frac{1-t}{p-t}} B^p \le B \le A$ *holds for $0 \le t < p \le 1$ with $p \ge \frac{1}{2}$.*

(ii) $A^t \natural_{\frac{2p-t}{p-t}} B^p \le B \le A$ *holds for $0 \le t < p \le \frac{1}{2}$.*

By replacing A, B, p, t by $A^q, B^q, \frac{p}{q}, \frac{t}{q}$ respectively, we have the following proposition.

Proposition 4.3. *Let $A > 0$ and $B \geq 0$. If $A^q \geq B^q$ for $q > 0$, then the following inequalities hold.*

(i) $A^t \natural_{\frac{q-t}{p-t}} B^p \leq B^q \leq A^q$ *holds for* $0 \leq t < p \leq q$ *with* $p \geq \frac{q}{2}$.

(ii) $A^t \natural_{\frac{2p-t}{p-t}} B^p \leq B^q \leq A^q$ *holds for* $0 \leq t < p \leq \frac{q}{2}$.

Proof of Theorem 4.1. By Furuta inequality, $A \geq B > 0$ implies

$$A^{1+t} \geq (A^{\frac{t}{2}} B^p A^{\frac{t}{2}})^{\frac{1+t}{p+t}} \tag{4.1}$$

for $p \geq 1$ and $t \geq 0$. Put $A_1 = A^{1+t}$ and $B_1 = (A^{\frac{t}{2}} B^p A^{\frac{t}{2}})^{\frac{1+t}{p+t}}$. Then $A_1^q \geq B_1^q$ holds for $0 \leq q \leq 1$ by (4.1) and Loewner-Heinz theorem. Then by putting $p_1 = \frac{(p+t)s}{1+t}$, $t_1 = \frac{r}{1+t}$ and $q = \frac{1-t+2r}{1+t}$, (i) in Proposition 4.3 ensures that

$$A_1^{t_1} \natural_{\frac{q-t_1}{p_1-t_1}} B_1^{p_1} \leq B_1^q \leq A_1^q$$

holds for $0 \leq t_1 < p_1 \leq q \leq 1$ with $p_1 \geq \frac{q}{2}$, that is,

$$A^r \natural_{\frac{1-t+r}{(p+t)s-r}} (A^{\frac{t}{2}} B^p A^{\frac{t}{2}})^s \leq (A^{\frac{t}{2}} B^p A^{\frac{t}{2}})^{\frac{1-t+2r}{p+t}} \leq A^{1-t+2r}$$

holds for $0 \leq r \leq t$, $p \geq 1$ and $\max\left\{\frac{1-t+2r}{2(p+t)}, \frac{r}{p+t}\right\} \leq s \leq \frac{1-t+2r}{p+t}$ with $(p+t)s - r \neq 0$. Therefore we have the desired result. □

Proof of Theorem 4.2. By Theorem 3.A, $\log A \geq \log B$ implies

$$A^t \geq (A^{\frac{t}{2}} B^p A^{\frac{t}{2}})^{\frac{t}{p+t}} \tag{4.2}$$

for $p > 0$ and $t \geq 0$. Put $A_1 = A^t$ and $B_1 = (A^{\frac{t}{2}} B^p A^{\frac{t}{2}})^{\frac{t}{p+t}}$. Then $A_1^q \geq B_1^q$ holds for $0 \leq q \leq 1$ by (4.2) and Loewner-Heinz theorem. Then by putting $p_1 = \frac{(p+t)s}{t}$, $t_1 = \frac{r}{t}$ and $q = \frac{1-t+2r}{t}$, (i) in Proposition 4.3 ensures that

$$A_1^{t_1} \natural_{\frac{q-t_1}{p_1-t_1}} B_1^{p_1} \leq B_1^q \leq A_1^q$$

holds for $0 \leq t_1 < p_1 \leq q \leq 1$ with $p_1 \geq \frac{q}{2}$, that is,

$$A^r \natural_{\frac{1-t+r}{(p+t)s-r}} (A^{\frac{t}{2}} B^p A^{\frac{t}{2}})^s \leq (A^{\frac{t}{2}} B^p A^{\frac{t}{2}})^{\frac{1-t+2r}{p+t}} \leq A^{1-t+2r}$$

holds for $0 \leq r \leq t$, $p > 0$, $t - r \geq \frac{1}{2}$ and $\max\left\{\frac{1-t+2r}{2(p+t)}, \frac{r}{p+t}\right\} \leq s \leq \frac{1-t+2r}{p+t}$ with $(p+t)s - r \neq 0$. Therefore we have the desired result. □

Theorems 3.1 and 4.1 ensure the following, which is a slight extension of [9, Theorems 3.6 and 3.8].

Theorem 4.4. *Let* $A \geq B > 0$ *and* $0 \leq r < t$ *with* $-1 \leq 1-2(t-r) \leq t$. *Then*

$$A^{r-t} \natural_{\frac{1-t+r}{(p+t)s-r}} (A^{-t} \sharp_s B^p) \leq A^{1-2(t-r)} \tag{4.3}$$

holds for $p \geq 1$ *and* $s = \frac{t}{p+t}$.

Proof. By putting $s = \frac{t}{p+t}$ in Theorem 4.1, (4.3) holds for $0 \leq 1-2(t-r) \leq t$. By putting $s = \frac{t}{p+t}$ in (i) of Theorem 3.1, (4.3) holds for $-1 \leq 1-2(t-r) \leq 0$. □

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