POWER MONOTONICITY FOR A PATH OF OPERATOR MEANS

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Abstract. We discuss the power monotonicity of the family $\{F_{p,q}| p, r \in \mathbb{R}\}$ of parametrized representing functions of Kubo-Ando operator means, which is introduced in our preceding paper. It includes several important representing functions, for example, arithmetic, geometric, harmonic, logarithmic, power and Stolarsky means. We shall discuss conditions of power monotonicity of functions.

1 Introduction. The theory of operator means is established by Kubo and Ando [4]: An *operator mean* $A \,\mathrm{m}\, B$ for positive invertible operators A, B is defined by a positive normalized operator monotone function f on $(0, \infty)$ by

$$A \,\mathrm{m}\, B = A^{\frac{1}{2}} f\left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}\right) A^{\frac{1}{2}}.$$

Here the *normalization* corresponds to $f(1) = 1$. One of the result of the Kubo-Ando theory is to give a bijection between an operator mean and a positive normalized operator monotone function on $(0, \infty)$ as above. In this bijection, f is often called the *representing function* of an operator mean m; $f(x) = 1 \,\mathrm{m}\, x$.

Recently Wada [7] introduced the power monotonicity of the representing function f, and showed the relation to the Ando-Hiai inequality [8, 1]: f is called *PMI (power monotone increasing)* (resp. *PMD (power monotone decreasing)*) if f satisfies

$$f(x)^r \le f(x^r) \qquad (\text{ resp. } f(x)^r \ge f(x^r)) \qquad \text{for all } r \ge 1 \text{ and } x > 0.$$

It has not been known any characterization of a function satisfying PMI or PMD. But we know some examples of PMI or PMD functions: For each $p > 0$ and $\lambda \in (0, 1)$

$$(1 - \lambda + \lambda x^p)^{\frac{1}{p}}$$

is PMI, and it is PMD for the case $p < 0$. Moreover,

$$\lim_{p \to 0} (1 - \lambda + \lambda x^p)^{\frac{1}{p}} = x^\lambda$$

is both PMI and PMD.

Especially,

$$f(x) = \frac{x - 1}{\log x} = \int_0^1 x^\lambda d\lambda,$$

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the representing function of the logarithmic mean is PMI. The above functions can be unified into the following function $F_{p,q}$:

$$F_{p,q}(x) := \left(\int_0^1 [1-\lambda+\lambda x^p]^{\frac{q}{p}} d\lambda \right)^{\frac{1}{q}} = \left(\frac{p}{p+q} \cdot \frac{x^{p+q}-1}{x^p-1} \right)^{\frac{1}{q}}.$$

This extension is discussed in [6]. It is known that for $p,q \in [-1,1]$, $F_{p,q}$ is operator monotone on $(0,\infty)$ and monotone increasing on $p,q \in [-1,1]$ [6]. More precisely, it is increasing on $p,q \in \mathbb{R}$, but it is not operator monotone if $p,q \notin [-1,1]$. Moreover $F_{p,q}$ is symmetric:

$$A \, \mathrm{m}_{F_{p,q}} B = B \, \mathrm{m}_{F_{p,q}} A, \qquad \text{that is,} \qquad F_{p,q}(x) = xF_{p,q}\left(\frac{1}{x}\right).$$

It includes several famous functions as in the following table.

(p,q)	$(-1,-1)$	$(-1,0)$	$(0,0)$	$(0,1)$	$(1,1)$	(p,p)
$F_{p,q}$	$\frac{2x}{1+x}$	$\frac{x\log x}{x-1}$	$\sqrt{x}$	$\frac{x-1}{\log x}$	$\frac{1+x}{2}$	$\left(\frac{1+x^p}{2}\right)^{\frac{1}{p}}$

Here we consider the limit in the cases of $(p,q) = (-1,0),(0,0),(0,1)$. We can also get important functions from $F_{p,q}$, too.

(1) (Power difference mean, [3]) $F_{p,1}(x) = \dfrac{p}{p+1}\dfrac{x^{p+1}-1}{x^p-1}$.

(2) (Normalized power difference mean, [2, 5]) $F_{\frac{3p-1}{2},1}(x) = \dfrac{3p-1}{3p+1}\dfrac{x^{\frac{3p+1}{2}}-1}{x^{\frac{3p-1}{2}}-1}$.

In this note, we shall show power monotonicity of $F_{p,q}$, firstly. Then we give another proof of power monotonicity of $F_{p,q}$ via a lower bound of $F_{p,q}(x^n) - F_{p,q}(x)^n$ in a restricted case. Lastly, we shall discuss conditions of power monotonicity of each differentiable function.

2 Main result.

In this section, we shall show power monotonicity of $F_{p,q}$.

Theorem 1. *$F_{p,q}$ is PMI for $2p+q \geq 0$ and is PMD for $2p+q \leq 0$*

It is easy that $F_{p,q}(x) = \sqrt{x}$, the representing function of the geometric mean, if $2p+q = 0$. Hence by the monotonicity of p,q, we can rewrite Theorem 1 into the following form.

Theorem 1'. *$F_{p,q}$ is PMI (resp. PMD) if and only if $\sqrt{x} \leq F_{p,q}(x)$ (resp. $\sqrt{x} \geq F_{p,q}(x)$) for all $x > 0$.*

To show Theorem 1, we note the following lemma. It follows from the definition of $F_{p,q}$, easily.

Lemma 2. *$F_{p,q}(x^{-1})^{-1} = F_{-p,-q}(x)$ holds for all $p,q \in \mathbb{R}$ and $x > 0$.*

Proof of Theorem 1. We shall divide 8 cases to prove Theorem 1 (see Figure 1, below).

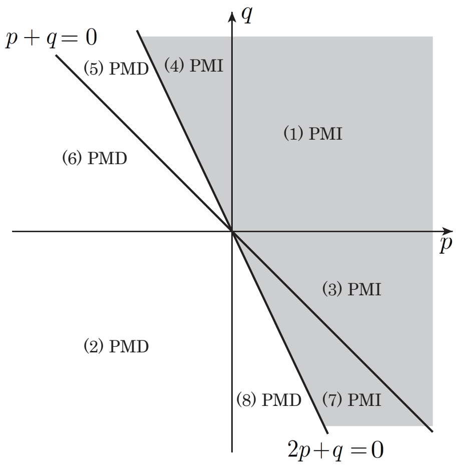

Figure 1: We divided 8 cases to prove Theorem 1.

(1) The case of $p, q > 0$. We notice that $f(x) = x^r$ is a convex function for $r \geq 1$. For $r \geq 1$,

$$\begin{aligned}
F_{p,q}(x)^r &= \left(\int_0^1 [1-\lambda+\lambda x^p]^{\frac{q}{p}} d\lambda\right)^{\frac{r}{q}} \\
&\leq \left(\int_0^1 [1-\lambda+\lambda x^p]^{\frac{qr}{p}} d\lambda\right)^{\frac{1}{q}} \quad \text{by } q > 0 \\
&\leq \left(\int_0^1 [1-\lambda+\lambda x^{pr}]^{\frac{q}{p}} d\lambda\right)^{\frac{1}{q}} \quad \text{by } p, q > 0 \\
&= F_{p,q}(x^r).
\end{aligned}$$

Hence $F_{p,q}$ is PMI for $p, q > 0$.
(2) The case of $p, q < 0$. By Lemma 2 and (1), we have

$$F_{p,q}(x^r) = F_{-p,-q}(x^{-r})^{-1} \leq F_{-p,-q}(x^{-1})^{-r} = F_{p,q}(x)^r.$$

Hence $F_{p,q}$ is PMD for $p, q < 0$.

(3) The case of $p + q > 0$ and $q < 0$.

$$\begin{aligned}
F_{p,q}(x) &= \left(\frac{p}{p+q} \cdot \frac{x^{p+q}-1}{x^p-1}\right)^{\frac{1}{q}} \\
&= \left(\frac{p+q}{p+q+(-q)} \cdot \frac{x^{p+q+(-q)}-1}{x^{p+q}-1}\right)^{\frac{1}{-q}} = F_{p+q,-q}(x)
\end{aligned}$$

Hence by (1), $F_{p,q}$ is PMI for $p + q > 0$ and $q < 0$.

(4) The case of $p < 0$ and $2p+q > 0$. We notice that $q > 0$.

$$
\begin{aligned}
F_{p,q}(x) &= \left(\frac{-p}{p+q} \cdot \frac{x^{p+q}-1}{1-x^p}\right)^{\frac{1}{q}} \\
&= x^{-\frac{p}{q}} \left(\frac{-p}{p+q} \cdot \frac{x^{p+q}-1}{x^{-p}-1}\right)^{\frac{1}{q}} \\
&= x^{-\frac{p}{q}} \left(\frac{-p}{-p+2p+q} \cdot \frac{x^{-p+2p+q}-1}{x^{-p}-1}\right)^{\frac{1}{2p+q}\cdot\frac{2p+q}{q}} \\
&= x^{-\frac{p}{q}} \left\{F_{-p,2p+q}(t)\right\}^{\frac{2p+q}{q}}.
\end{aligned}
$$

Hence by (1), $F_{p,q}$ is PMI for $p < 0$ and $2p+q > 0$.

(5) The case of $p+q > 0$ and $2p+q < 0$. We notice that $q > 0$.

$$
\begin{aligned}
F_{p,q}(x) &= x^{-\frac{p}{q}} \left(\frac{-p}{p+q} \cdot \frac{x^{p+q}-1}{x^{-p}-1}\right)^{\frac{1}{q}} \\
&= x^{-\frac{p}{q}} \left(\frac{p+q}{(p+q)+(-2p-q)} \cdot \frac{x^{(p+q)+(-2p-q)}-1}{x^{p+q}-1}\right)^{\frac{1}{-(2p+q)}\cdot\frac{2p+q}{q}} \\
&= x^{-\frac{p}{q}} \left\{F_{p+q,-2p-q}(x)\right\}^{\frac{2p+q}{q}}.
\end{aligned}
$$

Hence by (1), $F_{p,q}$ is PMD for $p+q > 0$ and $2p+q < 0$.

(6) The case of $p+q < 0$ and $q > 0$. By Lemma 2, we have $F_{p,q}(x^{-1})^{-1} = F_{-p,-q}(x)$. Since $-p-q > 0$ and $-q < 0$, we have that $F_{-p,-q}$ is PMI by (3), and therefore $F_{p,q}$ is PMD.

(7) The case of $p+q < 0$ and $2p+q > 0$. By Lemma 2, we have $F_{p,q}(x^{-1})^{-1} = F_{-p,-q}(x)$. Since $-p-q > 0$ and $-2p-q < 0$, we have that $F_{-p,-q}$ is PMD by (5). Therefore $F_{p,q}$ is PMI.

(8) The case of $2p+q < 0$ and $p > 0$. By Lemma 2, we have $F_{p,q}(x^{-1})^{-1} = F_{-p,-q}(x)$. Since $-2p-q > 0$ and $-p < 0$, $F_{-p,-q}$ is PMI by (4), therefore $F_{p,q}$ is PMD.

Thus we have the conclusion by combining 8 cases. □

From Theorem 1, we have the following power monotonicity of well-known functions.

(1) The representing function $s_\alpha(x)$ of the *Stolarsky mean* is defined by

$$
s_\alpha(x) = \left(\frac{x^\alpha - 1}{\alpha(x-1)}\right)^{\frac{1}{\alpha-1}}.
$$

Wada [7, Proposition 3.2] showed that s_α is PMD for $\alpha \in [-2,-1]$ and PMI for $\alpha \in [-1,2]$. It is obtained by Theorem 1 since $F_{1,\alpha-1}(x) = s_\alpha(x)$. More precisely, $s_\alpha(x)$ is PMI for $2+\alpha-1 \geq 0$ (i.e., $\alpha \geq -1$) and PMD for $2+\alpha-1 \leq 0$ (i.e., $\alpha \leq -1$). It is a generalization of Wada's result. Here we remark that s_α is operator monotone for $\alpha \in [-2,2]$. Namely Wada considered only the case $\alpha \in [-2,2]$. But Theorem 1 says that we can consider power monotonicity independent to the operator monotonicity.

(2) The representing function of the power difference mean

$$F_{p,1}(x) = \frac{p}{p+1}\frac{x^{p+1}-1}{x^p-1}$$

is PMI for $2p+1 \geq 0$ (i.e., $p \geq -\frac{1}{2}$), and PMD for $2p+1 \leq 0$ (i.e., $p \leq -\frac{1}{2}$). In other words, the representing function of the *normalized power difference mean*

$$F_{\frac{3p-1}{2},1}(x) = \frac{3p-1}{3p+1}\frac{x^{\frac{3p+1}{2}}-1}{x^{\frac{3p-1}{2}}-1}$$

is PMI for $2\frac{3p-1}{2}+1 \geq 0$ (i.e., $p \geq 0$), and PMD for $2\frac{3p-1}{2}+1 \leq 0$ (i.e., $p \leq 0$).

(3) The representing function of the *identric mean*

$$F_{0,1}(x) = \lim_{\alpha \to 1} s_\alpha(x) = \frac{1}{e}x^{\frac{x}{x-1}}$$

is PMI since $2 \cdot 0 + 1 \geq 0$.

(4) The representing function of the *logarithmic mean*

$$F_{1,0}(x) = \frac{x-1}{\log x}$$

is PMI since $2 \cdot 1 + 0 \geq 0$.

(5) The representing function of the *power mean*

$$F_{p,p}(x) = \left(\frac{1+x^p}{2}\right)^{\frac{1}{p}}$$

is PMI for $p \geq 0$, and PMD for $p \leq 0$.

Remark. Suppose $0 < p < r \leq q$ for a fixed q. Then the Jensen inequality shows

$$(1-\lambda+\lambda x^p)^{\frac{1}{p}} < (1-\lambda+\lambda x^r)^{\frac{1}{r}} \leq (1-\lambda+\lambda x^q)^{\frac{1}{q}}.$$

Thus we have the following monotonicity for PMI functions:

$$F_{p,q}(x) \nearrow F_{q,q}(x) = \left(\frac{1+x^q}{2}\right)^{\frac{1}{q}} \qquad \text{as} \qquad 0 < p \nearrow q.$$

Contrastively we have the monotonicity for PMD functions:

$$F_{p,q}(x) \searrow F_{q,q}(x) = \left(\frac{1+x^q}{2}\right)^{\frac{1}{q}} \qquad \text{as} \qquad 0 > p \searrow q.$$

3 Difference between $F_{p,q}(x^n)$ and $F_{p,q}(x)^n$. Restricting ourselves to the case $r = n$, integers. Then, based on the above remark, we show the following partial result of Theorem 1 via the power means:

Theorem 3. *Let n be a positive integer. For $0 \le p \le q$,*

$$F_{p,q}(x^n) - F_{p,q}(x)^n \ge F_{p,q}(x)\left(F_{q,q}(x)^{n-1} - F_{p,q}(x)^{n-1}\right) \ge 0$$

holds for all $x > 0$. For $q \le p \le 0$,

$$F_{p,q}(x)^n - F_{p,q}(x^n) \ge F_{p,q}(x)\left(F_{p,q}(x)^{n-1} - F_{q,q}(x)^{n-1}\right) \ge 0$$

holds for all $x > 0$.

To see this, we give a lemma:

Lemma 4. *For a fixed $q \in \mathbb{R}$ and an positive integer n, a function $g_n(p) = \frac{\sum_{\ell=0}^{n-1} x^{\ell(p+q)}}{\sum_{k=0}^{n-1} x^{kp}} = \left(\frac{F_{p,q}(x^n)}{F_{p,q}(x)}\right)^q$ is monotone increasing if $q \ge 0$, and monotone decreasing if $q \le 0$.*

Proof. At first we have

(1)

$$\begin{aligned} g'_{n+1}(p) &= \frac{\left(\sum_{k=0}^n x^{kp}\right)\left(\sum_{\ell=0}^n \ell \log x \cdot x^{\ell(p+q)}\right) - \left(\sum_{\ell=0}^n x^{\ell(p+q)}\right)\left(\sum_{k=0}^n k \log x \cdot x^{kp}\right)}{\left(\sum_{k=0}^n x^{kp}\right)^2} \\ &= \frac{x-1}{\left(\sum_{k=0}^n x^{kp}\right)^2}\frac{\log x}{x-1}\left\{\left(\sum_{k=0}^n x^{kp}\right)\left(\sum_{\ell=0}^n \ell \cdot x^{\ell(p+q)}\right) - \left(\sum_{\ell=0}^n x^{\ell(p+q)}\right)\left(\sum_{k=0}^n k \cdot x^{kp}\right)\right\} \\ &:= \frac{x-1}{\left(\sum_{k=0}^n x^{kp}\right)^2}\frac{\log x}{x-1} G_{n+1}(p), \end{aligned}$$

in which $G_{n+1}(p)$ becomes

$$\begin{aligned} G_{n+1}(p) &= \left(\sum_{k=0}^n x^{kp}\right)\left(\sum_{\ell=0}^n \ell \cdot x^{\ell(p+q)}\right) - \left(\sum_{\ell=0}^n x^{\ell(p+q)}\right)\left(\sum_{k=0}^n k \cdot x^{kp}\right) \\ &= \sum_{k,\ell=0}^n \ell\left(x^{\ell(p+q)+kp} - x^{k(p+q)+\ell p}\right) \\ &= \sum_{\ell>k}^n \ell\left(x^{\ell(p+q)+kp} - x^{k(p+q)+\ell p}\right) + \sum_{k>\ell}^n \ell\left(x^{\ell(p+q)+kp} - x^{k(p+q)+\ell p}\right) \\ &= \sum_{\ell>k}^n \ell\left(x^{\ell(p+q)+kp} - x^{k(p+q)+\ell p}\right) - \sum_{k>\ell}^n \ell\left(x^{k(p+q)+\ell p} - x^{\ell(p+q)+kp}\right). \end{aligned} \tag{2}$$

Here let $q \ge 0$ and $x > 1$. We note that

$$k(p+q) + \ell p - \{\ell(p+q) + kp\} = (k-\ell)q \ge 0$$

for $k > \ell$. Then (2) gives that

$$\begin{aligned} G_{n+1}(p) &= \sum_{\ell>k}^n \ell\left(x^{\ell(p+q)+kp} - x^{k(p+q)+\ell p}\right) - \sum_{k>\ell}^n \ell\left(x^{k(p+q)+\ell p} - x^{\ell(p+q)+kp}\right) \\ &\ge \sum_{\ell>k}^n \ell\left(x^{\ell(p+q)+kp} - x^{k(p+q)+\ell p}\right) - \sum_{k>\ell}^n k\left(x^{k(p+q)+\ell p} - x^{\ell(p+q)+kp}\right) = 0 \end{aligned}$$

for $q \geq 0$ and $x > 1$. If $q \geq 0$ and $0 < x < 1$. Then $G_{n+1}(p) \leq 0$ but $(x-1)G_{n+1}(p) \geq 0$. Since $\frac{\log x}{x-1} \geq 0$ for all $x > 0$, $g'_{n+1}(p) \geq 0$ and $g_{n+1}(p)$ is increasing for $q \geq 0$.

Next, let $q \leq 0$ and $x > 1$. Since

$$\begin{aligned} G_{n+1}(p) &= \sum_{\ell > k}^{n} \ell \left(x^{\ell(p+q)+kp} - x^{k(p+q)+\ell p} \right) - \sum_{k > \ell}^{n} \ell \left(x^{k(p+q)+\ell p} - x^{\ell(p+q)+kp} \right) \\ &= -\sum_{\ell > k}^{n} \ell \left(x^{k(p+q)+\ell p} - x^{\ell(p+q)+kp} \right) + \sum_{k > \ell}^{n} \ell \left(x^{\ell(p+q)+kp} - x^{k(p+q)+\ell p} \right) \end{aligned}$$

and

$$k(p+q) + \ell p - \{\ell(p+q) + kp\} = (k-\ell)q \geq 0$$

for $\ell > k$, we have

$$\begin{aligned} G_{n+1}(p) &= -\sum_{\ell > k}^{n} \ell \left(x^{k(p+q)+\ell p} - x^{\ell(p+q)+kp} \right) + \sum_{k > \ell}^{n} \ell \left(x^{\ell(p+q)+kp} - x^{k(p+q)+\ell p} \right) \\ &\leq -\sum_{\ell > k}^{n} k \left(x^{k(p+q)+\ell p} - x^{\ell(p+q)+kp} \right) + \sum_{k > \ell}^{n} \ell \left(x^{\ell(p+q)+kp} - x^{k(p+q)+\ell p} \right) = 0 \end{aligned}$$

for $q \leq 0$ and $x > 1$. By the same argument as above, we have $g'_{n+1}(p) \leq 0$ for $q \leq 0$ and $0 < x < 1$. Hence $g_{n+1}(p)$ is increasing on $p \in \mathbb{R}$ if $q > 0$, and decreasing on $p \in \mathbb{R}$ if $q < 0$.

Another formula is:

$$\left(\frac{F_{p,q}(x^n)}{F_{p,q}(x)} \right)^q = \frac{(x^{n(p+q)} - 1)/(x^{p+q} - 1)}{(x^{np} - 1)/(x^p - 1)} = \frac{\sum_{\ell=0}^{n-1} x^{\ell(p+q)}}{\sum_{k=0}^{n-1} x^{kp}} = g_n(p). \qquad \square$$

Proof of Theorem 3. Since $F_{0,0}(x) = \sqrt{x}$, Theorem 3 holds for $p = q = 0$. Then we can omit the case $q = 0$. First we see the case $0 \leq p \leq q$. Noting that

$$F_{p,q}(x^n) - F_{p,q}(x)^n = F_{p,q}(x) \left(g_n(p)^{\frac{1}{q}} - F_{p,q}(x)^{n-1} \right).$$

By $g_1(p) = 1$, we have only to show the case $n \geq 2$. Monotonicity of $F_{p,q}$ on $p, q \in \mathbb{R}$ and Lemma 4 show

$$\begin{aligned} g_n(p) \geq g_n(0) &= \frac{1 + x^q + \cdots + x^{q(n-1)}}{n} = \frac{x^{qn} - 1}{n(x^q - 1)} = F_{1,n-1}(x^q)^{n-1} \\ &\geq F_{1,1}(x^q)^{n-1} = \left(\frac{1 + x^q}{2} \right)^{n-1} = F_{q,q}(x)^{q(n-1)}. \end{aligned}$$

Thus we have

$$\begin{aligned} F_{p,q}(x^n) - F_{p,q}(x)^n &= F_{p,q}(x) \left(g_n(p)^{\frac{1}{q}} - F_{p,q}(x)^{n-1} \right) \\ &\geq F_{p,q}(x) \left(F_{q,q}(x)^{n-1} - F_{p,q}(x)^{n-1} \right) \geq 0 \end{aligned}$$

for $q \geq p \geq 0$ by monotonicity of $F_{p,q}$ on $p, q \in \mathbb{R}$. Next we show the case $q \leq p \leq 0$. In this case,

$$\begin{aligned} F_{p,q}(x)^n - F_{p,q}(x^n) &= F_{p,q}(x)\left(F_{p,q}(x)^{n-1} - g_n(p)^{\frac{1}{q}}\right) \\ &\geq F_{p,q}(x)\left(F_{p,q}(x)^{n-1} - g_n(0)^{\frac{1}{q}}\right) \quad \text{(by } g_n \text{ is decreasing)} \\ &= F_{p,q}(x)\left(F_{p,q}(x)^{n-1} - F_{1,n-1}(x^q)^{\frac{n-1}{q}}\right) \\ &\geq F_{p,q}(x)\left(F_{p,q}(x)^{n-1} - F_{1,1}(x^q)^{\frac{n-1}{q}}\right) \quad \text{(by } F_{p,q} \text{ is increasing on } p,q \in \mathbb{R}) \\ &= F_{p,q}(x)\left(F_{p,q}(x)^{n-1} - F_{q,q}(x)^{n-1}\right) \geq 0 \end{aligned}$$

for $q \leq p \leq 0$. It completes the proof. □

4 Conditions to power monotonicity. In this section, we shall discuss some conditions of a function f to satisfy power monotonicity. In the previous sections, we discussed power monotonicity of $F_{p,q}$, and obtain that if $F_{p,q}(x) \geq \sqrt{x}$ (resp. $F_{p,q}(x) \leq \sqrt{x}$), then it is PMI (resp. PMD). In other word $F_{p,q}(x)$ is PMI (resp. PMD) if and only if $F_{p,q}(x) \geq \sqrt{x}$ (resp. $F_{p,q}(x) \leq \sqrt{x}$). One might expect that power monotonicity of a function is closely related to comparison of $\sqrt{x}$.

First of all, we shall show the following proposition.

Proposition 5. *Let f be a differential function on $(0,\infty)$, such that, $f(1) = 1$, $f'(1) = \lambda \in [0,1]$. If f is PMI (resp. PMD) on $(0,\infty)$, then $x^\lambda \leq f(x)$ (resp. $x^\lambda \geq f(x)$) holds for all $x \in (0,\infty)$.*

Proof. Suppose that f is PMI on $(0,\infty)$. Then $G(r) = f(x^r)^{\frac{1}{r}}$ is an increasing function on $r > 0$, and

$$\begin{aligned} \log f(x) = \log G(1) &\geq \lim_{r\to+0} \log G(r) \\ &= \lim_{r\to+0} \frac{\log f(x^r)}{r} \\ &= \lim_{r\to+0} \frac{f'(x^r)x^r \log x}{f(x^r)} \quad \text{(by the L'Hospital's rule)} \\ &= \log x^\lambda. \end{aligned}$$

Hence $x^\lambda \leq f(x)$ holds for all $x \in (0,\infty)$. If f is PMD, then we can show $f(x) \leq x^\lambda$ by the same way. □

However, it has not known whether the converse implication holds or not, yet. Instead of this discussion, we can get a small contribution.

Proposition 6. *Let f be a differential function on $(0,\infty)$, such that, $f(1) = 1$, $f'(1) = \lambda \in [0,1]$. If $x^\lambda \leq f(x)$ (resp. $x^\lambda \geq f(x)$) holds for all $x \in (0,\infty)$, then f is PMI (resp. PMD) on a neighborhood of $x = 1$.*

Proof. Let $H(t) := \log f(e^t)$. Then $t^\lambda \leq f(t)$ is equivalent to

$$\lambda t \leq H(t). \tag{3}$$

Since $H(0) = 0$ and $H'(0) = \lambda$, $y = \lambda t$ is a tangent line of $H(t)$ at $t = 0$. Hence by (3), $H(t)$ is convex on a neighborhood of $t = 0$. Hence for $r \geq 1$,

$$H\left(\frac{1}{r}t\right) = H\left(\left(1 - \frac{1}{r}\right)0 + \frac{1}{r}t\right) \leq \left(1 - \frac{1}{r}\right)H(0) + \frac{1}{r}H(t) = \frac{1}{r}H(t) \tag{4}$$

holds for all t in a neighborhood of $t = 0$. Put $x = \frac{1}{r}t$. We have

$$rH(x) \leq H(rx).$$

Since $H(x) := \log f(e^x)$, $f(x)$ is PMI on the neighborhood of $x = 1$. The remained part can be proven by a similar way. $\square$

By Propositions 5 and 6, the following two statements are equivalent: (i) f is PMI on a neighborhood of $x = 0$ and (ii) $x^\lambda \leq f(x)$ for all $x > 0$. In addition, f is PMI on $(0, \infty)$ if and only if $rH(x) \leq H(rx)$ holds for all $r \geq 1$ and $x > 0$. It is a weaker condition than the convexity of H which follows from (4). Moreover as in the proof of Proposition 6, (ii) is equivalent to the convexity of H at $x = 0$. Hence we can conclude the following theorem.

Theorem 7. *Let f be a differential function on $(0, \infty)$, such that, $f(1) = 1$, $f'(1) = \lambda \in [0, 1]$. Then the following statements hold:*

(i) *If $H(x) := \log f(e^x)$ is convex (resp. concave) on $\mathbb{R}$, then f is PMI (resp. PMD),*

(ii) *if f is PMI (resp. PMD), then $x^\lambda \leq f(x)$ (resp. $f(x) \leq x^\lambda$) holds for all $x \in (0, \infty)$.*

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