KATO'S INEQUAILITIES UP TO THE BOUNDARY FOR A QUASILINEAR ELLIPTIC OPERATOR

Toshio Horiuchi, Peter Kumlin

Received June 29, 2019

Abstract

Let Ω be a bounded smooth domain of $\mathbb{R}^N$. By Δ_p with $1 < p < \infty$ we denote p-Laplacian. We prove that if $\Delta_p u$ is a finite measure in Ω, then under suitable assumptions on u, $\Delta_p u^+$ is also a finite measure in Ω up to the boundary $\partial\Omega$. *

1 Introduction

Let Ω be a bounded smooth domain of $\mathbb{R}^N$. By Δ_p for $p \in (1, +\infty)$ we denote p-Laplacian. The classical Kato's inequality for a Laplacian in [12] asserts that given any function $u \in L^1_{\text{loc}}(\Omega)$ such that $\Delta u \in L^1_{\text{loc}}(\Omega)$, then $\Delta(u^+)$ is a Radon measure and the following holds:

$$\Delta(u^+) \geq \chi_{[u \geq 0]} \Delta u \qquad \text{in } D'(\Omega), \tag{1.1}$$

where $u^+ = \max\{u, 0\}$. In [5, 6], H.Brezis and A.Ponce intensively studied Kato's inequalities with Δu being a Radon measure and established the strong maximum principle, the improved Kato's inequality and the inverse maximum principle (See also [8, 10]). Then, in [13, 14] Kato's inequality was further studied for $\Delta_p u$ with $p \in (1, \infty)$ and most of the counter-parts were established under the assumption that u is admissible in $W^{1,p^*}_{\text{loc}}(\Omega)$, where $p^* := \max\{1, p-1\}$. For the admissibility in $W^{1,p^*}_{\text{loc}}(\Omega)$, see Definition 4.1 in Appendix and see also [15]. We remark that when $p = 2$, the notion of admissibility becomes trivial. On the other hand, H.Brezis and A. Ponce in [7] and A. Ancona in [1] studied Kato's inequality (1.1) up to the boundary for $p = 2$.

The purpose in the present paper is to study Kato's inequality for Δ_p up to the boundary of Ω. As a result, we will show that $\Delta_p u^+$ is also a finite measure under suitable assumptions on u. In these arguments it is crucial to introduce a class $\mathbb{X}_p$ in Definition 1.1, which was originally introduced in Brezis, Ponce [7] for Δ, and to use effectively a notion of admissibility in $\mathbb{X}_p$ for Δ_p.

Definition 1.1. We say $u \in \mathbb{X}_p$ if $u \in W^{1,p^*}(\Omega)$ and if there exists a constant $C > 0$ such that

$$\left| \int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi \right| \leq C ||\varphi||_{L^\infty(\Omega)}, \qquad \text{for any } \varphi \in C^1(\overline{\Omega}), \tag{1.2}$$

in which case we set

$$[u]_{\mathbb{X}_p} = \sup_{\substack{\psi \in C^1(\bar{\Omega}) \\ \|\psi\|_{L^\infty} \leq 1}} \int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla \psi. \tag{1.3}$$

If $u \in \mathbb{X}_p$, then there exists a unique bounded linear functional $T \in [C(\bar{\Omega})]^* = \mathscr{M}_b(\bar{\Omega})$ such that

$$\langle T, \psi \rangle = \int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla \psi \quad (\forall \psi \in C^1(\bar{\Omega})).$$

*Keywords: Kato's inequality, p-Laplace operator
2000 mathematics Subject Classification: Primary 35J70, Secondary 35J60
This research was partially supported by Grant-in-Aid for Scientific Research (No. 16K05189) and (No. 15H03621).

On the other hand, by the Riesz Representation Theorem any $T \in \mathscr{M}_b(\bar{\Omega})$ admits a unique decomposition

$$\langle T, \psi \rangle = \int_{\partial\Omega} \psi \, d\nu + \int_{\Omega} \psi \, d\mu \quad (\forall \psi \in C(\bar{\Omega})),$$

where $\mu \in \mathscr{M}_b(\Omega)$ and $\nu \in \mathscr{M}_b(\partial\Omega)$. By $\mathscr{M}_b(\Omega)$ and $\mathscr{M}_b(\partial\Omega)$ we denote the space of all bounded measures in Ω and $\partial\Omega$, equipped with the standard norms $\|\cdot\|_{\mathscr{M}_b(\Omega)}$ and $\|\cdot\|_{\mathscr{M}_b(\partial\Omega)}$ respectively. We remark that measures in $\mathscr{M}_b(\Omega)$ are identified with measures in Ω which do not charge $\partial\Omega$. More precisely we have

$$||\mu||_{\mathscr{M}_b(\Omega)} = \sup \left\{ \int_{\Omega} \varphi \, d\mu ; \varphi \in C_0(\bar{\Omega}) \text{ and } ||\varphi||_{L^\infty(\Omega)} \le 1 \right\},$$

where by $C_0(\bar{\Omega})$ we denote the space of all continuous functions on $\bar{\Omega}$ vanishing on $\partial\Omega$. On the other hand $\mathscr{M}(\Omega)$ denotes the space of all Radon measures in Ω. In other words $\mu \in \mathscr{M}(\Omega)$ if and only if, for every $\omega \subset\subset \Omega$, there is $C_\omega > 0$ such that $|\int_\Omega \varphi \, d\mu| \le C_\omega ||\varphi||_\infty$ for all $\varphi \in C_0(\overline{\omega})$. When $u \in \mathbb{X}_p$, we will denote

$$\mu = -\Delta_p u \,, \quad \nu = |\nabla u|^{p-2} \frac{\partial u}{\partial n},$$

where n denotes the outer normal. In this paper, for $u \in \mathbb{X}_p$ we always use the notations $\Delta_p u$ and $|\nabla u|^{p-2} \frac{\partial u}{\partial n}$ in the above sense. Hence if $u \in \mathbb{X}_p$, then we have

$$\int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla \psi = \int_{\partial\Omega} \psi |\nabla u|^{p-2} \frac{\partial u}{\partial n} - \int_{\Omega} \psi \Delta_p u \quad (\forall \psi \in C^1(\bar{\Omega})).$$

It follows from Theorem 3.1 that for every $u \in \mathbb{X}_p$

$$[u]_{\mathbb{X}_p} = \int_{\Omega} |\Delta_p u| + \int_{\partial\Omega} |\nabla u|^{p-2} \left| \frac{\partial u}{\partial n} \right|$$

and if u is admissible in $\mathbb{X}_p$, then $[u]_{\mathbb{X}_p} = 0$ if and only if $u = const.$ in Ω.

2 Preliminaries: Admissibilities in $\mathbb{X}_p$ and $W_0^{1,p^*}(\Omega)$

We will work with the standard Sobolev spaces; $W^{1,p}(\Omega)$ and $W_0^{1,p}(\Omega)$, where the space $W^{1,p}(\Omega)$ is equipped with the norm

$$||u||_{W^{1,p}(\Omega)} = |||\nabla u|||_{L^p(\Omega)} + ||u||_{L^p(\Omega)}, \tag{2.1}$$

and by $W_0^{1,p}(\Omega)$ we denote the completion of $C_c^\infty(\Omega)$ in the norm $||\cdot||_{W^{1,p}(\Omega)}$. Now we introduce two admissiblities for Δ_p to deal with Kato's inequalities up to the boundary. We note that these notions become trivial if $p = 2$ and a local version was already introduced in [14].

Definition 2.1. (**Admissibility in** $\mathbb{X}_p$) Let $1 < p < \infty$ and $p^* := \max\{1, p-1\}$. A function u is said to be admissible in $\mathbb{X}_p$ if $u \in \mathbb{X}_p$ and there exists a sequence $\{u_k\}_{k=1}^\infty \subset W^{1,p}(\Omega) \cap L^\infty(\Omega)$ such that:

1. $u_k \to u$ a.e. in Ω and $u_k \to u$ in $W^{1,p^*}(\Omega)$ as $k \to \infty$.
2. $\Delta_p u_k \in L^1(\Omega)$ and $|\nabla u_k|^{p-2} \frac{\partial u_k}{\partial n} \in L^1(\partial\Omega)$ $(k = 1, 2, \cdots)$ and

$$\sup_k ||\Delta_p u_k||_{\mathscr{M}_b(\Omega)} = \sup_k \int_{\Omega} |\Delta_p u_k| < \infty \tag{2.2}$$

$$\sup_k \left|\left| |\nabla u_k|^{p-2} \frac{\partial u_k}{\partial n} \right|\right|_{\mathscr{M}_b(\partial\Omega)} = \sup_k \int_{\partial\Omega} \left| |\nabla u_k|^{p-2} \frac{\partial u_k}{\partial n} \right| < \infty. \tag{2.3}$$

Definition 2.2. (**Admissibility in** $W_0^{1,p^*}(\Omega)$) Let $1 < p < \infty$ and $p^* := \max\{1, p-1\}$. A function u is said to be admissible in $W_0^{1,p^*}(\Omega)$ if $u \in W_0^{1,p^*}(\Omega)$, $\Delta_p u \in \mathscr{M}_b(\Omega)$ and there exists a sequence $\{u_k\}_{k=1}^{\infty} \subset W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ such that:

1. $u_k \to u$ a.e. in Ω and $u_k \to u$ in $W_0^{1,p^*}(\Omega)$ as $k \to \infty$.
2. $\Delta_p u_k \in L^1(\Omega)$ $(k = 1, 2, \cdots)$ and

$$\sup_k ||\Delta_p u_k||_{\mathscr{M}_b(\Omega)} = \sup_k \int_\Omega |\Delta_p u_k| < \infty. \tag{2.4}$$

Roughly speaking, if u is admissible in one of these definitions, then u can be approximated by a sequence of good functions not only in the sense of the distributions but also in the sense of measures. Moreover it is possible to approximate u by a sequence of C^1-functions provided that u is admissible. In fact in Proposition 4.1 in Appendix we collect such nice properties of admissible functions together with a local version of the admissibility in $W_{\mathrm{loc}}^{1,p^*}(\Omega)$. In the subsequent we describe more remarks.

Remark 2.1.

1. For a general class of uniformly elliptic operators with a divergence form, one can define the admissibility and establish similar results in parallel to the present paper (c.f. [15]). Further it is possible to construct non-admissible functions in such cases. When $p = 2$, the existence of pathological solution, which is non-admissible, was initially shown by J Serrin in the famous paper [20] (See also [11]).

2. If $u \in W_{\mathrm{loc}}^{1,p^*}(\Omega)$, then $\Delta_p u$, $\Delta_p(u^+)$ and $\Delta_p(u^-)$ are well-defined in $D'(\Omega)$. Let $\{u_k\}$ be the sequence in one of the definitions. It follows from the condition 1 that $\Delta_p u_k = \Delta_p(u_k^+) - \Delta_p(u_k^-)$ and $\Delta_p u_k \to \Delta_p u$ (i.e. $\Delta_p(u_k^\pm) \to \Delta_p(u^\pm)$) in $D'(\Omega)$ as $k \to \infty$. Moreover, it follows from the condition 2 and the weak compactness of measures that we have $\Delta_p u_k \to \Delta_p u$ (i.e. $\Delta_p(u_k^\pm) \to \Delta_p(u^\pm)$) in the sense of measures as $n \to \infty$. (In the case of Definition 2.1, $|\nabla u_k|^{p-2} \frac{\partial u_k}{\partial n} \to |\nabla u|^{p-2} \frac{\partial u}{\partial n}$ in the sense of measures as well.) Therefore if u is admissible, then u^+ and u^- are so as well.

3. Let Ω be a unit ball $B_1(0)$ of R^N. Let $u = |x|^\alpha - 1$ for $\alpha = (p-N)/(p-1)$ and $p \in (1, N)$. Then u satisfies
$$\Delta_p u = \alpha|\alpha|^{p-2} c_N \delta \qquad \text{in } D'(\Omega),$$
where δ denotes a Dirac mass and c_N denotes the surface area of the N-dimensional unit ball B_1. Then u is admissible in $W_0^{1,p^*}(\Omega)$ if $p \in (2 - 1/N, N)$ with $N \geq 2$. We note that when $1 < p < 2 - \frac{1}{N}$, u is not admissible but regarded as a renormalized solution. For the detail see [2, 4, 17, 18, 19]

3 Main results

Given $M > 0$, we denote a truncation function T_M: $R \to R$ by

$$T_M(s) = \max\{-M, \min\{M, s\}\}. \tag{3.1}$$

Theorem 3.1. If $u \in \mathbb{X}_p$, then we have:

1.
$$[u]_{\mathbb{X}_p} = \int_\Omega |\Delta_p u| + \int_{\partial\Omega} |\nabla u|^{p-2} \Big| \frac{\partial u}{\partial n} \Big|. \tag{3.2}$$

2. If u is admissible in $\mathbb{X}_p$, then for every $M > 0$ $T_M u \in W^{1,p}(\Omega)$ and we have

$$\int_\Omega |\nabla T_M(u)|^p \leq M[u]_{\mathbb{X}_p}. \tag{3.3}$$

3. If u is admissible in $\mathbb{X}_p$, then $[u]_{\mathbb{X}_p} = 0$ if and only if $u = const.\ in\ \Omega$.

Theorem 3.2. If u is admissible in $\mathbb{X}_p$, then $u^+ \in \mathbb{X}_p$ and we have

$$[u^+]_{\mathbb{X}_p} \leq [u]_{\mathbb{X}_p}. \tag{3.4}$$

Theorem 3.3. Assume that u is admissible in $W_0^{1,p^*}(\Omega)$. Then we have the followings:

1. u is admissible in $\mathbb{X}_p$ (hence, $u^+ \in \mathbb{X}_p$).
2. $$\int_\Omega |\Delta_p u^+| \leq \int_\Omega |\Delta_p u|. \tag{3.5}$$

Remark 3.1. If u does not vanish on $\partial\Omega$, then the assertion (3.5) fails. To see this it suffices to take a linear function u.

Theorem 3.4. Assume that u is admissible in $\mathbb{X}_p$. Moreover assume that $\Delta_p u \in L^1(\Omega)$, $|\nabla u|^{p-2}\frac{\partial u}{\partial n} \in L^1(\partial\Omega)$. Then

$$\int_\Omega |\nabla u|^{p-2}\nabla u^+ \cdot \nabla\psi \leq \int_{\partial\Omega} H\psi - \int_\Omega G\psi \quad (\forall\psi \in C^1(\bar{\Omega}), \psi \geq 0\ in\ \Omega). \tag{3.6}$$

Here $G \in L^1(\Omega)$ and $H \in L^1(\partial\Omega)$ are given by

$$G = \begin{cases} \Delta_p u & on\ [u>0] \\ 0 & on\ [u \leq 0] \end{cases}, \quad H = \begin{cases} |\nabla u|^{p-2}\frac{\partial u}{\partial n} & on\ [u>0] \\ 0 & on\ [u<0] \\ \min\{|\nabla u|^{p-2}\frac{\partial u}{\partial n}, 0\} & on\ [u=0]. \end{cases} \tag{3.7}$$

Thus, we have

$$\begin{cases} \Delta_p u^+ \geq G & in\ \Omega \\ |\nabla u|^{p-2}\frac{\partial u^+}{\partial n} \leq H & on\ \partial\Omega. \end{cases} \tag{3.8}$$

3.1 Proof of Theorem 3.1

Proof of Theorem 3.1 (1). This is a standard argument. Since $u \in \mathbb{X}_p$, we have

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla\psi = \int_{\partial\Omega} \psi\nu + \int_{\partial\Omega} \psi\mu \quad (\forall\psi \in C^1(\bar{\Omega})), \tag{3.9}$$

where $\mu = -\Delta_p u \in \mathscr{M}_b(\Omega)$ and $\nu = |\nabla u|^{p-2}\frac{\partial u}{\partial n} \in \mathscr{M}_b(\partial\Omega)$. From the definition we have

$$[u]_{\mathbb{X}_p} = \sup_{\substack{\psi\in C^1(\bar{\Omega}) \\ \|\psi\|_{L^\infty}\leq 1}} \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla\psi \leq \int_\Omega |\Delta_p u| + \int_{\partial\Omega} |\nabla u|^{p-2}\Big|\frac{\partial u}{\partial n}\Big|.$$

To see the opposite inequality, without the loss of generality we assume that $\mu \in C_c^\infty(\Omega)$ and $\nu \in C_c^\infty(\mathbb{R}^N)$ with $\operatorname{supp}\mu \cap \operatorname{supp}\nu = \phi$. Define $\psi = \operatorname{sgn}(\mu) + \operatorname{sgn}(\nu)$, where $\operatorname{sgn}(t) = 1, t>0; 0, t=0; -1, t<0$. Let ψ_ε be a mollification of ψ such that $\psi_\varepsilon \in C_c^\infty(\mathbb{R}^N)$, $|\psi_\varepsilon| \leq 1$ and $\psi_\varepsilon \to \psi$ as $\varepsilon \downarrow 0$. Then for any $\eta > 0$ there exists a $\varepsilon > 0$ such that we have

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla\psi_\varepsilon \geq \int_\Omega |\Delta_p u| + \int_{\partial\Omega} |\nabla u|^{p-2}\Big|\frac{\partial u}{\partial n}\Big| - \eta.$$

Since η is an arbitrary positive number, the desired inequality holds. □

Proofs of (2) **and** (3)**.** The assertion (3) clearly follows from (2), we hence prove (2). Assume that u is admissible in $\mathbb{X}_p$. Then from Definition 2.1 there exists a sequence $\{u_k\} \subset W^{1,p}(\Omega) \cap L^\infty(\Omega)$ satisfying the properties 1 and 2. Noting that $\nabla(T_M u_k) = \chi_{|u_k| \le M} \nabla u_k$, we have

$$\begin{aligned}\int_\Omega |\nabla T_M(u_k)|^p \, dx &= \int_\Omega |\nabla u_k|^{p-2} \nabla u_k \cdot \nabla T_M(u_k) \\ &= \int_{\partial\Omega} |\nabla u_k|^{p-2} \frac{\partial u_k}{\partial n} T_M u_k - \int_\Omega \Delta_p u_k T_M u_k \\ &\le M[u_k]_{\mathbb{X}_p}.\end{aligned}$$

From the property 1 we see that $\Delta_p u_k \to \Delta_p u$ in $D'(\Omega)$ as $k \to \infty$. From the property 2 together with the weak compactness of Radon measures and the uniqueness of weak limit (see also Remark 2.1.2), $\lim_{k\to\infty} \Delta_p u_k = \Delta_p u$ in the sense of measures. Then by Fatou's lemma the assertion is proved. □

3.2 Proof of Theorem 3.2

First we prove Theorem 3.2 assuming that $u \in C^1(\overline{\Omega})$ and $\Delta_p u \in L^1(\Omega)$. Then we treat the general case.

Lemma 3.1. Assume that $u \in C^1(\bar{\Omega})$ and $\Delta_p u \in L^1(\Omega)$ (in the sense of distribution). Then

$$\int_\Omega |\nabla u|^{p-2} \nabla u^+ \cdot \nabla\phi \le \int_{\substack{\partial\Omega \\ [u \ge 0]}} \phi |\nabla u|^{p-2} \frac{\partial u}{\partial n} - \int_{\substack{\Omega \\ [u \ge 0]}} \phi \Delta_p u \quad (\forall \phi \in C^1(\bar{\Omega}), \phi \ge 0 \; in \; \bar{\Omega}). \tag{3.10}$$

Proof. Let Φ is a C^2 convex function in $\mathbb{R}$, $\Phi' \ge 0$ in $\mathbb{R}$ and $\Phi' \in L^\infty(\mathbb{R})$.

First we assume that $p \ge 2$.

By a direct calculation we see that

$$\Delta_p \Phi(u) = \Phi'(u)^{p-1} \Delta_p u + (p-1)\Phi'(u)^{p-2}\Phi''(u)|\nabla u|^p \qquad \text{in } D'(\Omega). \tag{3.11}$$

Since $\Phi'' \ge 0$, we have

$$\Delta_p \Phi(u) \ge \Phi'(u)^{p-1} \Delta_p u \qquad \text{in } D'(\Omega), \tag{3.12}$$

in particular, $\Delta_p \Phi(u) \in L^1(\Omega)$. Hence, for any $\phi \in C^1(\bar{\Omega}), \phi \ge 0 \; in \; \bar{\Omega}$ we have

$$\begin{aligned}\int_\Omega |\nabla\Phi(u)|^{p-2} \nabla\Phi(u) \cdot \nabla\phi &= \int_{\partial\Omega} |\nabla\Phi(u)|^{p-2} \Phi'(u) \frac{\partial u}{\partial n} \phi - \int_\Omega \Delta_p \Phi(u) \phi \\ &\le \int_{\partial\Omega} \phi |\Phi'(u)|^{p-2} \Phi'(u) |\nabla u|^{p-2} \frac{\partial u}{\partial n} - \int_\Omega \phi |\Phi'(u)|^{p-2} \Phi'(u) \Delta_p u.\end{aligned} \tag{3.13}$$

By the approximation argument, this is still valid for C^1 convex function Φ. Now we take a Φ in $\mathbb{R}$ such that $\Phi(t) = t$ if $t \ge 0$, $|\Phi(t)| < 1$ if $t < 0$, $0 \le \Phi' \le 1$ in $\mathbb{R}$ and $lim_{t\to-\infty}\Phi'(t) = 0$. Set $\Phi_n(t) = \Phi(nt)/n$ for $t \in \mathbb{R}$ and $n = 1, 2, \ldots$. Then we see that $\{\Phi_n\}$ is a sequence of C^1convex functions in $\mathbb{R}$ such that $\Phi_n(t) = t$ if $t \ge 0$, $|\Phi_n(t)| < \frac{1}{n}$ if $t < 0$, $0 \le \Phi_n' \le 1$ in $\mathbb{R}$. Then we see that $\Phi_n(t) \to t^+$ as $n \to \infty$. Replacing Φ by Φ_n in (3.13) and letting $n \to \infty$, we have the desired inequality by the dominated convergence theorem.

We proceed to the case where $1 < p < 2$. We set $\Phi^\eta(t) := \Phi(t) + \eta t$ for $t \in \mathbb{R}$ with $\eta > 0$. Then we see that for each $\eta > 0$

$$\sup_{t\in R} (\Phi^\eta)'(t)^{p-2} (\Phi^\eta)''(t) = \sup_{t\in R} (\Phi'(t) + \eta)^{p-2} \Phi''(t) \le \eta^{p-2} \sup_{t\in R} \Phi''(t) < \infty. \tag{3.14}$$

Hence we can apply he previous argument with Φ^η instead of Φ, so that in a similar way we reach to the inequality (3.13) replaced Φ by Φ^η. Letting $\eta \to 0$, we have (3.10) and this completes the proof. □

Lemma 3.2. Assume that $u \in C^1(\bar{\Omega})$ and $\Delta_p u \in L^1(\Omega)$ (in the sense of distribution). Then $u^+ \in \mathbb{X}_p$ and

$$[u^+]_{\mathbb{X}_p} \le [u]_{\mathbb{X}_p} \, . \tag{3.15}$$

Proof. We note that $u^+ \in W^{1,p^*}(\Omega)$. For the proof of Lemma it suffices to show the following.

$$\Big| \int_\Omega |\nabla u|^{p-2} \nabla u^+ \cdot \nabla \psi \Big| \le [u]_{\mathbb{X}_p} \|\psi\|_{L^\infty} \quad (\forall \psi \in C^1(\bar{\Omega})) \, . \tag{3.16}$$

For $\tilde{\psi} \in C^1(\bar{\Omega})$, we apply (3.10) with $\psi = \|\tilde{\psi}\|_{L^\infty} + \tilde{\psi}$. Then

$$\begin{aligned} \int_\Omega |\nabla u|^{p-2} \nabla u^+ \cdot \nabla \tilde{\psi} \le \Big(\int_{\substack{\partial\Omega \\ [u\ge 0]}} |\nabla u|^{p-2} \frac{\partial u}{\partial n} - \int_{\substack{\Omega \\ [u \ge 0]}} \Delta_p u \Big) \|\tilde{\psi}\|_{L^\infty} \\ + \int_{\substack{\partial\Omega \\ [u\ge 0]}} \tilde{\psi} |\nabla u|^{p-2} \frac{\partial u}{\partial n} - \int_{\substack{\Omega \\ [u\ge 0]}} \tilde{\psi} \Delta_p u \end{aligned} \tag{3.17}$$

Noting that

$$\int_{\substack{\partial\Omega \\ [u\ge 0]}} |\nabla u|^{p-2} \frac{\partial u}{\partial n} - \int_{\substack{\Omega \\ [u\ge 0]}} \Delta_p u = - \int_{\substack{\partial\Omega \\ [u<0]}} |\nabla u|^{p-2} \frac{\partial u}{\partial n} + \int_{\substack{\Omega \\ [u<0]}} \Delta_p u$$

we have

$$\begin{aligned} \int_\Omega |\nabla u|^{p-2} \nabla u^+ \cdot \nabla \tilde{\psi} &\le - \Big(\int_{\substack{\partial\Omega \\ [u<0]}} |\nabla u|^{p-2} \frac{\partial u}{\partial n} - \int_{\substack{\Omega \\ [u<0]}} \Delta_p u \Big) \|\tilde{\psi}\|_{L^\infty} + \int_{\substack{\partial\Omega \\ [u\ge 0]}} \tilde{\psi} |\nabla u|^{p-2} \frac{\partial u}{\partial n} - \int_{\substack{\Omega \\ [u\ge 0]}} \tilde{\psi} \Delta_p u \\ &\le \Big(\int_{\partial\Omega} |\nabla u|^{p-2} \Big| \frac{\partial u}{\partial n} \Big| + \int_\Omega |\Delta_p u| \Big) \|\tilde{\psi}\|_{L^\infty} = [u]_{\mathbb{X}_p} \|\tilde{\psi}\|_{L^\infty} \, . \end{aligned}$$

By replacing $\tilde{\psi}$ by $-\tilde{\psi}$, we have the desired inequality (3.15). □

Secondly we assume that u is admissible in $\mathbb{X}_p$. We recall a lemma on Neumann boundary problem for a monotone operator Δ_p.

Lemma 3.3. Let $\mu \in C_c^\infty(\Omega)$ and $\nu \in C_c^\infty(\mathbb{R}^N)$. Assume that $\int_\Omega \mu + \int_{\partial\Omega} \nu = 0$.
Then there exists a unique function $u \in C^{1,\sigma}(\bar{\Omega})$ for some $\sigma \in (0,1)$ such that

$$\begin{cases} -\Delta_p u = \mu & \text{in } \Omega \\ |\nabla u|^{p-2} \frac{\partial u}{\partial n} = \nu & \text{on } \partial\Omega, \\ \int_\Omega u = 0. \end{cases} \tag{3.18}$$

Proof. It follows from the standard theory that we have the unique solution u in $W^{1,p}(\Omega)$. For the detail, refer to [16]; theorems 2.1 and 2.2 for example. Since μ and ν smooth, we see that $u \in C^{1,\sigma}(\bar{\Omega})$ for some $\sigma \in (0,1)$ (See e.g. DiBenedetto [9]). Here we note that u is p-harmonic near the boundary as well. □

By Definition 2.1 of the admissibility in $\mathbb{X}_p$ we have for each $k \ge 1$ that

$$\int_\Omega |\nabla u_k|^{p-2} \nabla u_k \cdot \nabla \psi = \int_{\partial\Omega} \psi |\nabla u_k|^{p-2} \frac{\partial u_k}{\partial n} - \int_{\partial\Omega} \psi \Delta_p u_k \quad (\forall \psi \in C^1(\bar{\Omega})) \, . \tag{3.19}$$

It follows from Remark 2.1(2) that in the sense of weak* topology as $n \to \infty$

$$\Delta_p u_k \overset{*}{\rightharpoonup} \Delta_p u \ \mathit{in}\ \mathscr{M}_b(\Omega), \quad \|\Delta_p u_k\|_{L^1(\Omega)} \to \|\Delta_p u\|_{\mathscr{M}_b(\Omega)} \, , \tag{3.20}$$

$$|\nabla u_k|^{p-2} \frac{\partial u_k}{\partial n} \overset{*}{\rightharpoonup} |\nabla u|^{p-2} \frac{\partial u}{\partial n} \ \mathit{in}\ \mathscr{M}_b(\partial\Omega), \quad \| |\nabla u_k|^{p-2} \frac{\partial u_k}{\partial n} \|_{L^1(\partial\Omega)} \to \Big\| |\nabla u|^{p-2} \frac{\partial u}{\partial n} \Big\|_{\mathscr{M}_b(\partial\Omega)} \, . \tag{3.21}$$

By choosing $\psi = 1$ in (3.19), we have

$$\int_\Omega \Delta_p u_k = \int_{\partial\Omega} |\nabla u_k|^{p-2} \frac{\partial u_k}{\partial n}. \tag{3.22}$$

Let us set $\mu_k = -\Delta_p u_k$ and $\nu_k = |\nabla u_k|^{p-2} \frac{\partial u_k}{\partial n}$. Let $\{\mu_k^j\} \subset C_c^\infty(\bar\Omega)$ and $\{\nu_k^j\} \subset C_c^\infty(\mathbb{R}^N)$ be two sequences such that as $j \to \infty$

$$\mu_k^j \overset{*}{\rightharpoonup} -\Delta_p u_k \ weak^* \ in \ L^1(\Omega), \qquad \|\mu_k^j\|_{L^1(\Omega)} \to \|\Delta_p u_k\|_{L^1(\Omega)}\ , \tag{3.23}$$

$$\nu_k^j \overset{*}{\rightharpoonup} |\nabla u_k|^{p-2} \frac{\partial u_k}{\partial n} \ weak^* \ in \ L^1(\partial\Omega), \qquad \|\nu_k^j\|_{L^1(\partial\Omega)} \to \left\| |\nabla u_k|^{p-2} \frac{\partial u_k}{\partial n} \right\|_{L^1(\partial\Omega)}. \tag{3.24}$$

From (3.22) we assume that

$$\int_{\partial\Omega} \nu_k^j = -\int_\Omega \mu_k^j \quad (^\forall j,k \geq 1).$$

It follows from Lemma 3.3 that for any $n \geq 1$ and $k \geq 1$, there exists $w_n^k \in C^{1,\sigma}(\bar\Omega)$ such that

$$\begin{cases} -\Delta_p w_k^j & = \mu_k^j \quad in \ \Omega \\ |\nabla w_k^j|^{p-2} \frac{\partial w_k^j}{\partial n} & = \nu_k^j \quad on \ \partial\Omega, \end{cases} \tag{3.25}$$

or equivalently

$$\int_\Omega |\nabla w_k^j|^{p-2} \nabla w_k^j \cdot \nabla \psi = \int_\Omega \psi \, d\mu_k^j + \int_{\partial\Omega} \psi \, d\nu_k^j, \quad \text{for any } \psi \in C^1(\bar\Omega), \tag{3.26}$$

and without the loss of generality we also assume that for any $j,k \geq 1$

$$\int_\Omega w_k^j = \int_\Omega u_k\ . \tag{3.27}$$

Under these preparations we have

Lemma 3.4. For each $n \geq 1$, there exists a function $w_k \in W^{1,q}(\Omega)$ for every $q \in [1, \frac{N(p-1)}{N-1})$ such that w_k^j converges to w_k in $w_k \in W^{1,q}(\Omega)$ as $k \to \infty$ and w_k satisfies (3.19).

Proof. Since for each $k \geq 1$, $\{\mu_k^j\}_{j=1}^\infty$ and $\{\nu_k^j\}_{j=1}^\infty$ are bounded in $L^1(\Omega)$ and $L^1(\partial\Omega)$ respectively, this assertion follows from the same argument in the proof of Theorem 1 in [3] with an obvious modification. In fact, one can show that $\{w_k^j\}_{j=1}^\infty$ is bounded in $W^{1,q}(\Omega)$, using similar test functions for ψ. Then by the weak compactness, Poincaré's inequality and the Rellich type theorem, one can see that there exists a function $w_k \in W^{1,q}(\Omega)$ such that

$$\begin{aligned} &\nabla w_k^j \to \nabla w_k \quad \text{in } L^q \quad \text{(weak)} \\ &w_k^j \to w_k \quad \text{in } L^q \\ &w_k^j \to w_k \quad \text{a.e..} \end{aligned}$$

Moreover one can see that $\nabla w_k^j \to \nabla w_k$ in $L^1(\Omega)$. Then by the dominated convergence theorem the conclusion follows in a quite similar way. For the detail see [3]. □

Lemma 3.5. We have $w_k = u_k$ a.e. for $k = 1, 2, \cdots$.

Proof. We claim that $w_k = u_k \in W^{1,q}(\Omega)$ for $q \in [1, \frac{N(p-1)}{N-1})$. Choose any $M > 0$. Recalling that $u_k \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$, we use $T_M(w_k^j - u_k) \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ as a test function in (3.19) and (3.26). By a subtraction

$$\begin{aligned} &\int_\Omega (|\nabla w_k^j|^{p-2} \nabla w_k^j - |\nabla u_k|^{p-2} \nabla u_k) \cdot \nabla (T_M(w_k^j - u_k) \\ &\qquad = \int_\Omega T_M(w_k^j - u_k) \, d(\mu_k^j - \mu_k) + \int_{\partial\Omega} T_M(w_k^j - u_k) \, d(\nu_k^j - \nu_k). \end{aligned}$$

The left hand side is estimated from below in the following way,

$$\int_\Omega (|\nabla w_k^j|^{p-2}\nabla w_k^j - |\nabla u_k|^{p-2}\nabla u_k)\cdot\nabla T_M(w_k^j - u_k) \geq C\int_\Omega |\nabla T_M(w_k^j - u_k)|^p \tag{3.28}$$

for some positive number C independent of each j, and the right hand side goes to 0 as $j\to\infty$. Since this holds for all $M>0$, we conclude by the monotonicity of Δ_p that $\nabla w_k = \nabla u_k$ a.e. Taking into account that $w_k \in W^{1,q}(\Omega)$, $u_k\in W^{1,p}(\Omega)$ and (3.27), we conclude that $u_k = w_k$ a.e.. □

End of proof of Theorem 3.2. By applying Lemma 3.2 we have

$$\left|\int_\Omega |\nabla (w_k^j)^+|^{p-2}\nabla (w_k^j)^+\cdot\nabla\psi\right| \leq [w_k^j]_{\mathbb{X}_p}\|\psi\|_{L^\infty} \quad (\forall\psi\in C^1(\bar\Omega)) \,. \tag{3.29}$$

From Lemma 3.4 and Lemma 3.5 we have, up to subsequence, that $w_k^j \to u_k$ a.e. and $(w_k^j)_+ \to (u_k)_+$ in $W^{1,q}(\Omega)$ as $j\to\infty$. Letting $j\to\infty$, we have

$$\left|\int_\Omega |\nabla u_k^+|^{p-2}\nabla u_k^+\cdot\nabla\psi\right| \leq [u_k]_{\mathbb{X}_p}\|\psi\|_{L^\infty} \quad (\forall\psi\in C^1(\bar\Omega)).$$

Finally letting $k\to\infty$ we have the conclusion. □

3.3 Proof of Theorem 3.3

Proof of the assertion 1.
1st step. Assume that u is admissible in $W_0^{1,p^*}(\Omega)$, and hence both u^+ and u^- are admissible $W_0^{1,p^*}(\Omega)$. From the statement 4 of Proposition 4.1, we can assume that $\{u_k\}_{k=1}^\infty \subset W_0^{1,p}(\Omega)\cap C_0^1(\Omega)$ in Definition 2.2. We decompose $u_k\in W_0^{1,p}(\Omega)\cap C_0^1(\Omega)$ to obtain $u_k = u_k^+ - u_k^-$, where $u_k^+ = \max\{u_k,0\}$, $u_k^- = \max\{-u_k,0\}$. Then each $u_k^\pm \in W_0^{1,p}(\Omega)\cap C_0^{1,0}(\bar\Omega)$. Since $u_k^+\geq 0$ in Ω and $u_k^+=0$ on $\partial\Omega$, we see that $\frac{\partial u_k^+}{\partial n}\leq 0$ on $\partial\Omega$. Similarly we have $\frac{\partial u_k^-}{\partial n}\leq 0$ on $\partial\Omega$. Therefore

$$-\int_{\partial\Omega}|\nabla u_k^+|^{p-2}\left|\frac{\partial u_k^+}{\partial n}\right| = \int_{\partial\Omega}|\nabla u_k^+|^{p-2}\frac{\partial u_k^+}{\partial n} = \int_\Omega \Delta_p u_k^+,$$
$$-\int_{\partial\Omega}|\nabla u_k^-|^{p-2}\left|\frac{\partial u^-}{\partial n}\right| = \int_{\partial\Omega}|\nabla u_k^-|^{p-2}\frac{\partial u_k^-}{\partial n} = \int_\Omega \Delta_p u_k^-.$$

Hence

$$\int_{\partial\Omega}|\nabla u_k^+|^{p-2}\left|\frac{\partial u_k^+}{\partial n}\right| \leq \left|\int_\Omega \Delta_p u_k^+\right|, \quad \int_{\partial\Omega}|\nabla u_k^-|^{p-2}\left|\frac{\partial u_k^-}{\partial n}\right| \leq \left|\int_\Omega \Delta_p u_k^-\right|.$$

After all we have

$$\int_{\partial\Omega}|\nabla u_k|^{p-2}\left|\frac{\partial u_k}{\partial n}\right| \leq \int_\Omega |\Delta_p u_k|, \tag{3.30}$$

in particular $|\nabla u_k|^{p-2}\frac{\partial u_k}{\partial n}\in L^1(\partial\Omega)$. Hence we have

$$[u_k]_{\mathbb{X}_p} \leq \int_{\partial\Omega}|\nabla u_k|^{p-2}\left|\frac{\partial u_k}{\partial n}\right| + \int_\Omega |\Delta_p u_k| \leq 2\int_\Omega|\Delta_p u_k| < \infty. \tag{3.31}$$

2nd step. Again assume that $\{u_k\}_{n=1}^\infty \subset W_0^{1,p}(\Omega)\cap C_0^1(\Omega)$ in Definition 2.2. By Definition 2.2 (1) we have

$$\int_\Omega |\nabla u_k|^{p-2}\nabla u_k\cdot\nabla\psi \to \int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla\psi \quad \text{for any } \psi\in C_c^1(\Omega). \tag{3.32}$$

It follows from the weak compactness of bounded measures and the uniqueness of weak limit that $\Delta_p u_k \to \Delta_p u$ strongly in $\mathscr{M}(\Omega)$. By the previous step we have

$$|u_k|_{\mathbb{X}_p} \le 2\int_\Omega |\Delta_p u_k| \qquad \text{for } k = 1, 2, \cdots . \tag{3.33}$$

Hence we see that $|\nabla u_k|^{p-2}\frac{\partial u_k}{\partial n} \in L^1(\partial\Omega)$ converge to some measure ν in $M(\partial\Omega)$ up to subsequences. Therefore by the lower semicontinuity of the norm $||\cdot||_{M(\Omega)}$ with respect to the weak* convergence as $n \to \infty$, we have

$$[u]_{\mathbb{X}_p} \le 2\int_\Omega |\Delta_p u|.$$

Therefore u is admissible in $\mathbb{X}_p$, and hence $u^+ \in \mathbb{X}_p$ by Theorem 3.2. □

Proof of the assertion 2. We claim that $\int_\Omega |\Delta_p u^+| \le \int_\Omega |\Delta_p u|$.

Lemma 3.6. Assume that $u \in C_0^1(\bar{\Omega})$ and $\Delta_p u \in L^1(\Omega)$. Then $\Delta u^+ \in \mathscr{M}_b(\Omega)$ and

$$\|\Delta u^+\|_{\mathscr{M}_b(\Omega)} \le \|\Delta u\|_{L^1(\Omega)} \,. \tag{3.34}$$

Proof. By applying Lemma 3.2 with $u+\varepsilon$, where $\varepsilon > 0$, we deduce that

$$|(u+\varepsilon)^+|_{\mathbb{X}_p} \le |u+\varepsilon|_{\mathbb{X}_p} = |u|_{\mathbb{X}_p}. \tag{3.35}$$

Since $(u+\varepsilon)^+ = u+\varepsilon$ in a nelghborhood of $\partial\Omega$,

$$\frac{\partial}{\partial n}(u+\varepsilon)^+ = \frac{\partial u}{\partial n} \quad \text{ on } \partial\Omega. \tag{3.36}$$

Noting that

$$|(u+\varepsilon)^+|_{\mathbb{X}_p} = ||\Delta_p(u+\varepsilon)^+||_{\mathscr{M}(\Omega)} + |||\nabla(u+\varepsilon)^+|^{p-2}\frac{\partial}{\partial n}(u+\varepsilon)^+||_{L^1(\partial\Omega)}$$
$$|u|_{\mathbb{X}_p} = ||\Delta_p u||_{L^1(\Omega)} + |||\nabla u|^{p-2}\frac{\partial u}{\partial n}||_{L^1(\partial\Omega)},$$

we immediately have

$$||\Delta_p(u+\varepsilon)^+||_{\mathscr{M}(\Omega)} \le ||\Delta_p u||_{L^1(\Omega)} \quad \text{ for any } \varepsilon > 0. \tag{3.37}$$

The results follows from the lower semicontinuity of the norm $||\cdot||_{\mathscr{M}(\Omega)}$ with respect to the weak* convergence as $\varepsilon \to 0$. □

3.4 Proof of Theorem 3.4

We prepare some fundamental lemmas.

Lemma 3.7. Let $u \in W^{1,p^*}(\Omega)$. Assume that for some $h \in L^1(\partial\Omega)$ and $g \in L^1(\Omega)$ we have

$$\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla\varphi \le \int_{\partial\Omega} h\varphi + \int_\Omega g\varphi \quad \text{ for any } \varphi \in C^1(\bar{\Omega}), \varphi \ge 0. \tag{3.38}$$

Then $u \in \mathbb{X}_p$. Moreover $-\Delta_p u \le g$ in $\mathscr{M}(\Omega)$ and $|\nabla u|^{p-2}\frac{\partial u}{\partial n} \le h$ in $\mathscr{M}(\partial\Omega)$.

Proof. By (3.38) we have

$$\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla\varphi \le \int_{\partial\Omega} h^+\varphi + \int_\Omega g^+\varphi \quad \text{for any } \varphi\in C^1(\bar\Omega), \varphi\ge 0. \tag{3.39}$$

Using nonnegative test functions $||\varphi||_{L^\infty}\pm\varphi$ as the argument in the proof of Lemma 3.2, it is easy to see that

$$\left|\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla\varphi\right| \le 2(||h^+||_{L^1(\partial\Omega)}+||g^+||_{L^1(\Omega)})||\varphi||_{L^\infty(\Omega)}. \tag{3.40}$$

Then we see $u\in\mathbb{X}_p$. The rest of the assertions are clear. □

Lemma 3.8. In the previous Lemma 3.7, we further assume that u is admissible in $\mathbb{X}_p$. Then we have

$$\int_\Omega |\nabla u^+|^{p-2}\nabla u^+\cdot\nabla\varphi \le \int_{\substack{\partial\Omega\\ [u\ge 0]}} h\varphi + \int_{\substack{\Omega\\ [u\ge 0]}} g\varphi \quad \text{for any } \varphi\in C^1(\bar\Omega), \varphi\ge 0. \tag{3.41}$$

By the admissibility there exists a sequence $\{u_k\}\subset W^{1,p^*}(\Omega)$ having the properties in Definition 2.1. By virtue of Proposition 4.1 we can assume that $u_k\in C^1(\bar\Omega)$. Then it follows from Lemma 3.1 that

$$\int_\Omega |\nabla u_k|^{p-2}\nabla u_k^+\cdot\nabla\varphi \le \int_{\substack{\partial\Omega\\ [u_k\ge 0]}} \varphi|\nabla u_k|^{p-2}\frac{\partial u_k}{\partial n} - \int_{\substack{\Omega\\ [u_k\ge 0]}} \varphi\Delta_p u_k \quad (\forall\varphi\in C^1(\bar\Omega), \varphi\ge 0 \textit{ in } \bar\Omega) \tag{3.42}$$

Taking a limit as $k\to\infty$ we have

$$\int_\Omega |\nabla u|^{p-2}\nabla u^+\cdot\nabla\varphi \le \int_{\substack{\partial\Omega\\ [u\ge 0]}} \varphi|\nabla u|^{p-2}\frac{\partial u}{\partial n} - \int_{\substack{\Omega\\ [u\ge 0]}} \varphi\Delta_p u \quad (\forall\varphi\in C^1(\bar\Omega), \varphi\ge 0 \textit{ in } \bar\Omega) \tag{3.43}$$

Using Lemma 3.5 the conclusion holds. □

Lemma 3.9. Assume that $u\in C^1(\bar\Omega)$ is admissible in $\mathbb{X}_p$ and
$|\nabla u|^{p-2}\frac{\partial u}{\partial n}\in L^1(\partial\Omega)$. Then

$$|\nabla u^+|^{p-2}\frac{\partial u^+}{\partial n} \le \begin{cases} |\nabla u|^{p-2}\frac{\partial u}{\partial n} & \text{on } [u>0]\\ 0 & \text{on } [u<0]\\ \min\{|\nabla u|^{p-2}\frac{\partial u}{\partial n}, 0\} & \text{on } [u=0]. \end{cases} \tag{3.44}$$

Proof. Put $\mu=(-\Delta_p u)^+,\ h=|\nabla u|^{p-2}\frac{\partial u}{\partial n}$. Then

$$\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla\varphi \le \int_{\partial\Omega} h\varphi + \int_\Omega \varphi\, d\mu \quad (\forall\varphi\in C^1(\bar\Omega), \varphi\ge 0 \textit{ in } \bar\Omega)$$

It follows from Lemma 3.8 that u^+ satisfies

$$\int_\Omega |\nabla u|^{p-2}\nabla u^+\cdot\nabla\varphi \le \int_{\substack{\partial\Omega\\ [u\ge 0]}} h\varphi + \int_\Omega \varphi\, d\mu \quad (\forall\varphi\in C^1(\bar\Omega),\ \varphi\ge 0 \textit{ in } \bar\Omega) \tag{3.45}$$

By Theorem 3.2 we have $u^+\in\mathbb{X}_p$, hence

$$|\nabla u|^{p-2}\frac{\partial u^+}{\partial n} \le \chi_{[u\ge 0]}h = \chi_{[u\ge 0]}|\nabla u|^{p-2}\frac{\partial u}{\partial n} \quad \textit{on } \partial\Omega. \tag{3.46}$$

By using $u-\varepsilon$, where $\varepsilon>0$ instead of u we have in a similar way that

$$|\nabla u|^{p-2}\frac{\partial u^+}{\partial n} \le \chi_{[u>0]}h = \chi_{[u>0]}|\nabla u|^{p-2}\frac{\partial u}{\partial n} \quad \textit{on } \partial\Omega. \tag{3.47}$$

In particular,

$$|\nabla u|^{p-2}\frac{\partial u^+}{\partial n} \le 0 \quad \textit{on } [u=0]. \tag{3.48}$$

Hence the conclusion follows. □

Corollary 3.1. Assume that u is admissible in $\mathbb{X}_p$ and $u \in W_0^{1,p^*}(\Omega)$. If $u \geq 0$ *in* Ω, then

$$|\nabla u|^{p-2}\frac{\partial u}{\partial n} \leq 0 \quad on\ \partial\Omega.$$

Proof.
$u = u^+$ *in* Ω and $u = 0$ *on* $\partial\Omega$, hence applying the Lemma 3.9 we have

$$\frac{\partial u}{\partial n} = \frac{\partial u^+}{\partial n} \leq \min\{\frac{\partial u}{\partial n}, 0\} \leq 0 \quad on\ \partial\Omega.$$

□

Proof of Theorem 3.4. By Theorem 3.2 $u^+ \in \mathbb{X}_p$. By applying Kato's inequality (Corollary 1.1 in [13]) to $u - a \in \mathbb{X}_p$, we havre

$$\Delta_p(u-a)^+ \geq \chi_{[u\geq a]}\Delta_p u = G \quad in\ \Omega$$

for any $a \in \mathbf{R}$. Here we note thatt $(\Delta_p u)_d = \Delta_p u$, because $\Delta_p u \in L^1(\Omega)$. Letting $a \downarrow 0$ we have

$$\Delta_p u^+ \geq \chi_{[u>0]}\Delta_p u = G \quad in\ \Omega.$$

Combining this with Lemma 3.7, we have for any $\varphi \in C^1(\bar{\Omega}),\ \varphi \geq 0$ *in* Ω

$$\int_\Omega |\nabla u|^{p-2}\nabla u^+ \cdot \nabla\varphi = \int_{\partial\Omega} \varphi|\nabla u|^{p-2}\frac{\partial u^+}{\partial n} - \int_\Omega \varphi\Delta u^+ \leq \int_{\partial\Omega} H\varphi - \int_\Omega G\varphi.$$

□

4 Appendix (Proposition 4.1)

We begin with recalling a local version of Admissibility in [14].

Definition 4.1. (**Admissibility in** $W^{1,p^*}_{\mathrm{loc}}(\Omega)$) Let $1 < p < \infty$ and $p^* = \max\{1, p-1\}$. A function u is said to be admissible in in $W^{1,p^*}_{\mathrm{loc}}(\Omega)$, if $u \in W^{1,p^*}_{\mathrm{loc}}(\Omega)$, $\Delta_p u \in \mathscr{M}(\Omega)$; the total measure is not necessarily finite, and if there exists a sequence $\{u_k\}_{k=1}^\infty \subset W^{1,p}_{\mathrm{loc}}(\Omega) \cap L^\infty(\Omega)$ such that:

1. $u_k \to u$ a.e. in Ω and $u_k \to u$ in $W^{1,p^*}_{\mathrm{loc}}(\Omega)$ as $k \to \infty$.
2. $\Delta_p u_k \in L^1_{\mathrm{loc}}(\Omega)$ $(k = 1, 2, \cdots)$ and

$$\sup_k |\Delta_p u_k|(\omega) = \sup_k \int_\omega |\Delta_p u_k| < \infty \qquad \text{for all } \omega \subset\subset \Omega. \tag{4.1}$$

Here we describe the following fundamental results, parts of which are already known.

Proposition 4.1. Let Ω be a bounded smooth domain of $\mathbb{R}^N$.

1. Assume that u is admissible in $W^{1,p^*}_{\mathrm{loc}}(\Omega)$. Then, for every $M > 0$, $T_M u \in W^{1,p}_{\mathrm{loc}}(\Omega)$.
2. A function $u \in W_0^{1,p}(\Omega)$ is admissible in $W_0^{1,p^*}(\Omega)$, if $\Delta_p u \in \mathscr{M}_b(\Omega)$.
3. A function $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ is admissible in $W^{1,p^*}_{\mathrm{loc}}(\Omega)$, if $\Delta_p u \in \mathscr{M}(\Omega)$.
4. In Definition 2.1, the sequence $\{u_k\}$ can be taken in $C^1(\bar{\Omega})$.
5. In Definition 2.2, the sequence $\{u_k\}$ can be taken in $C_0^1(\bar{\Omega}) = \{\varphi \in C^1(\bar{\Omega}) : u = 0 \text{ on } \partial\Omega\}$.

The proof of assertion 1 for $p=2$ is seen in [5] and [6]) and for $p>1$ in [14], and the proof of assertion 2 is seen in Appendix of [14]. The assertion 4 is already verified in the proof of Theorem 3.2. Therefore we establish the assertions 3 and 5 in the rest of this section.

Proof of assertion 3. To use a diagonal argument, we choose and fix a family of open set $\{\omega_k\}$ such that

$$\omega_1 \subset\subset \omega_2 \subset\subset \cdots \subset\subset \omega_k \subset\subset \omega_{k+1} \subset\subset \cdots \subset\subset \Omega \text{ and } \Omega = \cup_{k=0}^{\infty} \omega_k. \tag{4.2}$$

Let $\rho \in C_0^\infty(B_1)$ be a radial, nonnegative and decreasing mollifier. By extending $v \in L^1(\Omega)$ to the whole space so that $v \equiv 0$ outside Ω, we define a mollification of v with $\varepsilon > 0$ by

$$v^\varepsilon(x) := \rho_\varepsilon * v(x) = \int_\Omega \rho_\varepsilon(x-y)v(y)dy \qquad \text{for } x \in \Omega. \tag{4.3}$$

First we prove that $u \in W_0^{1,p}(\Omega)$ is admissible in $W_{\mathrm{loc}}^{1,p^*}(\Omega)$, if $\Delta_p u$ is a Radon measure on Ω. Again by extending $u \in W_0^{1,p}(\Omega)$ and $\Delta_p u \in W^{-1,p'}$ to the whole space so that $u=0$ and $\Delta_p u = 0$ outside Ω respectively. Let $w_k \in W_0^{1,p}(\Omega) \cap C^1(\overline{\Omega})$ be the unique weak solution of the boundary value problem for the monotone operator Δ_p (see e.g. [16]): For $k = 1,2,\cdots$ and $\varepsilon_1 > \varepsilon_2 > \cdots \varepsilon_k > \cdots \to 0$, we set

$$\begin{cases} \Delta_p w_k = (\Delta_p u)^{\varepsilon_k} & \text{in } \Omega, \\ w_k = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.4}$$

where $|\nabla u|^{p-2}\nabla u \in (L^{p'}(\Omega))^N$ with $p' = p/(p-1)$, $(|\nabla u|^{p-2}\nabla u)^{\varepsilon_k} \in (C^\infty(\mathbb{R}^N))^N$ and $(|\nabla u|^{p-2}\nabla u)^{\varepsilon_k}$ is a mollification of $|\nabla u|^{p-2}\nabla u$ defined by (4.3). Let us set $\Delta_p u = \mu$. We note that $|\mu|(\omega) < \infty$ for any $\omega \subset\subset \Omega$. Then we have $\mathrm{div}\,(|\nabla u|^{p-2}\nabla u)^{\varepsilon_k} = (\mathrm{div}\,|\nabla u|^{p-2}\nabla u)^{\varepsilon_k} = (\Delta_p u)^{\varepsilon_k} = \mu^{\varepsilon_k}$ in ω provided that ε_k is sufficiently small. Hence we clearly have

$$|\Delta_p w_k|(\omega) = |\mu^{\varepsilon_k}|(\omega) \to |\mu|(\omega) \ \text{ as } \ k \to \infty.$$

Since μ does not charge $\partial\Omega$, this proves the condition 2. Next we show

$$w_k \to u \text{ in } W_0^{1,p}(\Omega) \ \text{ as } \ k \to \infty. \tag{4.5}$$

Then we can choose a subsequence so that the condition 1 is satisfied. By using $w_k - u \in W_0^{1,p}(\Omega)$ as a test function, we have

$$\begin{aligned} -\langle \Delta_p w_k - \Delta_p u, w_k - u \rangle &= \int_\Omega |(\nabla w_k|^{p-2}\nabla w_k - |\nabla u|^{p-2}\nabla u) \cdot \nabla(w_k - u) \\ &\geq c_2 \int_\Omega |\nabla(w_k - u)|^p. \end{aligned} \tag{4.6}$$

In the left-hand side, using Young's inequality for $\delta > 0$ we have

$$\begin{aligned} -\langle \Delta_p w_k - \Delta_p u, w_k - u \rangle &= \int_\Omega ((|\nabla u|^{p-2}\nabla u)^{\varepsilon_k} - |\nabla u|^{p-2}\nabla u) \cdot \nabla(w_k - u) \\ \leq C(\delta) \int_\Omega |(|\nabla u|^{p-2}\nabla u)^{\varepsilon_k} &- |\nabla u|^{p-2}\nabla u|^{p'} + \delta \int_\Omega |\nabla(w_k - u)|^p, \end{aligned} \tag{4.7}$$

where $C(\delta) > 0$ is a constant depending only on δ.

We note that $||(|\nabla u|^{p-2}\nabla u)^{\varepsilon_k} - |\nabla u|^{p-2}\nabla u||_{L^{p'}(\Omega)} \to 0$ as $k \to \infty$. It follows from (4.6) and (4.7) that $\nabla w_k \to \nabla u$ in $(L^p(\Omega))^N$ as $n \to \infty$, which implies (4.5). Then, taking a subsequence if necessary, $\{w_k\} \subset W_0^{1,p}(\Omega) \cap C^1(\overline{\Omega})$ satisfies the property $w_k \to u$ a.e. in Ω as $k \to \infty$.

Lastly we treat the case where $u \in W_{\mathrm{loc}}^{1,p}(\Omega)$. For each k we choose $\eta_k \in C_c^\infty(\omega_{k+1})$ such that $0 \leq \eta_k \leq 1$ and $\eta_k = 1$ in some neighborhood of $\overline{\omega_k}$. Let us set $v_k = \eta_k u\,(k = 1,2,3,\cdots)$. Then we see that $v_k \in W_0^{1,p}(\omega_{k+1})$, $v_k \to u$ in $W_{\mathrm{loc}}^{1,p}(\Omega)$ as $k \to \infty$ and $\Delta_p v_k \in W^{-1,p'}(\Omega) \cap M_b(\omega_k)$. Moreover we have

$|\Delta_p v_k|(\omega_j) = |\Delta_p u|(\omega_j)$ for any $k \geq j$. Hence u is admissible in $W^{1,p^*}_{\rm loc}(\omega_k)$ with $\Delta_p u \in \mathscr{M}_b(\omega_k)$ having an admissible sequence $\{v_k\}$. By the previous step with obvious modification, one can approximate each v_k inductively by $\xi_k \in W^{1,p}_0(\Omega) \cap C^1(\overline{\Omega})$ such that $\xi_k \to u$ in $W^{1,p^*}_{\rm loc}(\Omega)$ as $k \to \infty$ and $||\Delta_p \xi_k|(\omega_j) - |\Delta_p u|(\omega_j)| < \frac{1}{k}$ for $k \geq j$. Therefore the assertion is now proved. □

Proof of assertion 5. We assume that u is admissible in $W^{1,p^*}_0(\Omega)$. Then we have a sequence of functions $\{u_k\} \subset W^{1,p}_0(\Omega) \cap L^\infty(\Omega)$ $(k = 1, 2, \dots)$ satisfying the properties 1 and 2 in Definition 2.2. By the previous step, we see that each u_k is approximated as $j \to \infty$ by a sequence of functions $\{w^j_k\} \subset W^{1,p}_0(\Omega) \cap C^1(\overline{\Omega})$ defined by (4.4) with $w_k = w^j_k$, $u = u_k$ and $\varepsilon_k = \varepsilon_j$. Then we choose a suitable subsequence of $\{w^{j_k}_k\}$ as an approximation of u so that the assertion is verified. □

References

[1] A. Ancona, Inégalité de Kato et inégalité de Kato jusquáu bord. C. R. Math. Acad. Sci. Paris 346 (2008), 939-944

[2] P. Bénilan, L. Boccardo, T. Gallouët, R. Gariepy, M. Pierre and J.L. Vazquez, An L^1- theory of existence and uniqueness of solutions of nonlinear elliptic equations, Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4^e série, tome 22, No.2, (1995), 241-273.

[3] L. Boccardo, T. Gallouët, Nonlinear elliptic and parabolic equations involving measure data, J. Funct. Anal. vol. 87, 1989, 149-169

[4] L. Boccardo, T. Gallouët, L. Orsina, Existence and uniqueness of entropy solutions for nonlinear elliptic equations with measure data, Annales de l'I.H.P. section C, tome 13 no. 5 (1996), p. 539-551.

[5] H. Brezis , A. Ponce, Remarks on the strong maximum principle, Differential Integral Equations 16 (2003), 1-12.

[6] H. Brezis, A. Ponce, Kato's inequality when Δu is a measure, C. R. Acad. Sci. Paris, Ser. I 338 (2004), 599-604.

[7] H. Brezis and A. Ponce; Kato's inequality up to the boundary, Comm. Contemp. Math. 10 (2008), p. 1217-1241.

[8] H. Brezis, M. Marcus, A. Ponce, Nonlinear elliptic equations with measures revisited, in Mathematical Aspects of Nonlinear Dispersive Equations (J. Bourgain, C. Kenig, and S. Klainerman, eds.), Annals of Mathematics Studies, 163 Princeton University Press, Princeton, NJ, 2007, p. 55-110.

[9] E. DiBenedetto; $C^{1+\alpha}$ local regularity of weak solutions of degenerate elliptic equations. Nonlinear Anal. 7 (1983), no. 8, 827-850.

[10] L. Dupaigne and A. Ponce, Singularities of positive supersolutions in elliptic PDEs, Selecta Math. (N.S.) 10, (2004), 341-358.

[11] T. Jin, V. Maz'ya, J.V. Schaftingen, Pathological solutions to elliptic problems in divergence form with continuous coefficients, Comptes Rendus Mathematique, vol. 347, Issues 13-14, (2009), 773-778.

[12] T. Kato, Schrödinger operators with singular potentials.Israel J. Math.13 (1972), 135-148.

[13] X. Liu, T. Horiuchi, Remarks on Kato's inequality when $\Delta_p u$ is a measure, Mathematical Journal of Ibaraki University vol. 48 (2016), 45-61.

[14] X. Liu, T. Horiuchi, Remarks on the strong maximum principle involving p-Laplacian, Hiroshima Mathematical Journal Vol.46, vo. 3 (2016), 311-331.

[15] X. Liu, T. Horiuchi, Kato's inequalities for admissible functions to a quasilinear elliptic operator $\mathscr{A}$, Mathematical Journal of Ibaraki University vol. 51 (2019), 49-64.

[16] Leray, J. and Lions, J. L., Quelques résultats de Visik sur les problémes elliptiques nonlinéaires par lesméthodes de Minty-Browder, Bull. Soc. Math. Fr. 93 (1965), 97-107.

[17] G.D. Maso, F.Murat, L.Orsina and A.Prignet, Renormalized solutions of elliptic equations with general measure data, Annali della Scuola Normale Superiore di Pisa - Classe di Scienze, vol. 4 Tome 28, No. 4 (1999), 741-808.

[18] F. Maeda, Renormalized solutions of Dirichlet problems for quasilinear elliptic equations with general measure data, Hiroshima Mathematical Journal, vol. 38 (2008), 51-93.

[19] G.D. Maso, F. Murat, L. Orsina, A. Prignet, Renormalized solutions of elliptic equations with general measure data, Ann. Scuola Norm. Sup. Pisa, 28, (1999), 741-808.

[20] J. Serrin, Pathological solutions of elliptic differential equations, Annali della Scuola Normale Superiore di Pisa - Classe di Scienze vol. 8, Issue 3(1964), 385-387.

Communicated by Yoshitsugu Kabeya

Toshio Horiuchi:
Department of Mathematics, Faculty of Science, Ibaraki University
Bunkyo 2-1-1 Mito City Ibaraki Prefecture Japan e-mail; thoshio.horiuchi.math@vc.ibaraki.ac.jp

Peter Kumlin
Department of Mathematics, Chalmers University of Technology
S-41296 Goteborg, Sweden
e-mail; kumlin@chalmers.se