SEPARATION AXIOMS IN BI-ISOTONIC SPACES

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Abstract. The purpose of this study is to introduce and study the concept of bi-isotonic spaces. In this study, we introduce the notion of the continuous map between bi-isotonic spaces and give the characterizations of bi-isotonic maps. Moreover, we explore the topological concepts of separation axioms in bi-isotonic spaces.

1 Introduction A topological structure on a set is not only defined by the axioms for open sets but also by the collections of closed sets, neighborhood systems, closure operators or interior operators, etc. For instance, Day [6] and Hausdorff [15] developed the topological concepts from the notions of convergence, closure, and neighborhoods. Kuratowski [17] brought a different approach to construct a topological structure on a non-empty set X by defining closure operator $\mathrm{cl} : P(X) \to P(X)$ (where $P(X)$ is the power set of X) with the following properties for all $A, B \in P(X)$;

K0) $\mathrm{cl}(\emptyset) = \emptyset$ (grounded)

K1) $A \subseteq B \Rightarrow \mathrm{cl}(A) \subseteq \mathrm{cl}(B)$ (Isotony)

K2) $A \subseteq \mathrm{cl}(A)$ (Expansive)

K3) $\mathrm{cl}(A \cup B) = \mathrm{cl}(A) \cup \mathrm{cl}(B)$ (Preservation of binary union)

K4) $\mathrm{cl}(\mathrm{cl}(A)) = \mathrm{cl}(A)$ (Idempotency).

In this way, the closure operator satisfying the aforementioned axioms allows to define the topological space (X, cl) by taking closed sets as sets such as $\mathrm{cl}(A) = A$. Moreover, Kuratowski extended the topological spaces by removing the axiom $\mathrm{cl}(A \cup B) \subseteq \mathrm{cl}(A) \cup \mathrm{cl}(B)$ and defined closure spaces. On the other hand, the approach of Ćech in the definition of closure space excludes the idempotency axiom $\mathrm{cl}(\mathrm{cl}(A)) = \mathrm{cl}(A)$ [5]. In order to avoid confusion on the term of closure space, the terms of Kuratowski closure space and Ćech closure space can be seen in literature. Additionally, Gnilka [8], [9], [10] and Hammer [13], [14] preferred the term extended topological space instead of closure space. The basic concepts of compactness, quasi-metrizability, symmetry, continuity were investigated by the closure operators in these studies. In recent years, Stadler et al. [25], [26], and [27] have revealed a topological approach to chemical organizations, evolutionary theory, and combinatorial chemistry and exposed the relationships between the topological concepts of similarity, neighborhood, connectedness, and continuity with the chemical and biological situations. In these interdisciplinary studies, the authors have considered the basic concepts of the closure and isotonic spaces, such that a closure space (X, cl) satisfying only the grounded and the isotony closure axioms is called an isotonic space. The notions of the connectedness, lower and upper separation axioms in isotonic spaces have been studied

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by Habil and Elzenati [11], [12]. On the other hand, another essential construction in this realm is *Bitopological Spaces* defined by Kelly [16]. There have been a number of longitudinal studies involving bitopological spaces, Wilson [29], Weston [28], and Wiweger [30]. For instance, the separation axioms have been generalized in bitopological spaces and some related characterizations have been given by Lane [18], Marin and Romaguera [19], Murdeshwar and Naimpally [20], Patty [21], Ravi and Thivagar [22], and Reilly [24]. T_0-strongly nodec space has been introduced by using the quotient map in [23]. The definitions and relationships for pairwise T_1, T_2, T_3, $T_{3\frac{1}{2}}$ and T_4-spaces have been presented by Dvalishvili [7]. A great deal of research has been conducted on the bitopological spaces, but few studies have been carried out to discover the biclosure spaces [1]–[4] and to date, none has been discussed bi-isotonic spaces.

2 Bi-Isotonic Spaces

Definition 2.1 *A generalized bi-closure space is a triple* $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ *where the maps* $\mathrm{cl}_1 : P(X) \to P(X)$ *and* $\mathrm{cl}_2 : P(X) \to P(X)$ *defined on the power set* $P(X)$ *of a non-empty set* X *are two closure operators [1].*

If the closure operators $\mathrm{cl}_1 : P(X) \to P(X)$ and $\mathrm{cl}_2 : P(X) \to P(X)$ are isotonic operators satisfying only the grounded and isotony axioms given by (K0) and (K1), respectively, then the concepts to be studied will be more general than ones given in [1]-[4].

Definition 2.2 *Let* cl_1 *and* cl_2 *be two isotonic operators on* X*, then the triple* $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ *is called bi-isotonic space.*

Definition 2.3 *A subset* A *of a bi-isotonic space* $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ *is called closed if* $\mathrm{cl}_1\mathrm{cl}_2(A) = A$*. The complement of a closed set is called open.*

Under the light of this definition, the following proposition is obvious.

Proposition 2.4 *A subset* A *of a bi-isotonic space* $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ *is closed if and only if it is a closed subset of* (X, cl_1) *and* (X, cl_2)*.*

In other words, the followings are equivalent in bi-isotonic space $(X, \mathrm{cl}_1, \mathrm{cl}_2)$;

i) $\mathrm{cl}_1\mathrm{cl}_2(A) = A$.

ii) $\mathrm{cl}_1(A) = A$ and $\mathrm{cl}_2(A) = A$.

Example 2.5 *Let us consider that*

$$\mathrm{cl}_1(A) = \begin{cases} \emptyset\,, & A = \emptyset \\ (-\infty, a]\,, & \sup A = a \\ \mathbb{R}, & \sup A = \infty \end{cases}$$

and

$$\mathrm{cl}_2(A) = \begin{cases} \emptyset\,, & A = \emptyset \\ [b, \infty)\,, & \inf A = b \\ \mathbb{R}, & \inf A = -\infty \end{cases}$$

be two operators on $\mathbb{R}$ *then the bi-isotonic space* $(\mathbb{R}, \mathrm{cl}_1, \mathrm{cl}_2)$ *is a non-discrete space since* $\mathrm{cl}_1(\mathrm{cl}_2(A)) = \mathbb{R}$ *or* $\mathrm{cl}_1(\mathrm{cl}_2(A)) = \emptyset$ *for all* $A \subseteq \mathbb{R}$*.*

A bi-isotonic space $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ is not needed to be a topological space since the finite intersection of closed subsets is not need to be closed.

Example 2.6 *Let* $\mathrm{cl}_1 : P(X) \to P(X)$ *and* $\mathrm{cl}_2 : P(X) \to P(X)$ *be two maps on* $X = \{a,b,c\}$ *satisfying* $\mathrm{cl}_1(\emptyset) = \emptyset$, $\mathrm{cl}_1(\{b\}) = \{b\}$, $\mathrm{cl}_1(\{c\}) = \{c\}$, $\mathrm{cl}_1(\{a,b\}) = \{a,b\}$, $\mathrm{cl}_1(\{b,c\}) = \{b,c\}$, $\mathrm{cl}_1(X) = \mathrm{cl}_1(\{a\}) = \mathrm{cl}_1(\{a,c\}) = X$, *and* $\mathrm{cl}_2(\{b\}) = \{b\}$, $\mathrm{cl}_2(\{c\}) = \{c\}$, $\mathrm{cl}_2(\emptyset) = \emptyset$, $\mathrm{cl}_2(X) = \mathrm{cl}_2(\{a\}) = \mathrm{cl}_2(\{a,b\}) = \mathrm{cl}_2(\{a,c\}) = \mathrm{cl}_2(\{b,c\}) = X$.

In the bi-isotonic space $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ *although the subsets* $\{b\}$ *and* $\{c\}$ *are closed,* $\{b,c\}$ *is not closed since* $\mathrm{cl}_1(\{b,c\}) = \{b,c\}$ *and* $\mathrm{cl}_2(\{b,c\}) = X$.

Proposition 2.7 *Let* $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ *be a bi-isotonic space and* $A \subseteq X$. *Then*

- ***i)*** *A is an open set if and only if* $A = X - \mathrm{cl}_1\mathrm{cl}_2(X - A)$,
- ***ii)*** *If A is an open set and* $A \subseteq G$ *then* $A \subseteq X - \mathrm{cl}_1\mathrm{cl}_2(X - G)$.

Proof.

- **i)** It is obvious from the definition of an open set in bi-isotonic space.
- **ii)** Let A be open and $A \subseteq G$, then $X - G \subseteq X - A$. Thus, $\mathrm{cl}_1(\mathrm{cl}_2(X - G)) \subseteq \mathrm{cl}_1(\mathrm{cl}_2(X - A))$ since the isotonic operators cl_1 and cl_2 have the property (K1). It is clear that $X - \mathrm{cl}_1(\mathrm{cl}_2(X - A)) \subseteq X - \mathrm{cl}_1(\mathrm{cl}_2(X - G))$. The openness of A and the first assertion completes the proof.

The duals of the isotonic operators cl_i on a bi-isotonic space $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ are defined by

$$\mathrm{int}_i : P(X) \to P(X), \mathrm{int}_i(A) = X - (\mathrm{cl}_i(X - A))$$

and called interior operators. In that case, a subset A of a bi-isotonic space $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ is open if $X - A = \mathrm{cl}_i(X - A)$ or $\mathrm{int}(A) = A$ for all $i \in \{1, 2\}$.

The neighborhood operators for $x \in X$ are defined by

$$\nu_i : X \to P(P(X)) \quad , \quad \nu_i(x) = \{N \in P(X) : x \in \mathrm{int}_i(N)\}.$$

Proposition 2.8 *Let* $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ *be a bi-isotonic space and* $Y \subseteq X$ *if* $\mathrm{cl}_i^Y(A) = \mathrm{cl}_i(A) \cap Y$ *is satisfied for all* $A \subseteq Y$ *and* $i \in \{1, 2\}$, *then the operators* $\mathrm{cl}_i^Y : P(Y) \to P(Y)$ *are isotonic.*

*Proof.*Let us consider the subsets $A, B \subseteq Y$ such that $A \subseteq B$. Then $\mathrm{cl}_i(A) \subseteq \mathrm{cl}_i(B)$ for each $i \in \{1, 2\}$ since $\mathrm{cl}_i : P(X) \to P(X)$ are isotonic. Thereby, $\mathrm{cl}_i(A) \cap Y \subseteq \mathrm{cl}_i(B) \cap Y$, that is, $\mathrm{cl}_i^Y(A) \subseteq \mathrm{cl}_i^Y(B)$.

Definition 2.9 *Let* $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ *be a bi-isotonic space and* $Y \subseteq X$. *A bi-isotonic space* $\left(Y, \mathrm{cl}_1^Y, \mathrm{cl}_2^Y\right)$ *given with induced isotonic operators* cl_1^Y *and* cl_2^Y *is called a subspace of* $(X, \mathrm{cl}_1, \mathrm{cl}_2)$.

Definition 2.10 *If* $\left(Y, \mathrm{cl}_1^Y, \mathrm{cl}_2^Y\right)$ *is a subspace of a bi-isotonic space* $(X, \mathrm{cl}_1, \mathrm{cl}_2)$, *then the induced interior operators* int_i^Y *and induced neighborhood operators* ν_i^Y *on* $\left(Y, \mathrm{cl}_1^Y, \mathrm{cl}_2^Y\right)$ *are defined by*

$$\mathrm{int}_i^Y(A) = Y - \mathrm{cl}_i^Y(Y - A) = Y \cap \mathrm{int}_i(A \cup (X - Y))$$

and

$$\nu_i^Y(A) = \{N \cap Y : N \in \nu_i(A)\},$$

respectively, for any $A \subseteq Y$ *and* $i \in \{1, 2\}$.

Proposition 2.11 *Let $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ be a bi-isotonic space, Y be a closed subset of X, and $A \subseteq Y$. A is a closed subset of bi-isotonic space $\left(Y, \mathrm{cl}_1^Y, \mathrm{cl}_2^Y\right)$ if and only if A is a closed subset of bi-isotonic space $(X, \mathrm{cl}_1, \mathrm{cl}_2)$.*

Proof.($\Rightarrow$:) Let A be a closed subset in $\left(Y, \mathrm{cl}_1^Y, \mathrm{cl}_2^Y\right)$. In that case, there are closed subsets $\mathrm{cl}_i(A)$ in X for each $i \in \{1, 2\}$ such that $A = \mathrm{cl}_i^Y(A) = \mathrm{cl}_i(A) \cap Y$. Thus $A \subseteq \mathrm{cl}_i(A)$. Also $\mathrm{cl}_i(A) \subseteq \mathrm{cl}_i(Y)$, i.e., $\mathrm{cl}_i(A) \subseteq \mathrm{cl}_i(Y) \cap \mathrm{cl}_i(A)$ since $A \subseteq Y$. The closedness of Y gives us $\mathrm{cl}_i(A) \subseteq Y \cap \mathrm{cl}_i(A) = A$. As a consequence $A = \mathrm{cl}_i(A)$ is found.

($\Leftarrow$:) Let $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ be a bi-isotonic space, Y be a closed subset of X, and $A \subseteq Y$. Assume that A is closed in $(X, \mathrm{cl}_1, \mathrm{cl}_2)$, then $\mathrm{cl}_i(A) = A$ for each $i \in \{1, 2\}$. It is easily seen that $\mathrm{cl}_1^Y(A) = \mathrm{cl}_i(A) \cap Y = A \cap Y = A$. This means that A is closed in $\left(Y, \mathrm{cl}_1^Y, \mathrm{cl}_2^Y\right)$.

Example 2.12 *Let us consider the set $X = \{a, b, c\}$ and the isotonic operators $\mathrm{cl}_1 : P(X) \to P(X)$ and $\mathrm{cl}_2 : P(X) \to P(X)$, respectively, defined by $\mathrm{cl}_1(X) = \mathrm{cl}_1(\{b, c\}) = \mathrm{cl}_1(\{a, c\}) = X$, $\mathrm{cl}_1(\emptyset) = \emptyset$, $\mathrm{cl}_1(\{a\}) = \{a\}$, $\mathrm{cl}_1(\{b\}) = \{b\}$, $\mathrm{cl}_1(\{c\}) = \{c\}$, $\mathrm{cl}_1(\{a, b\}) = \{a, b\}$ and $\mathrm{cl}_2(\emptyset) = \emptyset$, $\mathrm{cl}_2(\{a\}) = \{a\}$, $\mathrm{cl}_2(\{a, b\}) = \{a, b\}$, $\mathrm{cl}_2(X) = \mathrm{cl}_2(\{c\}) = \mathrm{cl}_2(\{b\}) = \mathrm{cl}_2(\{a, c\}) = \mathrm{cl}_2(\{b, c\}) = X$. If the subset $Y = \{a, b\}$ of X is considered, then the bi-isotonic subspace $\left(Y, \mathrm{cl}_1^Y, \mathrm{cl}_2^Y\right)$ is constructed by the induced isotonic operators $\mathrm{cl}_1^Y : P(Y) \to P(Y)$ and $\mathrm{cl}_2^Y : P(Y) \to P(Y)$ such as $\mathrm{cl}_1^Y(Y) = \mathrm{cl}_1^Y(\{a, b\}) = Y$, $\mathrm{cl}_1^Y(\emptyset) = \emptyset$, $\mathrm{cl}_1^Y(\{a\}) = \{a\}$, $\mathrm{cl}_1^Y(\{b\}) = \{b\}$ and $\mathrm{cl}_2^Y(\emptyset) = \emptyset$, $\mathrm{cl}_2^Y(\{a\}) = \{a\}$, $\mathrm{cl}_2^Y(Y) = \mathrm{cl}_2^Y(\{b\}) = \mathrm{cl}_2^Y(\{a, b\}) = Y$.*

3 Bi-Continuous Maps in Bi-Isotonic Spaces

Definition 3.1 *Let $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ and $(Y, \mathrm{cl'}_1, \mathrm{cl'}_2)$ be generalized bi-closure spaces. If $f : (X, \mathrm{cl}_i) \to (Y, \mathrm{cl'}_i)$ continuous (open, closed or homeomorphism) then $f : (X, \mathrm{cl}_1, \mathrm{cl}_2) \to (Y, \mathrm{cl'}_1, \mathrm{cl'}_2)$ is called $i-$continuous ($i-$open, $i-$closed or $i-$homeomorphism).*
Also, the map f is called bi-continuous if it is $i-$continuous map for each $i \in \{1, 2\}$.

Example 3.2 *Let $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ and $(Y, \mathrm{cl'}_1, \mathrm{cl'}_2)$ be generalized bi-closure spaces. The identity map $\mathrm{I} : (X, \mathrm{cl}_1, \mathrm{cl}_2) \to (X, \mathrm{cl'}_1, \mathrm{cl'}_2)$ is bi-continuous if and only if the operator cl_i is coarser than $\mathrm{cl'}_i$ for each $i \in \{1, 2\}$, that is, $\mathrm{cl'}_i(A) \subseteq \mathrm{cl}_i(A)$ for all $A \in P(X)$ and $i \in \{1, 2\}$.*

Definition 3.3 *Let (X, cl) and $\left(Y, \mathrm{cl'}\right)$ be two spaces. A map $f : X \to Y$ is continuous if and only if for $f(\mathrm{cl}(A)) \subseteq \mathrm{cl'}(f(A))$ all $A \in P(X)$ [5].*

Hence by considering the continuity of the maps $f : (X, \mathrm{cl}_i) \to (Y, \mathrm{cl'}_i)$ for each $i \in \{1, 2\}$ the following proposition can be given.

Proposition 3.4 *Let $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ and $(Y, \mathrm{cl'}_1, \mathrm{cl'}_2)$ be two generalized bi-closure spaces. A map $f : (X, \mathrm{cl}_1, \mathrm{cl}_2) \to (Y, \mathrm{cl'}_1, \mathrm{cl'}_2)$ is bi-continuous if and only if $f(\mathrm{cl}_i(A)) \subseteq \mathrm{cl'}_i(f(A))$ for all $A \in P(X)$ and $i \in \{1, 2\}$.*

Proposition 3.5 *Let (X, cl) and $\left(Y, \mathrm{cl'}\right)$ be two isotonic spaces. A map $f : X \to Y$ is continuous if and only if $\mathrm{cl}\left(f^{-1}(B)\right) \subseteq f^{-1}\left(\mathrm{cl'}(B)\right)$ for all $B \in P(Y)$ [27].*

Thus the following proposition is obvious.

Proposition 3.6 *Let $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ and $(Y, \mathrm{cl'}_1, \mathrm{cl'}_2)$ be bi-isotonic spaces. A map $f : (X, \mathrm{cl}_1, \mathrm{cl}_2) \to (Y, \mathrm{cl'}_1, \mathrm{cl'}_2)$ is bi-continuous if and only if $\mathrm{cl}_i\left(f^{-1}(B)\right) \subseteq f^{-1}(\mathrm{cl'}_i(B))$ for all $B \in P(Y)$ and $i \in \{1, 2\}$.*

Proposition 3.7 *Let $(X, \mathrm{cl}_1, \mathrm{cl}_2)$, $(Y, \mathrm{cl}'_1, \mathrm{cl}'_2)$ and $(Z, \mathrm{cl}''_1, \mathrm{cl}''_1)$ be bi-isotonic spaces. $f : X \to Y$ and $g : Y \to Z$ are bi-continuous maps, then $g \circ f : X \to Z$ is bi-continuous.*

*Proof.*Consider a subset $B \in P(Z)$, then $\mathrm{cl}'_i \left(g^{-1}(B)\right) \subseteq g^{-1}(\mathrm{cl}''_i(B))$ for each $i \in \{1,2\}$ since g is bi-continuous.
Also, $\mathrm{cl}_i \left(f^{-1}\left(g^{-1}(B)\right)\right) \subseteq f^{-1}\left(\mathrm{cl}'_i\left(g^{-1}(B)\right)\right)$ since $g^{-1}(B) \in P(Y)$ and f is a bi-continuous map.
Moreover, $f^{-1}\left(\mathrm{cl}'_i\left(g^{-1}(B)\right)\right) \subseteq f^{-1}\left(g^{-1}(\mathrm{cl}''_i(B))\right)$ is satisfied. By these last two relations, we get $\mathrm{cl}_i\left(f^{-1}\left(g^{-1}(B)\right)\right) \subseteq f^{-1}\left(g^{-1}(\mathrm{cl}''_i(B))\right)$ and this completes the proof.

Proposition 3.8 *Let $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ and $(X, \mathrm{cl}'_1, \mathrm{cl}'_2)$ be bi-isotonic spaces. Then the following conditions (for bi-continuity) are equivalent:*

- ***i)*** *$f : (X, \mathrm{cl}_1, \mathrm{cl}_2) \to (Y, \mathrm{cl}'_1, \mathrm{cl}'_2)$ is bi-continuous.*
- ***ii)*** *$f^{-1}(\mathrm{int}'_i(B)) \subseteq \mathrm{int}_i\left(f^{-1}(B)\right)$ for all $B \in P(Y)$ and $i \in \{1,2\}$.*
- ***iii)*** *$f^{-1}(B) \in \nu_i(x)$ provided $B \in \nu'_i(f(x))$ for all $B \in P(Y)$ and $i \in \{1,2\}$.*

Proof.(i $\Rightarrow$ ii) Let f be a bi-continuous map. There is the equality

$$f^{-1}(\mathrm{int}'_i(B)) = f^{-1}(Y - \mathrm{cl}'_i(Y - B))$$

for all $B \in P(Y)$ and $i \in \{1,2\}$ since $\mathrm{cl}_i\left(f^{-1}(B)\right) \subseteq f^{-1}(\mathrm{cl}'_i(B))$. Hence $X - f^{-1}(\mathrm{cl}'_i(Y-B)) \subseteq X - \mathrm{cl}_i\left(f^{-1}(Y-B)\right)$. Also $X - f^{-1}(\mathrm{cl}'_i(Y-B)) \subseteq \mathrm{int}_i\left(f^{-1}(B)\right)$ since $X - \mathrm{cl}_i\left(f^{-1}(Y-B)\right) = \mathrm{int}_i\left(f^{-1}(B)\right)$

(ii $\Rightarrow$ iii) Assume that $f^{-1}(\mathrm{int}'_i(B)) \subseteq \mathrm{int}_i\left(f^{-1}(B)\right)$ for $\forall B \in P(Y)$ and $B \in \nu'_i(f(x))$. In that case, $f(x) \in \mathrm{int}'_i(B)$. Hence $x \in f^{-1}(\mathrm{int}'_i(B)) \subseteq \mathrm{int}_i\left(f^{-1}(B)\right)$, that is, $x \in \mathrm{int}_i\left(f^{-1}(B)\right) \Rightarrow f^{-1}(B) \in \nu_i(x)$.

(iii $\Rightarrow$ i) Assume that $f^{-1}(B) \in \nu_i(x)$ when $B \in \nu'_i(f(x))$ for $\forall B \in P(Y)$. If we take $x \in \mathrm{cl}_i\left(f^{-1}(B)\right)$, we find $x \in X - \mathrm{int}_i\left(X - f^{-1}(B)\right)$ since $\mathrm{cl}_i\left(f^{-1}(B)\right) = X - \mathrm{int}_i\left(X - f^{-1}(B)\right)$. Hereby $x \notin \mathrm{int}_i\left(X - f^{-1}(B)\right)$, i.e., $X - f^{-1}(B) \notin \nu_i(x)$. So $f^{-1}(B) \in \nu_i(x)$ is obtained. Also by the hypothesis $B \in \nu'_i(f(x))$, it is easy to derive $(Y - B) \notin \nu'_i(f(x))$ and we have $f(x) \notin \mathrm{int}'_i(Y - B)$. Thus $x \in f^{-1}(\mathrm{cl}'_i(B))$ because of $f(x) \in Y - (\mathrm{int}'_i(Y-B)) = \mathrm{cl}'_i(B)$. Finally, we obtain $\mathrm{cl}_i\left(f^{-1}(B)\right) \subseteq f^{-1}(\mathrm{cl}'_i(B))$.

Definition 3.9 *Let $f : (X, \mathrm{cl}_1, \mathrm{cl}_2) \to (Y, \mathrm{cl}'_1, \mathrm{cl}'_2)$ be a bijective map. If f is bi-continuous and f^{-1} bi-continuous, then it is called bi-homeomorphism.*

In our further discussion, we shall abbreviate "lower (upper) semicontinuous" to l.(u.)s.c.

Definition 3.10 *Let $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ be a bi-isotonic space and $(\mathbb{R}, \omega)$ be a usual topological space. A function $f : X \to \mathbb{R}$ is called*

* cl_i*−l.s.c. if and only if for any $a \in \mathbb{R}$, the subset $f^{-1}((a, \infty))$ is open in the isotonic space (X, cl_i),*
* cl_i*−u.s.c. if and only if for any $a \in \mathbb{R}$ the subset $f^{-1}((-\infty, a))$ is open in the isotonic space (X, cl_i).*

Proposition 3.11 *Let $(X, \mathrm{cl}_1, \mathrm{cl}_2)$, be a bi-isotonic space and $(\mathbb{R}, \omega)$ be a usual topological space. If a function $f : X \to \mathbb{R}$ is*

* cl_i-*l.s.c., then* $\mathrm{cl}_i\left(f^{-1}\left((-\infty,a)\right)\right)\subseteq f^{-1}\left((-\infty,a]\right)$,

* cl_i-*u.s.c., then* $\mathrm{cl}_i\left(f^{-1}\left((a,\infty)\right)\right)\subseteq f^{-1}\left([a,\infty)\right)$ *for any* $a\in\mathbb{R}$.

*Proof.*Let $f:X\to\mathbb{R}$ be cl_i-l.s.c., then for $a\in\mathbb{R}$, the subset $f^{-1}\left((a,\infty)\right)$ is open in the isotonic space (X,cl_i), then $f^{-1}\left((a,\infty)\right)=\mathrm{int}_i\left(f^{-1}\left((a,\infty)\right)\right)$. On the other hand $f^{-1}\left((a,\infty)\right)\subseteq f^{-1}\left([a,\infty)\right)$ since $(a,\infty)\subseteq[a,\infty)$. It is known that f is isotonic which means that $\mathrm{int}_i\left(f^{-1}\left((a,\infty)\right)\right)\subseteq\mathrm{int}_i\left(f^{-1}\left([a,\infty)\right)\right)$. From these relations, we get$f^{-1}\left((a,\infty)\right)\subseteq\mathrm{int}_i\left(f^{-1}\left([a,\infty)\right)\right)$ Hence, we obtain $X-f^{-1}\left((a,\infty)\right)\supseteq X-\mathrm{int}_i\left(f^{-1}\left([a,\infty)\right)\right)$. As a consequence

$$f^{-1}\left(\mathbb{R}-(a,\infty)\right)\supseteq\mathrm{cl}_i\left(f^{-1}\left(\mathbb{R}-[a,\infty)\right)\right)$$

i.e., $\mathrm{cl}_i\left(f^{-1}\left((-\infty,a)\right)\right)\subseteq f^{-1}\left((-\infty,a]\right)$ is satisfied. The other case can be seen in a similar manner.

Definition 3.12 *Let* (X,cl_1,cl_2) *be a bi-isotonic space and* $(\mathbb{R},\omega)$ *be a usual topological space. A function* $f:(X,cl_1,cl_2)\to(\mathbb{R},\omega)$ *is called* cl_icl_j-*l.u.s.c.if* f *is* cl_i-*l.s.c. and* cl_j-*u.s.c.*

Proposition 3.13 *Let* (X,cl_1,cl_2) *be a bi-isotonic space,* $(\mathbb{R},\omega)$ *be a usual topological space and* $(\mathbb{R},\omega_1,\omega_2)$ *be a bitopological space where* $\omega_1=\{\mathbb{R},\varnothing\}\cup\{(a,\infty):a\in\mathbb{R}\}$ *is the right ray topology and* $\omega_2=\{\mathbb{R},\varnothing\}\cup\{(-\infty,a):a\in\mathbb{R}\}$ *is the left ray topology on* $\mathbb{R}$*. A function* $f:(X,cl_1,cl_2)\to(\mathbb{R},\omega_1,\omega_2)$ *is bi-continuous if and only if* $f:(X,cl_1,cl_2)\to(\mathbb{R},\omega)$ *is* cl_icl_j-*l.u.s.c. for each* $i,j\in\{1,2\}$ *such that* $i\neq j$*.*

*Proof.*We shall consider only the case with $i=1$ and $j=2$ since the other case can be proved in a similar manner.

($\Rightarrow$:) Assume that $f:(X,\mathrm{cl}_1,\mathrm{cl}_2)\to(\mathbb{R},\omega_1,\omega_2)$ is a bi-continuous function then there is the relation $\mathrm{cl}_i\left(f^{-1}(B)\right)\subseteq f^{-1}\left(\mathrm{cl}_{\omega_i}(B)\right)$ for any $B\in P(\mathbb{R})$ and $i\in\{1,2\}$. If we assume $B=(-\infty,a)$ for any $a\in\mathbb{R}$ we see $\mathrm{cl}_{\omega_1}(B)=(-\infty,a]$ with respect to right ray topology ω_1. Hence we get $\mathrm{cl}_1\left(f^{-1}\left((-\infty,a)\right)\right)\subseteq f^{-1}\left((-\infty,a]\right)$ which means that $f:(X,\tau_1,\tau_2)\to(\mathbb{R},\omega)$ cl_1- l.s.c. Similarly, if suppose $B=(a,\infty)$ for any $a\in\mathbb{R}$, then the closure of B with respect to right ray topology ω_2 becomes $\mathrm{cl}_{\omega_2}(B)=[a,\infty)$. So $\mathrm{cl}_2\left(f^{-1}\left((a,\infty)\right)\right)\subseteq f^{-1}\left([a,\infty)\right)$ is obtained. This proves that the function $f:(X,\tau_1,\tau_2)\to(\mathbb{R},\omega)$ is cl_2-u.s.c. As a consequence $f:(X,\tau_1,\tau_2)\to(\mathbb{R},\omega)$ is $\mathrm{cl}_1\mathrm{cl}_2-$l.u.s.c.

($\Leftarrow$:) Let $f:(X,\mathrm{cl}_1,\mathrm{cl}_2)\to(\mathbb{R},\omega)$ be a$\mathrm{cl}_1\mathrm{cl}_2-$l.u.s.c. function. This means that it is cl_1-l.s.c. and cl_2- u.s.c. i.e., $\mathrm{cl}_1\left(f^{-1}\left((-\infty,a)\right)\right)\subseteq f^{-1}\left((-\infty,a]\right)$ and $\mathrm{cl}_2\left(f^{-1}\left((a,\infty)\right)\right)\subseteq f^{-1}\left([a,\infty)\right)$ is satisfied for all $a\in\mathbb{R}$. If we take any ω_1- open set B_1 and ω_2- open set B_2 , then $B_1=(a,\infty)$ and $B_2=(-\infty,a)$ for any, $a\in\mathbb{R}$ we get $\mathrm{cl}_1\left(f^{-1}(B_2)\right)\subseteq f^{-1}\left(\mathrm{cl}_{\omega_1}(B_2)\right)$ and. It is easily seen that $f:(X,\mathrm{cl}_i)\to(\mathbb{R},\omega_i)$ is $i-$continuous for each $i\in\{1,2\}$, that is, the function $f:(X,\tau_1,\tau_2)\to(\mathbb{R},\omega_1,\omega_2)$ is bicontinuous.

4 Separation Axioms in Bi-Isotonic Spaces

Definition 4.1 *A space* (X,cl) *is called* T_0-*space if there* $N_x\in\nu(x)$ *such that* $y\notin N_x$ *or there is* $N_y\in\nu(y)$ *such that* $x\notin N_y$ *for all distinct points* $x,y\in X$ *[27].*

Proposition 4.2 *An isotonic space* (X,cl) *is a* T_0-*space if and only if* $y\notin\mathrm{cl}(\{x\})$ *or* $x\notin\mathrm{cl}(\{y\})$ *for all distinct points* $x,y\in X$ *[27].*

Definition 4.3 *A generalized bi-closure* $(X,\mathrm{cl}_1,\mathrm{cl}_2)$ *is called pairwise* T_0-*space if there is* $N_x\in\nu_1(x)$ *such that* $y\notin N_x$ *or there is* $N_y\in\nu_2(y)$ *such that* $x\notin N_y$ *for all distinct points* $x,y\in X$*.*

Proposition 4.4 *A bi-isotonic space* $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ *is a pairwise* T_0-*space if and only if* $y \notin \mathrm{cl}_1(\{x\})$ *or* $x \notin \mathrm{cl}_2(\{y\})$ *for all distinct points* $x, y \in X$.

*Proof.*Proposition 4.2 and Definition 4.3 require that there is require that there is $N_x \in \nu_1(x)$ such that $y \notin N_x$ iff $y \notin \mathrm{cl}_1(\{x\})$ for all distinct points $x, y \in X$. Similarly $N_y \in \nu_2(y)$ such that $x \notin N_y$ iff $x \notin N_y$. These complete the proof.

Definition 4.5 *A generalized bi-closure space* $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ *is called*

- *pairwise* S_T_1-*space if there is* $N_x \in \nu_1(x)$ *such that* $y \notin N_x$ *and there is* $N_y \in \nu_2(y)$ *such that* $x \notin N_y$ *for all distinct points* $x, y \in X$,
- *pairwise* R_T_1-*space if* (X, cl_1) *and* (X, cl_2) *are both* T_1-*spaces.*

Proposition 4.6 *An isotonic space* (X, cl) *is* T_1-*space iff* $\mathrm{cl}(\{x\}) \subseteq \{x\}$ *for all* $x \in X$ *[27].*

Proposition 4.7 *If a bi-isotonic space* $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ *is a pairwise* R_T_1-*space then it is pairwise* S_T_1-*space.*

*Proof.*Let bi-isotonic space $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ be a pairwise R_T_1-space. Then from Proposition 4.7 $\mathrm{cl}_1(\{x\}) \subseteq \{x\}$ and $\mathrm{cl}_2(\{y\}) \subseteq \{y\}$ for $x \neq y$, $x, y \in X$ since the isotonic spaces (X, cl_1) and (X, cl_2) are T_1- spaces. Hence $X - \{x\} \subseteq \mathrm{int}_1(X - \{x\})$ and $X - \{y\} \subseteq \mathrm{int}_2(X - \{y\})$. It is easily seen $y \in X - \{x\} \subseteq \mathrm{int}_1(X - \{x\})$ from $y \in X - \{x\}$ and $x \notin X - \{x\}$ because $x \neq y$. So we find $X - \{x\} \in \nu_1(y)$. Similarly $x \in \mathrm{int}_2(X - \{y\})$ which means that $X - \{y\} \in \nu_2(x)$. As a consequence, the bi-isotonic space X is pairwise S_T_1-space.

Definition 4.8 *A generalized bi-closure space* $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ *is called pairwise Hausdorff space, if there are* $U \in \nu_1(x)$ *and* $V \in \nu_2(y)$ *such that* $U \cap V = \emptyset$ *for all distinct points* $x, y \in X$.

Proposition 4.9 *A bi-isotonic space* $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ *is pairwise Hausdorff space if and only if there is* $U \in \nu_2(x)$ *such that* $y \notin \mathrm{cl}_1(U)$ *or there is* $V \in \nu_1(y)$ *such that* $x \notin \mathrm{cl}_2(V)$ *for all distinct points* $x, y \in X$.

Proof.

($\Rightarrow$:) Let bi-isotonic space $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ be pairwise Hausdorff. Then there are $U \in \nu_1(x)$ and $U \in \nu_2(x)$ such that $U \cap V = \emptyset$ Hence $y \in \mathrm{int}_1(V)$, i.e., $y \notin X - \mathrm{int}_1(V) = \mathrm{cl}_1(X - V)$ since $V \in \nu_1(y)$. Also $\mathrm{cl}_1(U) \subseteq \mathrm{cl}_1(X - V)$ because cl_1 is the isotonic operator and $U \subseteq X - V$. So there is $U \in \nu_2(x)$ such that $y \notin \mathrm{cl}_1(U)$. Similarly, there is $V \in \nu_1(y)$ such that $x \notin \mathrm{cl}_2(V)$.

($\Leftarrow$:) Suppose that there is $U \in \nu_2(x)$ such that $y \notin \mathrm{cl}_1(U)$ for all distinct points $x, y \in X$. Then $y \in X - \mathrm{cl}_1(U) = \mathrm{int}_1(X - U)$ and this means that $X - U \in \nu_1(y)$. If we call $X - U = V$ we find $V \in \nu_1(y)$ and $U \in \nu_2(x)$ satisfying $U \cap V = \emptyset$. As a consequence X becomes a pairwise Hausdorff space.

Proposition 4.10 *If bi-isotonic space* $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ *is a pairwise Hausdorff space then* (X, cl_1) *and* (X, cl_2) *are* T_1-*spaces.*

*Proof.*Let bi-isotonic space $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ be pairwise Hausdorff. $x \notin \{y\}$ for all $x, y \in X$ such that $x \neq y$. Moreover $x \in \mathrm{int}_1(U)$ since there are $U \in \nu_1(x)$ and $V \in \nu_2(y)$ such that $U \cap V = \emptyset$. Form here $x \notin X - \mathrm{int}_1(U) = \mathrm{cl}_1(X - U)$. Also $\mathrm{cl}_1(V) \subseteq \mathrm{cl}_1(X - U)$ is satisfied since $V \subseteq X - U$. Accordingly we obtain $x \notin \mathrm{cl}_1(V)$. On the other hand $\mathrm{cl}_1\{y\} \subseteq \mathrm{cl}_1(V)$, because of $\{y\} \subseteq V$ and cl_1 is an isotonic operator. In conclusion we get

$x \notin \mathrm{cl}_1\{y\}$, i.e. $\mathrm{cl}_1\{y\} \subseteq \{y\}$ which means that (X, cl_1) is a T_1-space. Similarly it can be proved that (X, cl_2) is a T_1-space, too.

If we consider this last proposition associated with Definition 4.5 and Proposition 4.10, we can give the following result.

Corollary 4.11 *Each pairwise Hausdorff bi-isotonic space is a pairwise R_T_1-space, thereby a S_T_1-space.*

Definition 4.12 *A bi-isotonic space $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ is a pairwise $T_{2\frac{1}{2}}-$space if there are $U \in \nu_1(x)$ and $V \in \nu_2(y)$ such that $\mathrm{cl}_1(U) \cap \mathrm{cl}_2(V) = \emptyset$ for all distinct points $x, y \in X$.*

Definition 4.13 *Let $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ be a generalized bi-closure space. It is said that (X, cl_1) is regular with respect to (X, cl_2) if there are $U \in \nu_1(x)$ and $V \in \nu_2(F)$ such that $U \cap V = \emptyset$ for all $x \in X$ and $F \subseteq X$ where $x \notin \mathrm{cl}_1(F)$.*

If (X, cl_1) is regular with respect to (X, cl_2) and (X, cl_2) is regular with respect to (X, cl_1) then $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ is called pairwise regular.

Proposition 4.14 *Let $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ be a bi-isotonic space. The isotonic space (X, cl_1) is regular with respect to the isotonic space (X, cl_2) if and only if there is a $U \in \nu_1(x)$ such that $\mathrm{cl}_2(U) \subseteq N$ for all neighborhood $N \in \nu_1(x)$ of each $x \in X$.*

Proof.

($\Rightarrow$:) Let (X, cl_1) be regular with respect to the isotonic space (X, cl_2), then there are $U \in \nu_1(x)$ and $V \in \nu_2(F)$ such that $U \cap V = \emptyset$ for all $x \in X$ and $F \subseteq X$ where $x \notin \mathrm{cl}_1(F)$. Consider any $x \in X$ and $N \in \nu_1(x)$. In that case $x \notin X - \mathrm{int}_1(N) = \mathrm{cl}_1(X - N)$ since $x \in \mathrm{int}_1(N)$. So there is a $V \in \nu_2(X - N)$ such that $U \cap V = \emptyset$. Moreover, $X - \mathrm{cl}_2(U) \supseteq X - \mathrm{cl}_2(X - V) = \mathrm{int}_2(V)$ because of $\mathrm{cl}_2(U) \subseteq \mathrm{cl}_2(X - V)$. We get $X - N \subseteq \mathrm{int}_2(V) \subseteq X - \mathrm{cl}_2(U)$ since $V \in \nu_2(X - N)$ and $F \in \mathrm{int}_2(V)$. Finally there is a $U \in \nu_1(x)$ such that $\mathrm{cl}_2(U) \subseteq N$.

($\Leftarrow$:) Assume that there is a $U \in \nu_1(x)$ such that $\mathrm{cl}_2(U) \subseteq N$ for all neighborhood $N \in \nu_1(x)$ and consider a subset $F \subseteq X$ such that $x \notin \mathrm{cl}_1(F)$ for any $x \in X$. At that time $x \in X - \mathrm{cl}_1(F) = \mathrm{int}_1(X - F)$. Hence $X - F \in \nu_1(x)$. Under the assumption there is a $U \in \nu_1(x)$ such that $\mathrm{cl}_2(U) \subseteq X - F$. Then $F \subseteq X - \mathrm{cl}_2(U)$. If we call $X - \mathrm{cl}_2(U) = V$, then there are $U \in \nu_1(x)$ and $V \in \nu_2(F)$ such that $U \cap V = \emptyset$ and this completes the proof.

The following result can be given from Definition 4.13 and Proposition 4.14.

Corollary 4.15 *A bi-isotonic space $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ is a pairwise regular if and only if there is $U \in \nu_1(x)$ such that $\mathrm{cl}_2(U) \subseteq N'$ for all $N' \in \nu_1(x)$ and there is $V \in \nu_2(x)$ such that $\mathrm{cl}_1(V) \subseteq N''$ for all $N'' \in \nu_2(x)$.*

Definition 4.16 *A bi-isotonic space $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ is called pairwise T_3-space if it is both pairwise regular and pairwise R_T_1-space.*

Proposition 4.17 *If a bi-isotonic space $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ is a pairwise T_3-space then it is pairwise Hausdorff space.*

*Proof.*Let $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ be a pairwise T_3-space. Then $\mathrm{cl}_2(\{y\}) \subseteq \{y\}$ is satisfied for all $y \in X$ since it is pairwise R_T_1-space. From here $X - \{y\} \in \nu_2(x)$ for any $x \in X - \{y\}$ since $X - \{y\} \subseteq \mathrm{int}_2(X - \{y\})$.
On the other hand there is $\mathrm{cl}_1(V) \subseteq X - \{y\}$ such that $V \in \nu_2(x)$ for $X - \{y\} \in \nu_2(x)$ since $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ is pairwise regular. So we find a $V \in \nu_2(x)$ such that $y \notin \mathrm{cl}_1(V)$ and say X is pairwise Hausdorff space by Proposition 4.9.

Definition 4.18 *Let* $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ *be a bi-isotonic space and* $A, B \subset X$. *If there is a* $\mathrm{cl}_i\mathrm{cl}_j-$*l.u.s.c. function* $f : (X, \mathrm{cl}_1, \mathrm{cl}_2) \to (\mathbb{R}, \omega)$ *such that* $f(A) = 0$ *and* $f(B) = 1$, *then* A *is called* $(i, j)-$*completely separated from* B.

Definition 4.19 *A bi-isotonic space* $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ *is called*

* *$(i, j)-$completely regular if every closed F is $(i, j)-$completely separated from each point $x \notin F$.*
* *pairwise completely regular if it is both $(1, 2)-$completely regular and $(2, 1)-$completely regular.*
* *pairwise $T_{3\frac{1}{2}}-$space if it is both pairwise completely regular and pairwise R_T_1-space.*

Proposition 4.20 *If a bi-isotonic space* $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ *is pairwise completely regular then it is pairwise regular.*

*Proof.*Let $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ be a pairwise completely regular bi-isotonic space. Consider any closed subset F and any point x such that $x \notin F$ There is a $\mathrm{cl}_i\mathrm{cl}_j-$l.u.s.c. function $f : (X, \mathrm{cl}_1, \mathrm{cl}_2) \to (\mathbb{R}, \omega)$ such that $f(x) = 0$ and $f(F) = 1$. This means that the function $f : (X, \mathrm{cl}_1, \mathrm{cl}_2) \to (\mathbb{R}, \omega_1, \omega_2)$ is bi-continuous. On the other hand there are $\omega-$open neighborhoods U and V of the points 0 and 1, respectively, such that $U \cap V = \emptyset$ since the usual topological space $(\mathbb{R}, \omega)$ is Hausdorff. In that case $\mathrm{cl}_i\left(f^{-1}(U)\right) \subseteq f^{-1}(\mathrm{cl}_{\omega_i}(U))$, that is, $f^{-1}(0) = x \in f^{-1}(\mathrm{int}_{\omega_i}(U)) \subseteq \mathrm{int}_i\left(f^{-1}(U)\right)$. From here we can say that $f^{-1}(U) \in \nu_i(x)$. Analogously $f^{-1}(V) \in N_j\left(f^{-1}(1)\right) = N_j(F)$. As a consequence we obtain the subsets $f^{-1}(U) \in \nu_i(x)$ and $f^{-1}(V) \in N_j(F)$ such that $f^{-1}(U) \cap f^{-1}(V) = \emptyset$ and this is sufficient to say $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ is a pairwise regular bi-isotonic space.

If we consider the last proposition and the definitions of pairwise $T_{3\frac{1}{2}}-$space and pairwise T_3-space, then we can give the following corollary.

Corollary 4.21 *If a bi-isotonic space* $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ *is a pairwise* $T_{3\frac{1}{2}}-$*space then it is* T_3-*space.*

Definition 4.22 *Let* $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ *be a bi-isotonic space.*

(TN) *X is called pairwise t-normal, if there are $U \in \nu_1(F)$ and $V \in \nu_2(K)$ such that $U \cap V = \emptyset$ for all disjoint closed subsets $F, K \subseteq X$.*

(QN) *X is called pairwise quasi-normal, if there are $U \in \nu_1(F)$ and $V \in \nu_2(K)$ such that $U \cap V = \emptyset$ for all subsets $F, K \subseteq X$ such that $\mathrm{cl}_1(F) \cap \mathrm{cl}_2(K) = \emptyset$.*

(N) *X is called pairwise normal, if there are $U \in \nu_1(\mathrm{cl}_1(F))$ and $V \in \nu_2(\mathrm{cl}_2(K))$ such that $\mathrm{cl}_1(F) \cap \mathrm{cl}_2(K) = \emptyset$.*

It is easy to prove following proposition if we consider Definition 4.22 associated with the definition of closed sets in bi-isotonic spaces.

Proposition 4.23 *Let* $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ *be a bi-isotonic space.*

* *(N)$\Rightarrow$(TN).*
* *(QN)$\Rightarrow$(TN).*

Definition 4.24 *A bi-isotonic space* $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ *is called* T_4-*space if it is both pairwise quasi-normal and pairwise* R_T_1-*space.*

So the following proposition can be seen easily.

Proposition 4.25 *If $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ is a pairwise T_4-bi-isotonic space then it is pairwise $T_{3\frac{1}{2}}-$space.*

Definition 4.26 *Let $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ be a bi-isotonic space. The subsets $A, B \subseteq X$ are said to be semi- disjoint if $\mathrm{cl}_1(A) \cap B = A \cap \mathrm{cl}_2(B) = \emptyset$.*

Definition 4.27 *A bi-isotonic space $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ is called*

* *pairwise completely normal if there are $U \in \nu_1(A)$ and $V \in \nu_2(B)$ such that $U \cap V = \emptyset$ for all semi-disjoint subsets $A, B \subseteq X$,*
* *T_5-space if it is both pairwise completely normal and R_T_1-space.*

Proposition 4.28 *The properties of pairwise T_0, R_T_1, S_T_1, Hausdorff, $T_{2\frac{1}{2}}$, regular, T_3, completely regular, $T_{3\frac{1}{2}}$, t-normal, quasi-normal, normal, T_4, completely normal and T_5-spaces in bi-isotonic spaces are topological properties.*

*Proof.*Let $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ and $(Y, \mathrm{cl}'_1, \mathrm{cl}'_2)$ be two bi-isotonic spaces and $f : X \to Y$ be a bi-homeomorphism.

* Let X be a pairwise T_0-space. Consider any distinct points $x', y' \in Y$, then $f^{-1}(x') \neq f^{-1}(y')$. From here $f^{-1}(y') \notin \mathrm{cl}_1\left(\left\{f^{-1}(x')\right\}\right)$ $f^{-1}(x') \notin \mathrm{cl}_2\left(\left\{f^{-1}(y')\right\}\right)$. This gives us $y' = f\left(f^{-1}(y')\right) \notin f\left(\mathrm{cl}_1\left(\left\{f^{-1}(x')\right\}\right)\right) \subseteq \mathrm{cl}'_1\left(f\left(\left\{f^{-1}(x')\right\}\right)\right) = \mathrm{cl}'_1(\{x'\})$ or $x' = f\left(f^{-1}(x')\right) \notin f\left(\mathrm{cl}_2\left(\left\{f^{-1}(y')\right\}\right)\right) \subseteq \mathrm{cl}'_2\left(f\left(\left\{f^{-1}(y')\right\}\right)\right) = \mathrm{cl}'_2(\{y'\})$. That is $(Y, \mathrm{cl}'_1, \mathrm{cl}'_2)$ is a pairwise T_0-space.
* Let X be a pairwise R_T_1-space. Consider any $x' \in Y$, then there is a point $f^{-1}(x') = x \in X$. $\mathrm{cl}_i(\{x\}) \subseteq \{x\}$ is satisfied in (X, cl_i) for each $i \in \{1, 2\}$ since the isotonic spaces (X, cl_1) and (X, cl_2) are both T_1-spaces. Moreover $f(\mathrm{cl}_i(A)) = \mathrm{cl}'_i(f(A))$ for all $A \in P(X)$ because of the functions $f : (X, \mathrm{cl}_i) \to (Y, \mathrm{cl}'_i)$ are $i-$homeomorphisms for each her $i \in \{1, 2\}$. Thus,
$$\mathrm{cl}'_1(x') = \mathrm{cl}'_1(f(\{x\})) = f(\mathrm{cl}_1(\{x\})) \subseteq f(x) = \{x'\}$$
and
$$\mathrm{cl}'_2(x') = \mathrm{cl}'_2(f(\{x\})) = f(\mathrm{cl}_2(\{x\})) \subseteq f(x) = \{x'\}$$
is satisfied. This means that (Y, cl'_1) and (Y, cl'_2) are T_1-spaces. Consequently, the bi-isotonic space $(Y, \mathrm{cl}'_1, \mathrm{cl}'_2)$ is a pairwise R_T_1-space.
* Let X be a pairwise S_T_1-space. Consider any distinct points $x', y' \in Y$, then there are points $f^{-1}(x') = x$ and $f^{-1}(y') = y$ in X such that $x \neq y$. Hence there are $N_x \in \nu_1(x)$ and $N_y \in \nu_2(y)$ such that $x \notin N_y$ and $y \notin N_x$. If $A \in \nu_i(x)$ then $f(A) \in \nu'_i(f(x))$ for all $\forall A \in P(X)$ since the functions $f : (X, \mathrm{cl}_i) \to (Y, \mathrm{cl}'_i)$ are $i-$homeomorphisms for each $i \in \{1, 2\}$. In this way, there are $f(N_x) \in \nu'_1(x')$ and $f(N_y) \in \nu'_2(y')$ such that $x' \notin f(N_y)$ and $y' \notin f(N_x)$. This completes the proof.
* Let X be a pairwise Hausdorff space. Consider any distinct points $x', y' \in Y$, then there are points $f^{-1}(x') = x$ and $f^{-1}(y') = y$ in X such that $x \neq y$. So, there is $U \in \nu_2(x)$ such that $y \notin \mathrm{cl}_1(U)$ or there is $V \in \nu_1(y)$ such that $x \notin \mathrm{cl}_2(V)$. On the other hand $f(\mathrm{cl}_i(A)) \subseteq \mathrm{cl}'_i(f(A))$ for all $A \in P(X)$ and $i \in \{1, 2\}$. In this case, there is $f(U) \in \nu'_2(x')$ such that $y' \notin f(\mathrm{cl}_1(U)) \subseteq \mathrm{cl}'_1(f(U))$ or there is $f(V) \in \nu'_1(y')$ such that $x' \notin f(\mathrm{cl}_2(V)) \subseteq \mathrm{cl}'_2(f(V))$. This means that the bi-isotonic space $(Y, \mathrm{cl}'_1, \mathrm{cl}'_2)$ is a pairwise Hausdorff space.

* Let X be a pairwise $T_{2\frac{1}{2}}-$space and $x', y' \in Y$ such that $x' \neq y'$, then there are points $f^{-1}(x') = x$ and $f^{-1}(y') = y$ in X such that $x \neq y$. So, there are $U \in \nu_1(x)$ and $V \in \nu_2(y)$ such that $\mathrm{cl}_1(U) \cap \mathrm{cl}_2(V) = \emptyset$. Also

$$f(\mathrm{cl}_1(U) \cap \mathrm{cl}_2(V)) = f(\mathrm{cl}_1(U)) \cap f(\mathrm{cl}_2(V))$$

and

$$f(\mathrm{cl}_1(U)) \cap f(\mathrm{cl}_2(V)) = \mathrm{cl'}_1(f(U)) \cap \mathrm{cl'}_2(f(V))$$

are satisfied since the functions $f : (X, \mathrm{cl}_i) \to (Y, \mathrm{cl'}_i)$ are $i-$homeomorphisms for all $i \in \{1, 2\}$. Moreover, $f(A) \in \nu'_i(f(x))$, provided that $A \in \nu_i(x)$ for $\forall A \in P(X)$. Hence, there are $f(U) \in \nu'_1(x')$ and $f(V) \in \nu'_2(y')$ such that $\mathrm{cl'}_1(f(U)) \cap \mathrm{cl'}_2(f(V)) = \emptyset$. From here $(Y, \mathrm{cl'}_1, \mathrm{cl'}_2)$ is a pairwise $T_{2\frac{1}{2}}-$space.

* Let X be a pairwise regular. Consider any $x' \in Y$, then $f^{-1}(x') = x \in X$ and there is $U \in \nu_1(x)$ such that $\mathrm{cl}_2(U) \subseteq N'$ for all $N' \in \nu_1(x)$ and there is $V \in \nu_2(x)$ such that $\mathrm{cl}_1(V) \subseteq N''$ for all $N'' \in \nu_2(x)$.

* Let X be a pairwise T_3-space. Then it is pairwise regular and pairwise R_T_1-space. The aforementioned proposition requires that $(Y, \mathrm{cl'}_1, \mathrm{cl'}_2)$ is pairwise regular and pairwiseR_T_1-space. Thus, Y is a pairwise T_3-space, too.

* Let X be a pairwise completely regular space. Consider any $\mathrm{cl'}_1\mathrm{cl'}_2-$closed subset $F' \subset Y$and any point $x' \in Y$ such that $x' \notin F'$. Then there is $\mathrm{cl}_1\mathrm{cl}_2-$closed subset $f^{-1}(F') = F \subset X$ and a point $x \in X$ such that $x \notin F$ since $f : X \to Y$ is a bi-homeomorphism. There is a $\mathrm{cl}_i\mathrm{cl}_j-$l.u.s.c. a function $g : (X, \mathrm{cl}_1, \mathrm{cl}_2) \to (\mathbb{R}, \omega)$ such that $g(x) = 0$ and $g(F) = 1$ because X is pairwise completely regular space. From here $g : (X, \mathrm{cl}_1, \mathrm{cl}_2) \to (\mathbb{R}, \omega_1, \omega_2)$ is a bi-continuous function.

$$\begin{array}{ccc} (X, \mathrm{cl}_1, \mathrm{cl}_2) & \xrightarrow{g} & (\mathbb{R}, \omega_1, \omega_2) \\ f \searrow & & \nearrow g \circ f^{-1} \\ & (Y, \mathrm{cl'}_1, \mathrm{cl'}_2) & \end{array}$$

This diagram indicates that the function $g \circ f^{-1} : (Y, \mathrm{cl'}_1, \mathrm{cl'}_2) \to (\mathbb{R}, \omega_1, \omega_2)$ is bi-continuous and so $g \circ f^{-1} : (Y, \mathrm{cl'}_1, \mathrm{cl'}_2) \to (\mathbb{R}, \omega)$ is a $\mathrm{cl'}_i\mathrm{cl'}_j-$l.u.s.c. function. Also $g(x) = g\left(f^{-1}(x')\right) = g \circ f^{-1}(x') = 0$ and $g(F) = g\left(f^{-1}(F')\right) = g \circ f^{-1}(F') = 1$ are satisfied. That is $\mathrm{cl'}_1\mathrm{cl'}_2-$ closed subset$F'$of bi-isotonic space $(Y, \mathrm{cl'}_1, \mathrm{cl'}_2)$ is $(1, 2)$ and $(2, 1)-$completely separated from each point $x' \notin F'$ and this completes the proof.

* Let X be a pairwise $T_{3\frac{1}{2}}-$space. In this case, $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ is both pairwise completely regular and pairwise R_T_1-space. So, $(Y, \mathrm{cl'}_1, \mathrm{cl'}_2)$ is pairwise completely regular and pairwise R_T_1-space, too. Thus the bi-isotonic space Y is a pairwise $T_{3\frac{1}{2}}-$space.

* Let X be a pairwise t-normal bi-isotonic space. Let us take two separated closed subsets F' and K' in Y. Then $f^{-1}(F') = F$ and $f^{-1}(K') = K$ are disjoint closed subsets of X. There are $U \in \nu_1(F)$ and $V \in \nu_2(K)$ such that $U \cap V = \varnothing$ since X is a pairwise t-normal space. From Proposition 3.8, we see that there are $f(U) \in N'_1(F')$ and $f(V) \in N'_2(K')$ such that $f(U) \cap f(V) = \varnothing$. Consequently, Y is a pairwise t-normal bi-isotonic space.

* Let $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ be a pairwise quasi-normal bi-isotonic space. Consider two non-empty subsets F' and K' of Ysuch that $\mathrm{cl'}_1(F') \cap \mathrm{cl'}_2(K') = \varnothing$. It is easily seen

that$\mathrm{cl}_1\left(f^{-1}(F')\right) \cap \mathrm{cl}_2\left(f^{-1}(K')\right) = f^{-1}(\varnothing) = \varnothing$since$f$ is bi-homeomorphism. There are $U \in \nu_1\left(f^{-1}(F')\right)$ and $V \in \nu_2\left(f^{-1}(K')\right)$ such that $U \cap V = \varnothing$, since X is a pairwise quasi-normal space. In that way Y is a pairwise quasi-normal space from Proposition 3.8 .

* Let X be a pairwise normal space and consider the non-empty subsets F' and K' in Y such that $\mathrm{cl'}_1(F') \cap \mathrm{cl'}_2(K') = \varnothing$. Then there are the non-empty subsets F and K in X such that$\mathrm{cl}_1(F) \cap \mathrm{cl}_2(K) = \varnothing$ since f is a bi-homeomorphism satisfying $f(\mathrm{cl}_1(F)) = \mathrm{cl'}_1(F')$ and $f(\mathrm{cl}_2(K)) = \mathrm{cl'}_2(K')$. Also, there are $U \in \nu_1(\mathrm{cl}_1(F))$ and $V \in \nu_2(\mathrm{cl}_2(K))$ such that $U \cap V = \varnothing$ since X is a pairwise normal space. In these considerations, we find $f(U) \in N'_1(\mathrm{cl'}_1(F'))$ and $f(V) \in N'_2(\mathrm{cl'}_2(K))$ satisfying $f(U) \cap f(V) = \varnothing$. Finally, we see that the bi-isotonic space Y is a pairwise normal space.

* Let $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ be a pairwise T_4-space. This means that $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ is a pairwise normal and pairwise R_T_1-space. If $f : X \to Y$ is a bi-homeomorphism then $(Y, \mathrm{cl'}_1, \mathrm{cl'}_2)$ is pairwise normal and pairwise R_T_1-space. So, the bi-isotonic space Y is a pairwise T_4-space, too.

* Let X be a pairwise completely normal space and $A', B' \subset Y$ be separated sets, i.e., $\mathrm{cl'}_1(A') \cap B' = A' \cap \mathrm{cl}_2(B') = \varnothing$. Thus, $f^{-1}(\mathrm{cl'}_1(A')) \cap f^{-1}(B') = f^{-1}(B') \cap f^{-1}(\mathrm{cl'}_2(A')) = \varnothing$ is satisfied and there are two sets $f^{-1}(A') = A$ and $f^{-1}(B') = B$ in the bi-isotonic space X such that $\mathrm{cl}_1(A) \cap B = A \cap \mathrm{cl}_2(B) = \varnothing$ since the function f is bi-homeomorphism. From the hypothesis, we see $U \in \nu_1(A)$ and $V \in \nu_2(B)$ such that $U \cap V = \varnothing$. Then, we find the sets $f(U) \in N'_1(A')$ and$f(V) \in N'_2(B')$ satisfying $f(U) \cap f(V) = \varnothing$. As a consequence, the bi-isotonic space $(Y, \mathrm{cl'}_1, \mathrm{cl'}_2)$ is pairwise completely normal.

* Let X be a pairwise T_5-space. Then $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ is a pairwise completely normal and pairwise R_T_1-space. $(Y, \mathrm{cl'}_1, \mathrm{cl'}_2)$ pairwise completely normal and pairwise R_T_1-space since $f : X \to Y$ is a bi-homeomorphism. Finally, Y is a pairwise T_5-space, too.

Proposition 4.29 *In a bi-isotonic space, the properties of being a pairwise T_0, pairwise R_T_1, pairwise S_T_1, pairwise Hausdorff, pairwise $T_{2\frac{1}{2}}$, pairwise regular, pairwise T_3, pairwise completely regular, pairwise completely normal and pairwise T_5-space are hereditary properties.*

*Proof.*Let $\left(Y, \mathrm{cl}_1^Y, \mathrm{cl}_2^Y\right)$ and $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ be bi-isotonic spaces.

* Assume that the bi-isotonic space $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ is a pairwise T_0-space and $x, y \in Y$ such that $x \neq y$. Then $y \notin \mathrm{cl}_1(\{x\})$ or $x \notin \mathrm{cl}_2(\{y\})$ for $x, y \in X$. From Proposition 2.8, we can say $y \notin \mathrm{cl}_1(\{x\}) \cap Y = \mathrm{cl}_1^Y(\{x\})$ or $x \notin \mathrm{cl}_2(\{y\}) \cap Y = \mathrm{cl}_2^Y(\{x\})$. Thus, the subspace $(Y, \mathrm{cl'}_1, \mathrm{cl'}_2)$ is obtained as a pairwise T_0-space.

* Assume that the bi-isotonic space $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ is a pairwise R_T_1-space. In this case (X, cl_1) and (X, cl_2) spaces are T_1-spaces. Hence $\mathrm{cl}_1(\{x\}) \subseteq \{x\}$ and $\mathrm{cl}_2(\{x\}) \subseteq \{x\}$ for every $x \in Y \subseteq X$. It is found that $\mathrm{cl}_1^Y(\{x\}) = \mathrm{cl}_1(\{x\}) \cap Y \subseteq \{x\} \cap Y = \{x\}$ and $\mathrm{cl}_2^Y(\{x\}) = \mathrm{cl}_2(\{x\}) \cap Y \subseteq \{x\} \cap Y = \{x\}$ in $\left(Y, \mathrm{cl}_1^Y\right)$ and $\left(Y, \mathrm{cl}_2^Y\right)$, respectively. Then $\left(Y, \mathrm{cl}_1^Y\right)$ and $\left(Y, \mathrm{cl}_2^Y\right)$ spaces are T_1-spaces. Eventually, it is obtained that $(Y, \mathrm{cl'}_1, \mathrm{cl'}_2)$ is a pairwise R_T_1-space

* Assume that the bi-isotonic space $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ is a pairwise S_T_1-space and $x, y \in Y$ such that $x \neq y$. Then there are $N_x \in \nu_1(x)$ and $N_y \in \nu_2(y)$ such that $y \notin N_x$ and $x \notin N_y$, respectively, for the distinct points $x, y \in X$. There are $N_x^Y \in \nu_1^Y(x)$ and $N_y^Y \in \nu_2^Y(y)$ such that $y \notin N_x \cap Y = N_x^Y$ and $x \notin N_y \cap Y = N_y^Y$ since Y is a subspace. Thus, $(Y, \mathrm{cl'}_1, \mathrm{cl'}_2)$ is a pairwise S_T_1-space.

* Assume that the bi-isotonic space $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ is a pairwise Hausdorff space and $x, y \in Y$ such that $x \neq y$. By the hypothesis there is $U \in \nu_2(x)$ or $V \in \nu_1(y)$ such that $y \notin \mathrm{cl}_1(U)$ or $x \notin \mathrm{cl}_2(V)$, respectively, as $x \neq y$ for every $x, y \in X$. From Definition 2.5., it is seen that $U \in \nu_2^Y(x)$ or $V \in \nu_1^Y(x)$ since $x \notin \mathrm{cl}_2(V) \cap Y = \mathrm{cl}_2^Y(V)$ or $y \notin \mathrm{cl}_1(U) \cap Y = \mathrm{cl}_1^Y(U)$, respectively. So $(Y, \mathrm{cl'}_1, \mathrm{cl'}_2)$ becomes a pairwise Hausdorff space.

* Assume that the bi-isotonic space $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ is a pairwise $T_{2\frac{1}{2}}-$space and $x, y \in Y$ for $x \neq y$. By the hypothesis, there are $U \in \nu_1(x)$ and $V \in \nu_2(y)$ such that $\mathrm{cl}_1(U) \cap \mathrm{cl}_2(V) = \varnothing$. Then, there are $U \in \nu_2^Y(x)$ and $V \in \nu_2^Y(y)$ such that $\mathrm{cl}_1^Y(U) \cap \mathrm{cl}_2^Y(V) = (\mathrm{cl}_1(U) \cap \mathrm{cl}_2(V)) \cap Y = \varnothing$. As a result $(Y, \mathrm{cl'}_1, \mathrm{cl'}_2)$ is a pairwise $T_{2\frac{1}{2}}-$space.

* Assume that the bi-isotonic space $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ is a pairwise regular space and $x, y \in Y$ for $x \neq y$. Then, there is a set $U \in \nu_1(x)$ such that $\mathrm{cl}_2(U) \subseteq \nu_1$ for every neighborhood $\nu_1 \in \nu_1(x)$ and there is a set $V \in \nu_2(y)$ such that $\mathrm{cl}_1(V) \subseteq \nu_2$ for every neighborhood $\nu_2 \in \nu_2(x)$. From here, there are $U \in \nu_1^Y(x)$ and $V \in \nu_2^Y(x)$ such that $\mathrm{cl}_2^Y(U) = \mathrm{cl}_2(U) \cap Y \subseteq \nu_1 \cap Y = \nu_1^Y$ and $\mathrm{cl}_1^Y(V) = \mathrm{cl}_1(V) \cap Y \subseteq \nu_2 \cap Y = \nu_2^Y$ for the neighborhoods $\nu_1 \cap Y = \nu_1^Y \in \nu_1^Y(x)$ and $\nu_2 \cap Y = \nu_2^Y \in \nu_2^Y(x)$, respectively. It is seen that $(Y, \mathrm{cl'}_1, \mathrm{cl'}_2)$ is pairwise regular.

* Assume that the bi-isotonic space $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ is a T_3-space. Then X is both pairwise regular and pairwise R_T_1-space and so, $Y \subset X$ subspace is both pairwise regular and pairwise R_T_1-space. This means that the bi-isotonic subspace $(Y, \mathrm{cl'}_1, \mathrm{cl'}_2)$ is a T_3-space.

* Assume that the bi-isotonic space $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ a pairwise completely regular. Let us take any $\mathrm{cl}_1^Y \mathrm{cl}_2^Y-$closed set F and any point $x \in Y$ such that $x \notin F$. Then $x \notin \mathrm{cl}_1\mathrm{cl}_2(F)$ for the closed subset $\mathrm{cl}_1\mathrm{cl}_2(F)$ in X. By the hypothesis, there is a function $\mathrm{cl}_i\mathrm{cl}_j-$l.u.s.c. $f : (X, \mathrm{cl}_1, \mathrm{cl}_2) \to (\mathbb{R}, \omega)$ such that $f(x) = 0$ and $f(\mathrm{cl}_1\mathrm{cl}_2(F)) = 1$. If we denote the restriction function of f to Y with $f|_Y = g$, it is provided $g(x) = 0$ and $g(F) = 1$ since $g : \left(X, \mathrm{cl}_1^Y, \mathrm{cl}_2^Y\right) \to (\mathbb{R}, \omega)$ is $\mathrm{cl}_i^Y \mathrm{cl}_j^Y-$l.u.s.c. Thus, the bi-isotonic subspace $(Y, \mathrm{cl'}_1, \mathrm{cl'}_2)$ is pairwise completely regular space.

* Assume that the bi-isotonic space $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ is a $T_{3\frac{1}{2}}-$space. Then X is both pairwise completely regular and pairwise R_T_1-space. Thus, $Y \subset X$ subspace is both pairwise completely regular and pairwise R_T_1-space. So $(Y, \mathrm{cl'}_1, \mathrm{cl'}_2)$ bi-isotonic subspace is $T_{3\frac{1}{2}}-$space, too.

* Assume that the bi-isotonic space $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ is a pairwise completely normal. Let $A, B \subseteq Y$ be separated sets. Namely $\mathrm{cl}_1^Y(A) \cap B = A \cap \mathrm{cl}_2^Y(B) = \varnothing$. Then $\mathrm{cl}_1(A) \cap B = A \cap \mathrm{cl}_2(B) = \varnothing$. By the hypothesis, there are the neighborhoods $U \in \nu_1(A)$ and $V \in \nu_2(B)$ such that $U \cap V = \varnothing$. So, it is obtained that there are $U^Y \in \nu_1^Y(A)$ and $V^Y \in \nu_2^Y(B)$such that $U^Y \cap V^Y = \varnothing$. As a result, the bi-isotonic subspace $\left(Y, \mathrm{cl}_1^Y, \mathrm{cl}_2^Y\right)$ is found as pairwise completely normal space.

* Assume that the bi-isotonic space $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ is a T_5-space. Then X is pairwise completely normal and pairwise R_T_1-space. So, the bi-isotonic subspace $\left(Y, \mathrm{cl}_1^Y, \mathrm{cl}_2^Y\right)$ is T_5-space since it is pairwise completely normal and pairwise R_T_1-space.

Proposition 4.30 *Let (X, cl_1, cl_2) be a bi-isotonic space and $Y \subseteq X$ be a closed subset. If (X, cl_1, cl_2) is a pairwise t-normal, pairwise quasi-normal, pairwise normal space and pairwise T_4-space, then $\left(Y, cl_1^Y, cl_2^Y\right)$ is a pairwise t-normal, pairwise quasi-normal, pairwise normal and pairwise T_4-space, respectively.*

*Proof.*Let $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ be a bi-isotonic space and subset $Y \subseteq X$ be closed.

* Let the bi-isotonic space $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ be a pairwise t-normal and $F, K \subseteq Y$ be non-empty discrete closed subsets. Then the subsets F, K in $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ are closed from the Proposition 2.4. By the hypothesis, there are the neighborhoods $U \in \nu_1(F)$ and $V \in \nu_2(K)$ such that $U \cap V = \varnothing$. So there are the sets $U^Y \in \nu_1^Y(F)$, $V^Y \in \nu_2^Y(K)$ such that $U^Y \cap V^Y = (U \cap Y) \cap (V \cap Y) = \varnothing$. Namely, the bi-isotonic subspace $(Y, \mathrm{cl'}_1, \mathrm{cl'}_2)$ is a pairwise t-normal space.

* Let the bi-isotonic space $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ be a pairwise quasi-normal and $F, K \subseteq Y$ be non-empty discrete subsets such that $\mathrm{cl}_1^Y(F) \cap \mathrm{cl}_2^Y(K) = \varnothing$. Then there are non-empty discrete subsets $F, K \subseteq X$ satisfying $\mathrm{cl}_1(F) \cap \mathrm{cl}_2(K) = \varnothing$. By the hypothesis, there are $U \in \nu_1(F)$ and $V \in \nu_2(K)$ such that $U \cap V = \varnothing$. As a consequence, it is easy to find that there are $U^Y \in \nu_1^Y(F)$ and $V^Y \in \nu_2^Y(K)$ such that $U^Y \cap V^Y = \varnothing$, which means that the bi-isotonic subspace $\left(Y, \mathrm{cl}_1^Y, \mathrm{cl}_2^Y\right)$ is a pairwise quasi-normal space.

* Let the bi-isotonic space $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ be a pairwise quasi-normal and $F, K \subseteq Y$ be non-empty discrete subsets such that $\mathrm{cl}_1^Y(F) \cap \mathrm{cl}_2^Y(K) = \varnothing$. Then there are non-empty discrete subsets $F, K \subseteq X$ satisfying $\mathrm{cl}_1(F) \cap \mathrm{cl}_2(K) = \varnothing$. By the hypothesis, there are $U \in \nu_1(F)$ and $V \in \nu_2(K)$ such that $U \cap V = \varnothing$. As a consequence, it is easy to find that there are $U^Y \in \nu_1^Y(F)$ and $V^Y \in \nu_2^Y(K)$ such that $U^Y \cap V^Y = \varnothing$, which means that the bi-isotonic subspace $\left(Y, \mathrm{cl}_1^Y, \mathrm{cl}_2^Y\right)$ is a pairwise quasi-normal space.

* Let the bi-isotonic space $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ is a T_4-space. Then it is pairwise quasi-normal and pairwise R_T_1-space. So the bi-isotonic subspace $\left(Y, \mathrm{cl}_1^Y, \mathrm{cl}_2^Y\right)$ is pairwise quasi-normal and pairwise R_T_1-space provided that Y is closed subset of X . This completes the proof.

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