EXTENSIONS OF ANDO-HIAI INEQUALITY WITH NEGATIVE POWER

Dedicated to the 100th anniversary of the birth of
the late Professor Masahiro Nakamura

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Abstract. The Ando-Hiai inequality says that if $A\#_\alpha B \le 1$ for a fixed $\alpha \in [0,1]$ and positive invertible operators A, B on a Hilbert space, then $A^r\#_\alpha B^r \le 1$ for $r \ge 1$, where $\#_\alpha$ is the α-geometric mean defined by $A\#_\alpha B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\alpha A^{\frac{1}{2}}$. In this note, we generalize it as follows: If $A\natural_\alpha B \le 1$ for a fixed $\alpha \in [-1,0]$ and positive invertible operators A, B on a Hilbert space, then $A^r\#_\beta B^s \le 1$ for $r \in [0,1]$ and $s \in [\frac{-2\alpha r}{-\alpha}, 1]$, where $\beta = \frac{\alpha r}{\alpha r + (1-\alpha)s}$ and $A\natural_\alpha B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\alpha A^{\frac{1}{2}}$. As an application, we pose operator inequalities of type of Furuta inequality and grand Furuta inequality. For instance, if $A \ge B > 0$, then $A^{-r}\natural_{\frac{1+r}{p+r}} B^p \le A$ holds for $p \le -1$ and $r \in [-1,0]$, where $A\natural_\alpha B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\alpha A^{\frac{1}{2}}$.

1 Introduction

Throughout this note, an operator A means a bounded linear operator acting on a complex Hilbert space H. An operator A is positive, denoted by $A \ge 0$, if $(Ax, x) \ge 0$ for all $x \in H$. We denote $A > 0$ if A is positive and invertible. The α-geometric mean $\#_\alpha$ is defined by $A\#_\alpha B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\alpha A^{\frac{1}{2}}$ for $A > 0$ and $B \ge 0$.

A log-majorization theorem due to Ando-Hiai [1] is expressed as follows: For $\alpha \in [0,1]$ and positive definite matrices A and B,

$$(A\#_\alpha B)^r \succ_{(\log)} A^r\#_\alpha B^r \quad (r \ge 1).$$

The core in the proof is that $A\#_\alpha B \le 1$ implies $A^r\#_\alpha B^r \le 1$ for $r \ge 1$. It holds for positive operators A, B on a Hilbert space, and is called the Ando-Hiai inequality,

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simply (AH). Afterwards, it is generalized to two variable version: If $A\#_\alpha B \le 1$ for $\alpha \in [0,1]$ and positive operators A, B, then $A^r \#_\beta B^s \le 1$ for $r, s \ge 1$, where $\beta = \frac{\alpha r}{\alpha r + (1-\alpha)s}$. It is known that both one-sided versions are equivalent, and that they are alterantive expressions of the Furuta inequality, see [4, 5].

A binary operation $\natural_\alpha$ is defined by the same formula as the α-geometric mean for $\alpha \notin [0,1]$, that is,

$$A \natural_\alpha B = A^{\frac{1}{2}}(A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^\alpha A^{\frac{1}{2}} \quad \text{for } A, B > 0.$$

Very recently (AH) is extended by Seo [17] and [13] as follows: For $\alpha \in [-1,0]$, $A \natural_\alpha B \le 1$ for $A, B > 0$ implies $A^r \natural_\alpha B^r \le 1$ for $r \in [0,1]$.

In this note, we present two variable version of it, presicely we show that if $A \natural_\alpha B \le 1$ for $\alpha \in [-1,0]$ and positive invertible operators A, B, then $A^r \natural_\beta B^s \le 1$ for $r \in [0,1]$ and $s \in [\frac{-2\alpha r}{1-\alpha}, 1]$, where $\beta = \frac{\alpha r}{\alpha r + (1-\alpha)s}$. As an application, we pose operator inequalities of type of Furuta inequality and grand Furuta inequality.

2 Extensions of (AH) with negative power

In the beginning of this section, we mention the following useful identity on the binary operation $\natural$: For $\beta \in \mathbb{R}$ and positive invertible operators X and Y,

$$X \natural_\beta Y = X(X^{-1} \natural_{-\beta} Y^{-1})X. \tag{2.1}$$

Lemma 2.1. *If $A \natural_\alpha B \le 1$ for $\alpha \in [-1,0]$ and positive invertible operators A and B, then $A^r \natural_\beta B \le 1$ for $r \in [0,1]$, where $\beta = \frac{\alpha r}{\alpha r + (1-\alpha)}$.*

Proof. For convenience, we show that if $A^{-1} \natural_\alpha B \le 1$, then $A^{-r} \natural_\beta B \le 1$ for $r \in [0,1]$. Thus the assumption ensures that $C^\alpha \le A$, where $C = A^{\frac{1}{2}} B A^{\frac{1}{2}}$. Note that $\beta \in [-1,0]$.

Now we first assume that $r = 1 - \epsilon \in [\frac{1}{2}, 1]$, i.e., $\epsilon \in [0, \frac{1}{2}]$. Then we have

$$\begin{aligned} A^\epsilon \natural_\beta C &= A^\epsilon (A^{-\epsilon} \#_{-\beta} C^{-1}) A^\epsilon \\ &\le A^\epsilon (C^{-\alpha\epsilon} \#_{-\beta} C^{-1}) A^\epsilon \\ &= A^\epsilon C^{\alpha(1-2\epsilon)} A^\epsilon \\ &\le A^\epsilon A^{1-2\epsilon} A^\epsilon = A. \end{aligned}$$

Hence it follows that

$$A^{-r}\natural_\beta B = A^{-\frac{1}{2}}(A^\epsilon\natural_\beta C)A^{-\frac{1}{2}} \le A^{-\frac{1}{2}}AA^{-\frac{1}{2}} = 1.$$

In particular, we note that $A^r\natural_\beta B \le 1$ for $r = \frac{1}{2}$, that is, $A^{-\frac{1}{2}}\natural_{\alpha_1} B \le 1$ holds for $\alpha_1 = \frac{\alpha}{2-\alpha}$. Hence it follows from the preceding paragragh that for $r \in [\frac{1}{2}, 1]$,

$$1 \ge (A^{-\frac{1}{2}})^r\natural_{\beta_1} B = A^{-\frac{r}{2}}\natural_{\beta_1} B,$$

where $\beta_1 = \frac{\alpha_1 r}{\alpha_1 r+(1-\alpha_1)} = \frac{\alpha r/2}{\alpha r/2+(1-\alpha)}$. This means tht the desired inequality holds for $r \in [\frac{1}{4}, \frac{1}{2}]$. Finally we have the conclusion by the induction.

Lemma 2.2. *If $A\natural_\alpha B \le 1$ for $\alpha \in [-1, 0]$ and positive invertible operators A and B, then $A\natural_\beta B^s \le 1$ for $s \in [\frac{-2\alpha}{1-\alpha}, 1]$, where $\beta = \frac{\alpha}{\alpha+(1-\alpha)s}$.*

Proof. For convenience, we show that if $A\natural_\alpha B^{-1} \le 1$, then $A\natural_\beta B^{-s} \le 1$ for $s \in [\frac{-2\alpha}{1-\alpha}, 1]$. Thus the assumption is understood as $D^{1-\alpha} \le B$, where $D = B^{\frac{1}{2}}AB^{\frac{1}{2}}$. We first note that $\beta \in [-1, 0]$ by $s \in [\frac{-2\alpha}{1-\alpha}, 1]$. So we put $s = 1 - \epsilon$ for some $\epsilon \in [0, 1 - \frac{-2\alpha}{1-\alpha}]$. Then we have

$$D\natural_\beta B^\epsilon = D(D^{-1}\#_{-\beta}B^{-\epsilon})D \le D(D^{-1}\#_{-\beta}D^{-\epsilon(1-\alpha)})D = D^{1-\alpha} \le B,$$

so that

$$A\natural_\beta B^{-s} = B^{-\frac{1}{2}}(D\natural_\beta B^\epsilon)B^{-\frac{1}{2}} \le B^{-\frac{1}{2}}BB^{-\frac{1}{2}} = 1.$$

Theorem 2.3. *If $A\natural_\alpha B \le 1$ for $\alpha \in [-1, 0]$ and positive invertible operators A and B, then $A^r\natural_\beta B^s \le 1$ for $r \in [0, 1]$ and $s \in [\frac{-2\alpha r}{1-\alpha}, 1]$, where $\beta = \frac{\alpha r}{\alpha r+(1-\alpha)s}$.*

Proof. Suppose that $A\natural_\alpha B \le 1$. Then Lemma 2.1 says that $A^r\natural_\gamma B \le 1$ for $r \in [0, 1]$, where $\gamma = \frac{\alpha r}{\alpha r+(1-\alpha)}$. Next we apply Lemma 2.2 to this obtained inequality. Then we have

$$1 \ge A^r\natural_{\frac{\gamma}{\gamma+(1-\gamma)s}}B^s = A^r\natural_{\frac{\alpha r}{\alpha r+(1-\alpha)s}}B^s$$

for $s \in [\frac{-2\gamma}{1-\gamma}, 1] = [\frac{-2\alpha r}{1-\alpha}, 1]$.

As a special case $s = r$ in the above, we obtain Seo's original extension of (AH) because $\beta = \alpha$ (by $s = r$) and $r \in [\frac{-2\alpha r}{1-\alpha}, 1]$.

Corollary 2.4. *If $A\natural_\alpha B \le 1$ for $\alpha \in [-1, 0]$ and positive invertible operators A and B, then $A^r\natural_\beta B^r \le 1$ for $r \in [0, 1]$.*

Remark 2.5. We here consider the condition $s \in [\frac{-2\alpha}{1-\alpha}, 1]$ in Lemma 2.2. In particular, take $\alpha = -1$. Then the assumption $A\natural_\alpha B \leq 1$ means that $B \geq A^2$, and $\beta = \frac{\alpha}{\alpha+(1-\alpha)s} = \frac{1}{1-2s}$. Though $s = 1$ in this case by $s \in [\frac{-2\alpha}{1-\alpha}, 1]$, the inequality in Lemma 2.2 still holds for $s \in [\frac{3}{4}, 1]$. We use the formula $X\natural_\gamma Y = Y\natural_{1-\gamma}X = Y(Y^{-1}\natural_{\gamma-1}X^{-1})Y$. Note that $-\beta \in [1, 2]$. Therefore we have

$$\begin{aligned} A\natural_\beta B^s &= A(A^{-1}\natural_\beta B^{-s})A = AB^{-s}(B^s \#_{\beta-1} A)B^{-s}A \\ &\leq AB^{-s}(B^s \#_{-\beta-1} B^{\frac{1}{2}})B^{-s}A = AB^{-1}A \leq AA^{-2}A = 1. \end{aligned}$$

On the other hand, it is false for $s \in [0, \frac{1}{4}]$. Note that $\beta = \frac{1}{1-2s} \in [1, 2]$. Suppose to the contrary that $A\natural_\beta B^s \leq 1$ holds under the assumption $B \geq A^2$. Then it follows that $1 \leq A\natural_\beta B^s = B^s(B^{-s}\#_{\beta-1}A^{-1})B^s$ and so

$$B^{-2s} \geq B^{-s}\#_{\beta-1}A^{-1} \geq B^{-s}\#_{\beta-1}B^{-\frac{1}{2}} = B^{-2s},$$

so that $B = A^2$ follows, which is imposible in general.

3 Operator inequalities of Furuta type In this section, we discuss representations of Furuta type associtated with extensions of Ando-Hiai inequality obtained in the preceding section. For convenience for readers, we cite the Furuta inequality which is a remarkable and amazing extension of Löwner-Heinz inequality (LH) in [?], [?] and [?], i.e., if $A \geq B \geq 0$, then $A^\alpha \geq B^\alpha$ for $\alpha \in [0, 1]$.

Furuta Inequality (FI)

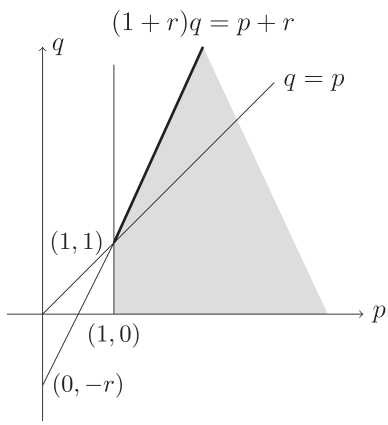
If $A \geq B \geq 0$, then for each $r \geq 0$,

(i) $(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{1}{q}} \geq (B^{\frac{r}{2}}B^pB^{\frac{r}{2}})^{\frac{1}{q}}$

and

(ii) $(A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{1}{q}} \geq (A^{\frac{r}{2}}A^pA^{\frac{r}{2}})^{\frac{1}{q}}$

hold for $p \geq 0$ and $q \geq 1$ with $(1+r)q \geq p + r$.



Related to Furuta inequality, see [2], [3], [6], [8], [9] and [18].

Especially the optimal case $(1+r)q = p+r$ is the most important, which is realized as a beautiful formula by the use of the α-geometric mean:

If $A \geq B \geq 0$, then for each $r \geq 0$

$$A^{-r} \, \#_{\frac{1+r}{p+r}} \, B^p \leq A$$

holds for $p \geq 1$.

More precisely, the conclusion in above is improved by

$$A^{-r} \, \#_{\frac{1+r}{p+r}} \, B^p \leq B \ (\leq A)$$

holds for $p \geq 1$, due to Kamei [12].
The following inequality is led by Lemma 2.1.

Theorem 3.1. *If $A \geq B > 0$, then*

$$A^{-r} \natural_{\frac{1+r}{p+r}} B^p \leq A$$

holds for $p \leq -1$ and $r \in [-1, 0]$.

Proof. As in the proof of Lemma 2.1, it says that if $A^{-1} \natural_\alpha B \leq 1$, then $A^{-r} \natural_\beta B \leq 1$ for $r \in [0, 1]$, where $\beta = \frac{\alpha r}{\alpha r + (1-\alpha)}$. Thus the assumption is that $C^\alpha \leq A$, where $C = A^{\frac{1}{2}} B A^{\frac{1}{2}}$. So we put $B_1 = C^\alpha \leq A$, and moreover $p = \frac{1}{\alpha}$, $r_1 = r - 1$. Then $p \leq -1$ and $r_1 \in [-1, 0]$ and $\beta = \frac{1+r_1}{p+r_1}$. Moreover the conclusion is rephrased as

$$A^{-r+1} \natural_\beta C \leq A, \text{ or } A^{-r_1} \natural_{\frac{1+r_1}{p+r_1}} {B_1}^p \leq A.$$

Now the Furuta inequality was generalized to so-called "grand Furuta inequality" by the appearence of Ando-Hiai inequality, which is due to Furuta [10], see also [5] and [6].

Grand Furuta inequality (GFI) *If $A \geq B > 0$ and $t \in [0, 1]$, then*

$$[A^{\frac{r}{2}} (A^{-\frac{t}{2}} B^p A^{-\frac{t}{2}})^s A^{\frac{r}{2}}]^{\frac{1-t+r}{(p-t)s+r}} \leq A^{1-t+r}$$

holds for $r \geq t$ and $p, s \geq 1$.

As a matter of fact, (GFI) interpolates (FI) with (AH), presicely

$$\text{(GFI) for } t = 1,\ r = s \iff \text{(AH)}$$

$$\text{(GFI) for } t = 0,\ (s = 1) \iff \text{(FI)}.$$

As well as (FI), (GFI) has also mean theoretic expression as follows:
If $A \geq B > 0$ and $t \in [0,1]$, then

$$A^{-r+t} \ \#_{\frac{1-t+r}{(p-t)s+r}} \ (A^t \ \#_s \ B^p) \leq A$$

holds for $r \geq t$ and $p, s \geq 1$.

In succession with the above discussion, Theorem 2.3 gives us the following inequality of (GFI)-type.

Theorem 3.2. *If $A \geq B > 0$, then*

$$A^{-r+1} \natural_{\frac{r}{r+(p-1)s}} (A \#_s B^p) \leq A$$

holds for $p \leq -1$, $r \in [0,1]$ and $s \in [\frac{-2r}{p-1}, 1]$.

Proof. Theorem 2.3 says that if $A^{-1} \natural_\alpha B \leq 1$, then $A^{-r} \natural_\beta B^s \leq 1$ for $r \in [0,1]$ and $s \in [\frac{-2\alpha r}{1-\alpha}, 1]$, where $\beta = \frac{\alpha r}{\alpha r + (1-\alpha)s}$. So the assumption is that $B_1 = C^\alpha \leq A$, where $C = A^{\frac{1}{2}} B A^{\frac{1}{2}}$. On the other hand, the conclusion is that, putting $\alpha = \frac{1}{p}$,

$$1 \geq A^{-r} \ \natural_{\frac{\alpha r}{\alpha r + (1-\alpha)s}} \ B^s = A^{-r} \ \natural_{\frac{r}{r+(p-1)s}} (A^{-\frac{1}{2}} {B_1}^p A^{-\frac{1}{2}})^s$$

or equivalently

$$A \geq A^{-r+1} \ \natural_{\frac{r}{r+(p-1)s}} (A \#_s {B_1}^p).$$

Furthermore, from the viewpoint of (GFI), the following generalization is expected:

Conjecture 3.3. *If $A \geq B > 0$ and $t \in [0,1]$, then*

$$A^{-r+t} \ \natural_{\frac{1-t+r}{r+(p-t)s}} (A^t \#_s B^p) \leq A$$

holds for $p \leq -1$, $r \in [0,t]$ and $s \in [\frac{-2r}{p-t}, 1]$.

At present, we can prove it under a restriction:

Theorem 3.4. *If* $A \geq B > 0$ *and* $t \in [0,1]$, *then*

$$A^{-r+t} \natural_{\frac{1-t+r}{r+(p-t)s}} (A^t \#_s B^p) \leq A$$

holds for $p \leq -1$, $r \in [0,t]$ *and* $s \in [\max\{\frac{-t}{p-t}, \frac{-2r-(1-t)}{p-t}\}, 1]$.

Proof. First of all, we note that $-1 \leq \frac{1-t+r}{r+(p-t)s} \leq 0$. Hence we have

$$A^{r-t} \#_{\frac{-(1-t+r)}{r+(p-t)s}} (A^{-t} \#_s B^{-p}) \leq A^{r-t} \#_{\frac{-(1-t+r)}{r+(p-t)s}} B^{-(p-t)s-t} \leq A^{2(r-t)+1}.$$

The second inequality in above is shown as follows: The exponent $-(p-t)s-t$ of B is nonnegative by $\frac{-t}{p-t} \leq s$. Thus, if $-(p-t)s-t \leq 1$, the second inequality holds. On the other hand, if $-(p-t)s-t \geq 1$, then the Furuta inequality assures that

$$(A^{\frac{t-r}{2}} B^{(-p+t)s-t} A^{\frac{t-r}{2}})^{\frac{1-t+r}{(-p+t)s-r}} \leq A^{1-t+r},$$

or equivalently

$$A^{r-t} \#_{\frac{1-t+r}{(-p+t)s-r}} B^{(-p+t)s-t} \leq A^{2(r-t)+1}.$$

Hence, noting that $X \natural_{-q} Y = X(X^{-1} \natural_q Y^{-1})X$, it follows that

$$\begin{aligned} A^{-r+t} \natural_{\frac{1-t+r}{r+(p-t)s}} (A^t \#_s B^p) &= A^{-r+t}\{A^{r-t} \#_{\frac{-(1-t+r)}{r+(p-t)s}} (A^{-t} \#_s B^{-p})\} A^{-r+t} \\ &\leq A^{-r+t} A^{2(r-t)+1} A^{-r+t} = A. \end{aligned}$$

Remark. On $\gamma = \max\{\frac{-t}{p-t}, \frac{-2r-(1-t)}{p-t}\}$ in the statement, $\gamma = \frac{-2r-(1-t)}{p-t}$ is equivalent to the condition $t - r \leq \frac{1}{2}$, which appears in Theorem 3.4.

The following two theorems show that Theorem 3.4 is true at the critical points $s = \frac{-t}{p-t}, \frac{-2r-(1-t)}{p-t}$.

Theorem 3.5. *If* $A \geq B > 0$ *and* $t \in [0,1]$, *then*

$$A^{-r+t} \natural_{\frac{1-t+r}{r+(p-t)s}} (A^t \#_s B^p) \leq A$$

holds for $p \leq -1$, $r \in [0,t]$ *and* $s = \frac{-2r-(1-t)}{p-t}$.

Proof. First of all, we note that $\frac{1-t+r}{r+(p-t)s} = -1$ and $X \natural_{-1} Y = XY^{-1}X$. Therefore the conclusion is arranged as

$$A^{-r+t} \natural_{-1} (A^t \#_s B^p) \leq A,$$

$$A^{-r+t}(A^{-t}\#_s B^{-p})A^{-r+t} \le A$$

and so

$$A^{-t}\#_s B^{-p} \le A^{1+2r-2t}. \quad (*)$$

To prove this, we recall the Furuta inequality, i.e., if $A \ge B \ge 0$, then

$$(A^{\frac{t}{2}} B^P A^{\frac{t}{2}})^{\frac{1}{q}} \le A^{\frac{P+t}{q}}$$

holds for $t, P \ge 0$ and $q \ge 1$ with $(1+t)q \ge P+t$. Taking $P = -p$ and $q = \frac{1}{s}$, the required condition $(1+t)q \ge P+t$ is enjoyed and we obtain

$$(A^{\frac{t}{2}} B^{-p} A^{\frac{t}{2}})^s \le A^{1+2r-t},$$

which is equivalent to (*).

In succession to Theorem 3.5, the other case $s = \frac{-t}{p-t}$ can be proved:

Theorem 3.6. *If $A \ge B > 0$ and $t \in [0,1]$, then*

$$A^{-r+t} \natural_{\frac{1-t+r}{r+(p-t)s}} (A^t \#_s B^p) \le A$$

holds for $p \le -1$, $r \in [0,t]$ and $s = \frac{-t}{p-t}$.

Since we have only to consider the case $\frac{-t}{p-t} < \frac{-2r-(1-t)}{p-t}$ by the above theorems, that is, $0 \le t-r < \frac{1}{2}$ can be assumed as cited in Remark of Theorem 3.4, we have

$$\frac{1-t+r}{r+(p-t)s} = 1 - \frac{1}{t-r} < -1.$$

As a special case, we take $t = \frac{2}{3}$, $r = \frac{1}{3}$ and $p = -2$. Then $s = \frac{1}{4}$ and $\frac{1-t+r}{r+(p-t)s} = -2$. Hence the statement in this case is arranged as follows:
If $A \ge B > 0$, then

$$A^{\frac{1}{3}} \natural_{-2} (A^{\frac{2}{3}} \#_{\frac{1}{4}} B^{-2}) \le A$$

holds? It is proved by using Furuta inequality twice: First of all, since $A \ge B > 0$, (FI) ensures that

$$(A^{\frac{1}{3}} B^2 A^{\frac{1}{3}})^{\frac{5}{8}} \le A^{\frac{5}{3}}.$$

So we have

$$\begin{aligned} A^{\frac{1}{3}} \natural_{-2}(A^{\frac{2}{3}} \#_{\frac{1}{4}} B^{-2}) &= A^{\frac{1}{6}}(A^{-\frac{1}{6}}(A^{\frac{2}{3}} \#_{\frac{1}{4}} B^{-2})A^{-\frac{1}{6}})^{-2}A^{\frac{1}{6}} \\ &= A^{\frac{1}{6}}(A^{\frac{1}{6}}(A^{-\frac{2}{3}} \#_{\frac{1}{4}} B^{2})A^{\frac{1}{6}})^{2}A^{\frac{1}{6}} \\ &= A^{\frac{1}{6}}(A^{-\frac{1}{3}} \#_{\frac{1}{4}} A^{\frac{1}{6}}B^{2}A^{\frac{1}{6}})^{2}A^{\frac{1}{6}} \\ &= A^{\frac{1}{6}}(A^{-\frac{1}{6}}(A^{\frac{1}{3}}B^{2}A^{\frac{1}{3}})^{\frac{1}{4}}A^{-\frac{1}{6}})^{2}A^{\frac{1}{6}} \\ &= (A^{\frac{1}{3}}B^{2}A^{\frac{1}{3}})^{\frac{1}{4}}A^{-\frac{1}{3}}(A^{\frac{1}{3}}B^{2}A^{\frac{1}{3}})^{\frac{1}{4}} \\ &\le (A^{\frac{1}{3}}B^{2}A^{\frac{1}{3}})^{\frac{1}{2}-\frac{1}{8}} \\ &\le (A^{\frac{1}{3}}B^{2}A^{\frac{1}{3}})^{\frac{3}{8}} \\ &\le A, \end{aligned}$$

as desired.

To prove Theorem 3.6, we cite a lemma obtained by the Furuta inequality.

Lemma 3.7. *If $A \ge B > 0$, $t \ge 0$ and $p \le -1$, then*

$$(A^{\frac{t}{2}}B^{-p}A^{\frac{t}{2}})^{\frac{1+t}{-p+t}} \le A^{1+t};$$

in particular, $(A^{\frac{t}{2}}B^{-p}A^{\frac{t}{2}})^{s} \le A^{t}$ holds for $s = \frac{t}{-p+t}$.

To show Theorem 3.6, we reformulate it as follows:

Theorem 3.8. *If $A \ge B > 0$, $t \ge \frac{c-1}{c+1}$ for some $c \ge 2$, $1 \ge t > r \ge 0$ with $t - r = \frac{1}{c+1}$ and $p \le -1$, then*

$$A^{\frac{1}{c+1}} \natural_{-c} (A^{t} \#_{s} B^{p}) \le A$$

holds for $s = \frac{t}{-p+t}$.

Proof. Put $\alpha = t - r$. Then $\alpha = \frac{1}{c+1} < \frac{1}{2}$, $c = \frac{1-\alpha}{\alpha}$ and the assumption $t \ge \frac{c-1}{c+1}$ means $\alpha(c-1) \le t$, which plays a role when we use the Löwner-Heinz inequality in the below. We put $X = A^{\frac{t}{2}}B^{-p}A^{\frac{t}{2}}$ and $Y = A^{-\frac{r}{2}}X^{s}A^{-\frac{r}{2}}$. Then $A^{\frac{1}{c+1}} \natural_{-c} (A^{t}\#_{s}B^{p}) = A^{\frac{\alpha}{2}}Y^{c}A^{\frac{\alpha}{2}}$, and $X^{\frac{s}{t}} = X^{\frac{1}{-p+t}} \le A$, in particular, $X^{s} \le A^{t}$ and $X^{\frac{st'}{t}} \le A^{t'}$ for $0 \le t' \le 1 + t$ by Lemma 3.7.

(1) First we suppose that $2n \le c < 2n+1$ for some n, i.e., $c = 2n + \epsilon$ for some $\epsilon \in [0, 1)$. Since $\alpha(c-2) \le t - \alpha = r$ by $\alpha(c-1) \le t$, we have $\alpha\epsilon \le \alpha(2(n-1)+\epsilon) = \alpha(c-2) \le r$ and so

$$-1 \le \frac{\alpha\epsilon - r}{t} \le \frac{\alpha(2(n-k)+\epsilon) - r}{t} \le 0$$

for $k = 1, 2, \cdots, n$. Noting that $0 \le 2s + [\alpha(2(n-1)+\epsilon) - r]\frac{s}{t} \le \frac{1+t}{-p+t}$ by $\frac{c-1}{c+1} \le 1$, it follows that

$$
\begin{aligned}
Y^c &= Y^n Y^\epsilon Y^n = Y^n (A^{-\frac{r}{2}} X^s A^{-\frac{r}{2}})^\epsilon Y^n \\
&\le Y^n (A^{-\frac{r}{2}} A^t A^{-\frac{r}{2}})^\epsilon Y^n = Y^n A^{\alpha\epsilon} Y^n \quad \text{by } X^s \le A^t \text{ and (LH)} \\
&= Y^{n-1} A^{-\frac{r}{2}} X^s A^{\alpha\epsilon - r} X^s A^{-\frac{r}{2}} Y^{n-1} \\
&\le Y^{n-1} A^{-\frac{r}{2}} X^{2s+(\alpha\epsilon-r)\frac{s}{t}} A^{-\frac{r}{2}} Y^{n-1} \quad \text{by } X^s \le A^t,\ \frac{\alpha\epsilon - r}{t} \in [-1, 0] \\
&\le Y^{n-1} A^{2t+\alpha\epsilon-2r} Y^{n-1} \quad \text{by putting } t' = 2t + \alpha\epsilon - r \le 1 + t \\
&= Y^{n-1} A^{\alpha(2+\epsilon)} Y^{n-1} \\
&\le Y^{n-2} A^{\alpha(4+\epsilon)} Y^{n-2} \\
&\cdots \\
&\le Y A^{\alpha(2(n-1)+\epsilon)} Y \\
&\le A^{\alpha(2n+\epsilon)} \\
&= A^{\alpha c}.
\end{aligned}
$$

Hence we have

$$A^{\frac{1}{c+1}} \natural_{-c} (A^t \#_s B^p) = A^{\frac{\alpha}{2}} Y^c A^{\frac{\alpha}{2}} \le A^{\alpha c + \alpha} = A,$$

as desired.

(2) Next we suppose that $2n + 1 \le c < 2n + 2$ for some n, i.e., $c = 2n + 1 + \epsilon$ for some $\epsilon \in [0, 1)$. For this case, we prepare the inequality

$$Y^{1+\epsilon} \le A^{\alpha(1+\epsilon)}.$$

It is proved as follows:

$$
\begin{aligned}
Y^{1+\epsilon} &= (A^{-\frac{r}{2}} X^s A^{-\frac{r}{2}})^{1+\epsilon} \\
&= A^{-\frac{r}{2}} X^{\frac{s}{2}} (X^{\frac{s}{2}} A^{-r} X^{\frac{s}{2}})^\epsilon X^{\frac{s}{2}} A^{-\frac{r}{2}} \\
&\le A^{-\frac{r}{2}} X^{\frac{s}{2}} (X^{\frac{s}{2}} X^{-\frac{sr}{t}} X^{\frac{s}{2}})^\epsilon X^{\frac{s}{2}} A^{-\frac{r}{2}} \\
&= A^{-\frac{r}{2}} X^{s+(s-\frac{sr}{t})\epsilon} A^{-\frac{r}{2}} \\
&\le A^{-\frac{r}{2}} A^{t+\alpha\epsilon} A^{-\frac{r}{2}} = A^{\alpha(1+\epsilon)}.
\end{aligned}
$$

Now, if $n = 0$, i.e., $c = 1 + \epsilon$, then

$$A^{\frac{\alpha}{2}} Y^{1+\epsilon} A^{\frac{\alpha}{2}} \le A^{\frac{\alpha}{2}} A^{\alpha(1+\epsilon)} A^{\frac{\alpha}{2}} = A^{\alpha(2+\epsilon)} = A.$$

Next, if $c = 2n + 1 + \epsilon$ for some $\epsilon \in [0, 1)$ with $n \neq 0$, then

$$\begin{aligned} Y^c &= Y^n Y^{1+\epsilon} Y^n \leq Y^n A^{\alpha(1+\epsilon)} Y^n \\ &= Y^{n-1} A^{-\frac{r}{2}} X^s A^{\alpha(1+\epsilon)-r} X^s A^{-\frac{r}{2}} Y^{n-1} \\ &\leq Y^{n-1} A^{-\frac{r}{2}} X^{2s+(\alpha(1+\epsilon)-r)\frac{s}{t}} A^{-\frac{r}{2}} Y^{n-1} \\ &\leq Y^{n-1} A^{2t+\alpha(1+\epsilon)-2r} Y^{n-1} \\ &= Y^{n-1} A^{\alpha(3+\epsilon)} Y^{n-1} \\ &\leq Y^{n-2} A^{\alpha(5+\epsilon)} Y^{n-2} \\ &\cdots \\ &\leq Y A^{\alpha(2(n-1)+1+\epsilon)} Y \\ &\leq A^{\alpha(2n+1+\epsilon)} = A^{\alpha c}, \end{aligned}$$

in which $(-1 \leq -r \leq)\ \alpha(2(n-1) + 1 + \epsilon) - r \leq 0$ is required in order to use the Löwner-Heinz inequality. (Fortunately it is assured by the assumption $t \geq \frac{c-1}{c+1}$.) Hence we have

$$A^{\frac{1}{c+1}} \natural_{-c} (A^t \#_s B^p) = A^{\frac{\alpha}{2}} Y^c A^{\frac{\alpha}{2}} \leq A^{\alpha c + \alpha} = A,$$

as desired.

4 Log-majorization

In this section, we express operator inequalities obtained in Section 2 as log-majorization inequalities.

Theorem 4.1. *For $\alpha \in [-1, 0]$ and positive invertible operators A and B,*

$$(A \natural_\alpha B)^{\frac{rs}{\alpha r + (1-\alpha)s}} \succ_{(\log)} A^r \natural_\beta B^s$$

holds for $r, s \in [0, 1]$, where $\beta = \frac{\alpha r}{\alpha r + (1-\alpha)s}$.

Theorem 4.2. *For $\alpha \in [-1, 0]$ and positive invertible operators A and B,*

$$(A \natural_\alpha B)^{\frac{(1-t+r)s}{\alpha r + (1-\alpha t)s}} \succ_{(\log)} A^r \natural_\beta B^s$$

holds for $r, s \in [0, 1]$, where $\beta = \frac{\alpha(1-t+r)}{\alpha r + (1-\alpha t)s}$.

References

[1] T. Ando and F. Hiai, *Log majorization and complementary Golden-Thompson type inequalities*, Linear Algebra Appl., **197, 198** (1994), 113-131.

[2] M. Fujii, *Furuta's inequality and its mean theoretic approach*, J. Operator Theory, **23** (1990), 67–72.

[3] M. Fujii, *Furutas inequality and its related topics*, Ann. Funct. Anal., **1** (2010), 28–45.

[4] M. Fujii, M. Ito, E. Kamei and A. Matsumoto, *Operator inequalities related to Ando-Hiai inequality*, Sci. Math. Japon., **70** (2009), 229-232.

[5] M. Fujii and E. Kamei, *Mean theoretic approach to the grand Furuta inequality*, Proc. Amer. Math. Soc., **124** (1996), 2751-2756.

[6] M. Fujii, J. Mićić Hot, J. Pečarić and Y. Seo, *Recent Developments of Mond-Pečarić Method in Operator Inequalities*, Element, Zagreb, Monographs in Inequalities **4**, 2012.

[7] M. Fujii and E. Kamei, *Ando-Hiai inequality and Furuta inequality*, Linear Algebra Appl., **416** (2006), 541-545.

[8] T. Furuta, $A \geq B \geq 0$ *assures* $(B^r A^p B^r)^{1/q} \geq B^{(p+2r)/q}$ *for* $r \geq 0,\ p \geq 0,\ q \geq 1$ *with* $(1+2r)q \geq p+2r$, Proc. Amer. Math. Soc. 101 (1987), 85–88.

[9] T. Furuta, *Elementary proof of an order preserving inequality*, Proc. Japan Acad. 65 (1989), 126.

[10] T. Furuta, *Extension of the Furuta inequality and Ando-Hiai log-majorization*, Linear Algebra Appl., **219** (1995), 139-155.

[11] E. Heinz, *Beitrage zur Storungstheorie der Spectral-zegung*, Math. Ann., **123** (1951), 415–438.

[12] E. Kamei, *A satellite to Furuta's inequality*, Math. Japon. 33 (1988), 883–886.

[13] M. Kian and Y. Seo, *Norm inequalities related to the matrix geometric mean of negative power*, Sci. Math. Japon. (in Editione Electronica), e-2018, article 2018-7.

[14] F. Kubo and T. Ando, *Means of positive linear operators*, Math. Ann., **246** (1980), 205-224.

[15] K. Löwner, *Über monotone Matrix function*, Math. Z., **38** (1934), 177-216.

[16] G. K. Pedersen, *Some operator monotone functions*, Proc. Amer. Math. Soc., **36** (1972), 309-310.

[17] Y. Seo, *Matrix trace inequalities related to the Tsallis relative entropy of negative order*, J. Math. Anal. Appl., **472** (2019), 1499-1508.

[18] K. Tanahashi, *Best possibility of the Furuta inequality*, Proc. Amer. Math. Soc., **124** (1996), 141-146.

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