GEOMETRIC DESCRIPTION OF SCHREIER GRAPHS OF B-S GROUPS

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Abstract. Let $BS(1,n) = \langle A, B \mid AB = B^n A\rangle$ be the Baumslag-Solitar group, where $n \geq 2$. This group has the natural action on the real line. In this paper we explicitly construct Schreier coset graphs of the group for stabilizers of all points in the real line under the action. As its consequence, we classify the Schreier coset graphs up to isomorphism, and obtain a relevance to presentations for the stabilizers.

1. Introduction

Let m and n be non-zero integers. The group which has the presentation $\langle A, B \mid AB^m = B^n A\rangle$ is called the *Baumslag-Solitar group* and denoted by $BS(m,n)$. In 1962, G. Baumslag and D. Solitar [1] introduced these groups and showed that $BS(3,2)$ is a non-Hopfian group with one defining relation. It is the first example having such property. Since then these groups have served as a proving ground for many new ideas in combinatorial and geometric group theory (see [2, 3] for examples).

Schreier coset graphs are generalizations of the Cayley graph of a group G, which are constructed for each choice of a subgroup of G and a generating set of G. The detail is given in Section 2. In general, given a group G and its subgroup H, it is difficult to construct the Cayley graph of G or the Schreier coset graph of all left cosets of H in G. However once we have the appropriate Cayley or Schreier graphs, we can use them as discrete models and may learn, from combinatorial and geometric viewpoints, some properties of the original group or its subgroups. Recently, in [5, 6], D. Savchuk constructed Schreier graphs of Thompson's group F from a motivation to study the amenability of the group.

In this paper we focus on the solvable group $BS(1,n)$ for $n \geq 2$. It is known that $BS(1,n)$ is isomorphic to some subgroup G_n with the generator S_n of the affine group $\mathrm{Aff}(\mathbb{R})$ of the real line $\mathbb{R}$, thus it has the natural action on $\mathbb{R}$ (see Section 2 for details). For any $x \in \mathbb{R}$, we explicitly construct the Schreier coset graph $(BS(1,n)/\mathrm{Stab}_{BS(1,n)}(x), \{A, B\}^{\pm})$ for the stabilizer $\mathrm{Stab}_{BS(1,n)}(x)$ of x under the

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action. First, we show that for any $x \in \mathbb{R}$, the Schreier graphs $(\mathrm{Orb}_{G_n}(x), S_n, x)$ and $(BS(1,n)/\mathrm{Stab}_{BS(1,n)}(x), \{A, B\}^{\pm}, \mathrm{Stab}_{BS(1,n)}(x))$ is isomorphic as marked labelled directed graphs, where $\mathrm{Orb}_{G_n}(x)$ is the orbit of x under the natural action on $\mathbb{R}$ (see Proposition 1 below). Hence, in most of this paper we consider the Schreier graph $(\mathrm{Orb}_{G_n}(x), S_n)$. Let $\mathbb{Z}_n^{\omega}$ be the set of all infinite words over the finite group $\mathbb{Z}_n$. The following theorem allows us to understand the structure of the Schreier graphs.

THEOREM 1. *Let $n \geq 2$ and x be a real number represented by $w \in \mathbb{Z}_n^{\omega}$. Then, there exists a homomorphism $h = (f, \psi, \gamma) : (\mathrm{Orb}_{G_n}(x), S_n) \to \mathit{\Gamma}_w$ such that for every $v \in V_w$, the subgraph $h^{-1}(v) = (D_v, D_v \times \{b\}^{\pm}, S_n, \alpha|, \beta|, l|)$ is isomorphic to $\mathit{\Gamma}_{\mathbb{Z}}$, where $h^{-1}(v) = (f^{-1}(v), \psi^{-1}(v), S_n, \alpha|, \beta|, l|)$.*

See Definition 3 below for $\mathit{\Gamma}_w$ and $\mathit{\Gamma}_{\mathbb{Z}}$. As its consequence, we classify the Schreier graphs up to isomorphism.

THEOREM 2. *Let $m, n \geq 2$ with $m \neq n$.*

(1) *For any $x, y \in \mathbb{R}$, the Schreier graph $(\mathrm{Orb}_{G_m}(x), S_m)$ is not isomorphic to the Schreier graph $(\mathrm{Orb}_{G_n}(y), S_n)$ as labelled directed graphs.*
(2) *For any $\alpha_1, \alpha_2 \in \mathbb{R} \setminus \mathbb{Q}$, the Schreier graph $(\mathrm{Orb}_{G_n}(\alpha_1), S_n, \alpha_1)$ is S_n-isomorphic to the Schreier graph $(\mathrm{Orb}_{G_n}(\alpha_2), S_n, \alpha_2)$ as marked labelled directed graphs.*
(3) *For any $q \in \mathbb{Q}$ and any $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, the Schreier graph $(\mathrm{Orb}_{G_n}(q), S_n)$ is not isomorphic to the Schreier graph $(\mathrm{Orb}_{G_n}(\alpha), S_n)$ as labelled directed graphs.*
(4) *Let q_1, $q_2 \in \mathbb{Q}$. Then, the following statements are equivalent.*
    (*a*) *The Schreier graph $(\mathrm{Orb}_{G_n}(q_1), S_n)$ is isomorphic to the Schreier graph $(\mathrm{Orb}_{G_n}(q_2), S_n)$ as labelled directed graphs.*
    (*b*) $\mathrm{Orb}_{G_n}(q_1) = \mathrm{Orb}_{G_n}(q_2)$ *or* $\mathrm{Orb}_{G_n}(-q_1) = \mathrm{Orb}_{G_n}(q_2)$.

This result leads to a relevance to presentations for the stabilizers which turn out to be infinite index subgroups in $BS(1, n)$(Theorem 5). Thus we expect that this idea may give a way to investigate infinite index subgroups in a suitable group.

In Section 2, we set up notation and terminology concerning Schreier graphs and Baumslag-Solitar groups. In Section 3, we start to construct Schreier graphs and give a complete description of Schreier graphs of $BS(1, n)$ with respect to any real numbers. In Section 4, we classify them up to isomorphism. In Section 5, by using the Schreier graphs we determine the group structure of the stabilizers and obtain a relevance to presentations for the stabilizers of rational numbers.

2. Schreier graphs and Baumslag-Solitar groups

A *labelled directed graph* denoted by $(V, E, L, \alpha, \beta, l)$ consists of a nonempty set V of vertices, a set E of edges, a set L of labels and three mappings $\alpha : E \to V$,

$\beta : E \to V$, and $l : E \to L$. The vertices $\alpha(e)$ and $\beta(e)$ are called the *initial* and the *terminal vertices* of the edge e, respectively.

A *marked labelled directed graph* denoted by $(V, E, L, \alpha, \beta, l, v_0)$ is a labelled directed graph with a distinguished vertex v_0 called the *marked vertex*.

For $i \in \{1, 2\}$ let $\Gamma_i = (V_i, E_i, L_i, \alpha_i, \beta_i, l_i)$ be a labelled directed graph. Let $f : V_1 \to V_2$, $\psi : E_1 \to E_2 \sqcup V_2$ and $\gamma : L_1 \to L_2$ be maps satisfying the following statements:

(1) If $\psi(e) \in E_2$, then $\alpha_2(\psi(e)) = f(\alpha_1(e))$, $\beta_2(\psi(e)) = f(\beta_1(e))$, and $l_2(\psi(e)) = \gamma(l_1(e)) \in L_2$.
(2) If $\psi(e) \in V_2$, then $\psi(e) = f(\alpha_1(e)) = f(\beta_1(e))$.

The triple (f, ψ, γ) of maps is called the *homomorphism from Γ_1 to Γ_2*. Labelled directed graphs Γ_1 and Γ_2 are *isomorphic* if there exists a homomorphism $(f, \psi, \gamma) : \Gamma_1 \to \Gamma_2$, called *an isomorphism*, such that both f and γ are bijections and ψ is a injection with $\psi(E_1) = E_2$. In particular, if $L_1 = L_2 = L$ and $\gamma = 1_L$, Γ_1 is said to be *L-isomorphic* to Γ_2.

For $i \in \{1, 2\}$ let Γ_i be a marked labelled directed graph. Γ_1 is said to be *isomorphic* to Γ_2 if Γ_1 is isomorphic to Γ_2 as labelled directed graphs and the mapping between vertices preserves the marked vertices.

Let S be a generating set of a group G. The generating set S is *symmetric* if $S = S^{-1}$.

Let G be a group with a symmetric finite generating set S, M be a set and $\varphi : G \to \mathrm{Aut}(M)$ be a homomorphism, where $\mathrm{Aut}(M)$ is the set of all bijections of M onto itself. The *orbit* of an element m of M is the set $\mathrm{Orb}_G(m) = \{\varphi(g)(m) \mid g \in G\}$. The *stabilizer* of an element m of M is the subgroup $\mathrm{Stab}_G(m) = \{g \in G \mid \varphi(g)(m) = m\}$.

Definition 1. Let G be a group with a symmetric finite generating set S, M be a set and $\varphi : G \to \mathrm{Aut}(M)$ be a homomorphism. The *Schreier graph* denoted by (M, S, φ) is a labelled directed graph $(M, M \times S, S, \alpha, \beta, l)$ such that $\alpha(m, s) = m$, $l(m, s) = s$, and $\beta(m, s) = \varphi(s)(m)$. The *Schreier graph with a marked vertex* denoted by (M, S, φ, m_0) is a Schreier graph with a marked vertex $m_0 \in M$.

Let G be a group with a symmetric finite generating set S, H be a subgroup of G and G/H be the set of all left cosets of H in G. The *Schreier coset graph* denoted by $(G/H, S)$ is a Schreier graph $(G/H, S, \varphi_H)$ where $\varphi_H : G \to \mathrm{Aut}(G/H)$ is the usual left action on G/H.

Remark 1. For $i \in \{1, 2\}$ let G_i be a group with a symmetric finite generateing set S_i. The Schreier graph (M_1, S_1, φ_1) is isomorphic to (M_2, S_2, φ_2) as labelled directed graphs if and only if there exist bijections $f : M_1 \to M_2$ and $\gamma : S_1 \to S_2$ such that $\varphi_1(s) = f^{-1}\varphi_2(\gamma(s))f$ for all $s \in S_1$. In particular, if $S_1 = S_2 = S$, (M_1, S, φ_1) is S-isomorphic to (M_2, S, φ_2) as labelled directed graphs if and only if there exists a bijection $f : M_1 \to M_2$ such that $\varphi_1(s) = f^{-1}\varphi_2(s)f$ for all $s \in S$.

The next proposition will help us to describe Schreier graphs explicitly in the later sections.

PROPOSITION 1. *Let G be a group with a symmetric finite generating set S, M be a set, $x_0 \in M$, and $\varphi : G \to \mathrm{Aut}(M)$ be a homomorphism. Then the Schreier graph $(\mathrm{Orb}_G(x_0), S, \varphi, x_0)$ with the marked vertex x_0 is S-isomorphic to the Schreier coset graph $(G/H, S, H)$ with the marked vertex $H = \mathrm{Stab}_G(x_0)$ as marked labelled directed graphs.*

PROOF. Define $f : G/H \to \mathrm{Orb}_G(x_0)$ by $f(gH) = \varphi(g)(x_0)$. Since $g^{-1}g' \in H = \mathrm{Stab}_G(x_0)$ implies $\varphi(g)(x_0) = \varphi(g')(x_0)$, its map is well-defined. Clearly f is a bijection. Since $f(\varphi_H(s)(gH)) = f(sgH) = \varphi(sg)(x_0) = \varphi(s)\varphi(g)(x_0) = \varphi(s)(f(gH))$, we have $\varphi_H(s) = f^{-1}\varphi(s)f$ for all $s \in S$, which is the desired conclusion by Remark 1. $\square$

Let m and n be nonzero integers. The group with the presentation $\langle A, B \mid AB^m = B^nA\rangle$ is called the *Baumslag-Solitar group* and it is denoted by $BS(m,n)$. For any $n \geq 2$, $BS(1,n)$ has a geometric representation. That is, we define two affine maps a and b of the real line $\mathbb{R}$ by $a(x) = nx$ and $b(x) = x + 1$ respectively. Let $n \geq 2$, $S_n = \{a, b\}^{\pm}$ and $G_n = \langle S_n\rangle$ be the subgroup of the affine group $\mathrm{Aff}(\mathbb{R})$. Then there exists the isomorphism $h_n : BS(1,n) \to G_n$ with $h_n(A) = a$ and $h_n(B) = b$ (see [4, p.100]). Thus, $BS(1,n)$ has the natural left action $\varphi_n : BS(1,n) \to G_n \hookrightarrow \mathrm{Aff}(\mathbb{R}) \hookrightarrow \mathrm{Aut}(\mathbb{R})$. By [4, p.102], we note that

$$(*)_n \qquad G_n = \{g : \mathbb{R} \to \mathbb{R} \mid g(x) = n^i x + j/n^k,\ i, j, k \in \mathbb{Z}\}.$$

3. Schreier graphs of all real numbers

Let $x \in \mathbb{R}$ and $\phi_x : G_n \to \mathrm{Aut}(\mathrm{Orb}_{G_n}(x))$ be the usual left action. By the isomorphism h_n and Proposition 1, the Schreier graph $(\mathrm{Orb}_{G_n}(x), S_n, \phi_x, x)$ and the Schreier coset graph $(BS(1,n)/\mathrm{Stab}_{BS(1,n)}(x), \{A, B\}^{\pm}, \mathrm{Stab}_{BS(1,n)}(x))$ with the marked vertexes are isomorphic, so we will consider the Schreier graph $(\mathrm{Orb}_{G_n}(x), S_n, \phi_x)$ for each $x \in \mathbb{R}$. For simplicity of notation, we write g and $(\mathrm{Orb}_{G_n}(x), S_n)$ instead of $\phi_x(g)$ and the Schreier graph $(\mathrm{Orb}_{G_n}(x), S_n, \phi_x)$, respectively.

REMARK 2. For any $x \in \mathbb{R}$ and any $f \in \mathrm{Stab}_{G_n}(x)$ with $f \neq 1_{\mathbb{R}}$, $bfb^{-1} \notin \mathrm{Stab}_{G_n}(x)$. Thus $\mathrm{Stab}_{G_n}(x)$ is not a normal subgroup of G_n.

We notice that the Schreier graph $(\mathrm{Orb}_{G_n}(\alpha), S_n)$ for $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ is S_n-isomorphic to the Cayley graph of $BS(1,n)$ relative to the generators $\{A, B\}^{\pm}$ by the above since the stabilizer $\mathrm{Stab}_{BS(1,n)}(\alpha)$ is trivial. However in this section we construct the Schreier graphs $(\mathrm{Orb}_{G_n}(q), S_n)$ for rational numbers q and will compare those descriptions in the later section (see Theorem 4). Therefore we employ the Schreier graph $(\mathrm{Orb}_{G_n}(\alpha), S_n)$. We construct the Schreier graph $(\mathrm{Orb}_{G_n}(\alpha), S_n)$ by an arrangement of elements in the orbit $\mathrm{Orb}_{G_n}(\alpha)$. The construction of the Cayley graph of $BS(1,n) \cong G_n$ given in [4] depends on the fact that the word problem for $BS(1,n)$ is solvable.

Let $\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$ be the finite group with the additive group structure. The set of all finite words over $\mathbb{Z}_n$ and the set of all infinite words over $\mathbb{Z}_n$ are denoted by $\mathbb{Z}_n^*$ and $\mathbb{Z}_n^\omega$ respectively. Let $\widetilde{\mathbb{Z}_n} = \mathbb{Z}_n^* \setminus \{\varepsilon\}$, where ε denotes the *empty word*. For every word $w = w_1 w_2 \ldots w_k$ in $\mathbb{Z}_n^*$, the *length* of w, denoted by $|w|$, is the number k. Note that $|\varepsilon|$ is zero.

Definition 2. An element w of $\mathbb{Z}_n^\omega$ is called a *rational element* in $\mathbb{Z}_n^\omega$ if there exist $u \in \mathbb{Z}_n^*$ and $v \in \widetilde{\mathbb{Z}_n}$ such that

(1) $w = uv^\infty$,
(2) $v \neq t^k$ whenever $k \geq 2$ and $t \in \widetilde{\mathbb{Z}_n}$, and
(3) $u_{|u|} \neq v_{|v|}$ whenever $u \neq \varepsilon$.

Then, we say that the pair (u, v) of words *satisfies* (A). An element w of $\mathbb{Z}_n^\omega$ which is not rational is called an *irrational element* in $\mathbb{Z}_n^\omega$. Let $x \in \mathbb{R}$. Then, there exists $w \in \mathbb{Z}_n^\omega$ such that $x - \lfloor x \rfloor = \sum_{i \geq 1} w_i / n^i$, where $\lfloor x \rfloor = \max\{k \in \mathbb{Z} \mid k \leq x\}$. We say that x is *represented by* $w \in \mathbb{Z}_n^\omega$. It is easy to see that x is a rational number if and only if it is represented by a rational element in $\mathbb{Z}_n^\omega$.

Lemma 1. *Let $x, x' \in \mathbb{Z}_n^*$ and y be an irrational element of $\mathbb{Z}_n^\omega$ with $xy = x'y$. Then $x = x'$.*

Proof. Without loss of generality, we can assume that $|x| \leq |x'|$. By assumption, $y_{|x'|-|x|+j} = y_j$ for each $j \geq 1$. Since y is an irrational element in $\mathbb{Z}_n^\omega$, $|x'| = |x|$. Therefore, $x = x'$. $\square$

Lemma 2. *Suppose that pairs (x, y) and (x', y') of words satisfy (A). Then $xy^\infty = x'y'^\infty$ if and only if $x = x'$ and $y = y'$.*

Proof. Suppose that $xy^\infty = x'y'^\infty$. It suffices to show that $x = x'$ and $y = y'$. First we show that $|x| = |x'|$. On the contrary, suppose that $|x| < |x'|$. For any $k \geq 1$, there exists a unique $\underline{k} \in \{1, \ldots, |y|\}$ such that $k \equiv \underline{k} \mod |y|$. Then

$$x'_{|x'|} = (x'y'^\infty)_{|x'|} = (xy^\infty)_{|x'|} = (y^\infty)_{|x'|-|x|} = y_{\underline{|x'|-|x|}}.$$

On the other hand,

$$y'_{|y'|} = (x'y'^\infty)_{|x'|+|y'|(|y|/g)} = (xy^\infty)_{|x'|+|y|(|y'|/g)} = (y^\infty)_{|x'|-|x|+|y|(|y'|/g)} = y_{\underline{|x'|-|x|}},$$

where $g = \gcd(|y'|, |y|)$. Since $x' \neq \varepsilon$, by the assumption of x', we see $x'_{|x'|} \neq y'_{|y'|}$, a contradiction. Thus $|x| = |x'|$. Hence we have that $x = x'$ and $y^\infty = y'^\infty$.

Next we show that $|y| = |y'|$. On the contrary, suppose that $|y| < |y'|$. There exist $\alpha \in \mathbb{Z}$ and $\beta \geq 0$ such that $|y'|\alpha + |y|\beta = g$. For any $i \geq 1$

$$(y'^{\infty})_{i+g} = (y'^{\infty})_{i+|y'|\alpha+|y|\beta} = (y'^{\infty})_{i+|y|\beta} = (y^{\infty})_{i+|y|\beta} = (y^{\infty})_i = (y'^{\infty})_i.$$

Since y'^{∞} has the period g, y' has the period $g \leq |y| < |y'|$. This contradicts the assumption of y'. Since $|y| = |y'|$, we conclude $y = y'$. □

LEMMA 3. *Let $x, y \in \widetilde{\mathbb{Z}_n}$. Suppose that $x_{|x|} = y_{|y|}$ and the word y satisfies the condition* (2) *in Definition 2. Then $xy^{\infty} = y^{\infty}$ if and only if $|x| \equiv 0 \bmod |y|$ and $x = y^{|x|/|y|}$.*

PROOF. Suppose that $xy^{\infty} = y^{\infty}$. It suffices to show that $|x| \equiv 0 \bmod |y|$ and $x = y^{|x|/|y|}$. Let $m \geq 0$ and $1 \leq r \leq |y|$ such that $|x| = |y|m + r$. Then for any $i \geq 1$

$$\begin{aligned}(y^{\infty})_{i+r} = (xy^{\infty})_{|x|+i+r} = (xy^{\infty})_{|x|+i+r+|y|m} = (xy^{\infty})_{|x|+i+|x|} &= (y^{\infty})_{i+|x|} \\ &= (xy^{\infty})_{i+|x|} \\ &= (y^{\infty})_i.\end{aligned}$$

Thus y^{∞} has the period r and $(y_1 \dots y_{|y|})^{\infty} = y^{\infty} = (y_1 \dots y_r)^{\infty}$. Since (ε, y) and $(\varepsilon, y_1 \dots y_r)$ satisfy (A), by Lemma 2, we have $|y| = r$. Therefore $|x| \equiv 0 \mod |y|$. Moreover, since $(xy^{\infty})_i = (y^{\infty})_i$ for all $1 \leq i \leq |x|$, we have $x = y^{|x|/|y|}$. □

Let $\sigma : \mathbb{Z}_n^{\omega} \to \mathbb{Z}_n^{\omega}$ be the sift map defined by $\sigma(w_1w_2w_3\dots) = w_2w_3w_4\dots$ Write $\sigma^{k-1} = \underbrace{\sigma\sigma\cdots\sigma}_{k-1}$ for each $k \geq 1$, where σ^0 is the identity map. We note that $\sigma^{k-1}(w)_i = w_{k-1+i}$ for any $k, i \geq 1$ and each $w \in \mathbb{Z}_n^{\omega}$.

LEMMA 4. *Let (x, y) be a pair of words satisfying (A). Then for $|x| \leq j < j'$, $\sigma^j(xy^{\infty}) = \sigma^{j'}(xy^{\infty})$ if and only if $j' - j \equiv 0 \mod |y|$.*

PROOF. For any $k \geq 1$, there exists a unique $\underline{k} \in \{1, \dots, |y|\}$ such that $k \equiv \underline{k} \mod |y|$. Then

$$\sigma^j(xy^{\infty}) = \sigma^{j-|x|}(y^{\infty}) = (y_{\underline{j-|x|+1}} \cdots y_{\underline{j'-|x|}})\,\sigma^{j'-|x|}(y^{\infty}), \text{ and}$$

$$\sigma^{j'}(xy^{\infty}) = \sigma^{j'-|x|}(y^{\infty}).$$

Thus $\sigma^j(xy^{\infty}) = \sigma^{j'}(xy^{\infty})$ if and only if $(y_{\underline{j-|x|+1}} \cdots y_{\underline{j'-|x|}})\,\sigma^{j'-|x|}(y^{\infty}) = \sigma^{j'-|x|}(y^{\infty})$. By Lemma 3, $(y_{\underline{j-|x|+1}} \cdots y_{\underline{j'-|x|}})\,\sigma^{j'-|x|}(y^{\infty}) = \sigma^{j'-|x|}(y^{\infty})$ if and only if $j' - j \equiv 0 \mod |y|$. □

For any $v \in \mathbb{Z}_n^\omega$ and any $t \in \mathbb{Z}_n$, set $D_v = \mathbb{Z} + \sum_{i\geq 1} v_i/n^i \subset \mathbb{R}$, and $D_v^t = n\mathbb{Z} + t + \sum_{i\geq 1} v_i/n^i \subset \mathbb{R}$. Note that $0 \leq \sum_{i\geq 1} v_i/n^i \leq 1$ and $D_v = \bigsqcup_{t\in X} D_v^t$.

LEMMA 5. *Let y and y' be irrational elements in $\mathbb{Z}_n^\omega$. Then, the following statements are equivalent.*

(1) $D_y \cap D_{y'} \neq \emptyset$.
(2) $\sum_{i\geq 1} y_i/n^i = \sum_{i\geq 1} y'_i/n^i$.
(3) $y = y'$.

PROOF. It suffices to show that (2) implies (3). On the contrary, suppose that there exists $i \geq 1$ such that $y_i \neq y'_i$. Let $i_0 = \min\{i \mid y_i \neq y'_i\}$. Then,

$$y_{i_0}/n^{i_0} + \sum_{i\geq i_0+1} y_i/n^i = y'_{i_0}/n^{i_0} + \sum_{i\geq i_0+1} y'_i/n^i.$$

Without loss of generality, we can assume that $y_{i_0} < y'_{i_0}$. Since y and y' are irrational elements,

$$1/n^{i_0} < y'_{i_0}/n^{i_0} - y_{i_0}/n^{i_0} + \sum_{i\geq i_0+1} y'_i/n^i = \sum_{i\geq i_0+1} y_i/n^i < 1/n^{i_0},$$

a contradiction. □

LEMMA 6. *Let (x, y) and (x', y') be pairs of words satisfying (A) such that $\min\{|y|, |y'|\} \geq 2$ whenever $y \neq y'$. Then, the following statements are equivalent.*

(1) $D_{xy^\infty} \cap D_{x'y'^\infty} \neq \emptyset$.
(2) $\sum_{i\geq 1} (xy^\infty)_i/n^i = \sum_{i\geq 1} (x'y'^\infty)_i/n^i$.
(3) $xy^\infty = x'y'^\infty$.

PROOF. Suppose that $\sum_{i\geq 1} (xy^\infty)_i/n^i = \sum_{i\geq 1} (x'y'^\infty)_i/n^i$. It suffices to prove that $xy^\infty = x'y'^\infty$. On the contrary, suppose that there exists $i \geq 1$ such that $(xy^\infty)_i \neq (x'y'^\infty)_i$. Let $i_0 = \min\{i \mid (xy^\infty)_i \neq (x'y'^\infty)_i\}$. Then

$$(xy^\infty)_{i_0}/n^{i_0} + \sum_{i\geq i_0+1} (xy^\infty)_i/n^i = (x'y'^\infty)_{i_0}/n^{i_0} + \sum_{i\geq i_0+1} (x'y'^\infty)_i/n^i.$$

Without loss of generality, we can assume that $(xy^\infty)_{i_0} < (x'y'^\infty)_{i_0}$.

If $\min\{|y|, |y'|\} \geq 2$, or if $y = y' \in \{1, \ldots, n-2\}$, then we have

$$1/n^{i_0} < (x'y'^\infty)_{i_0}/n^{i_0} - (xy^\infty)_{i_0}/n^{i_0} + \sum_{i\geq i_0+1} (x'y'^\infty)_i/n^i = \sum_{i\geq i_0+1} (xy^\infty)_i/n^i < 1/n^{i_0},$$

a contradiction.

If $y = y' = 0$, then $i_0 \leq |x'|$. Then

$$1/n^{i_0} \leq (x'y'^{\infty})_{i_0}/n^{i_0} - (xy^{\infty})_{i_0}/n^{i_0} + \sum_{i \geq i_0+1} (x'y'^{\infty})_i/n^i = \sum_{i \geq i_0+1} (xy^{\infty})_i/n^i < 1/n^{i_0},$$

a contradiction.

If $y = y' = n-1$, then $i_0 \leq |x|$. Then

$$1/n^{i_0} < (x'y'^{\infty})_{i_0}/n^{i_0} - (xy^{\infty})_{i_0}/n^{i_0} + \sum_{i \geq i_0+1} (x'y'^{\infty})_i/n^i = \sum_{i \geq i_0+1} (xy^{\infty})_i/n^i \leq 1/n^{i_0},$$

a contradiction. Therefore $xy^{\infty} = x'y'^{\infty}$. □

The proof of the following lemma is immediate, so the details are left to the reader.

Lemma 7. *Let $v \in \mathbb{Z}_n^{\omega}$ and $t \in \mathbb{Z}_n$. Then,*

(*a*) $a(D_v) = D^{v_1}_{\sigma(v)}$, $a^{-1}(D^t_v) = D_{tv}$, $a^{-1}(D_v) = \bigsqcup_{t \in \mathbb{Z}_n} D_{tv}$,

(*b*) $b^{\pm 1}(D^t_v) = D^{t \pm 1}_v$, *and* $b^{\pm 1}(D_v) = D_v$.

Definition 3. Let $w \in \mathbb{Z}_n^{\omega}$. Set $V_w = \{u\sigma^j(w) \,|\, j \geq 0, u \in \mathbb{Z}_n^*\}$, $E_w = V_w \times (\{a\} \sqcup \mathbb{Z}_n)$, and $L_w = \{a\}^{\pm}$. Define $\alpha_w : E_w \to V_w$, $\beta_w : E_w \to V_w$ and $l_w : E_w \to L_w$ by $\alpha_w(v,a) = \alpha_w(v,k) = v$, $\beta_w(v,a) = \sigma(v)$, $\beta_w(v,k) = kv$, $l_w(v,a) = a$ and $l_w(v,k) = a^{-1}$ for each $v \in V_w$ and each $k \in \mathbb{Z}_n$. The labelled directed graph $(V_w, E_w, L_w, \alpha_w, \beta_w, l_w)$ and the Schreier graph $(\mathbb{Z}, \{\pm 1\}, \phi)$ will be denoted by Γ_w and $\Gamma_{\mathbb{Z}}$ respectively, where $\phi : \mathbb{Z} \to \mathrm{Aut}(\mathbb{Z})$ is the usual action.

Lemma 8. (1) *If w is an irrational element in $\mathbb{Z}_n^{\omega}$, then*

$$V_w = \bigsqcup_{j \geq 1} \{\sigma^j(w)\} \sqcup \bigsqcup_{u \in \mathbb{Z}_n^*} \{uw\} \sqcup \bigsqcup_{j \geq 1, s \in \mathbb{Z}_n^*, t \in \mathbb{Z}_n, t \neq w_j} \{st\sigma^j(w)\}.$$

(2) *If $w = uv^{\infty}$ is a rational element in $\mathbb{Z}_n^{\omega}$ as in Definition 2, then*

$$V_w = \bigsqcup_{|u| \leq j < |u|+|v|} \{\sigma^j(w)\} \sqcup \bigsqcup_{|u| < j \leq |u|+|v|, s \in \mathbb{Z}_n^*, t \in \mathbb{Z}_n, t \neq w_j} \{st\sigma^j(w)\}.$$

Proof. By Lemmas 2 and 4, we can easily show (2). Thus we prove (1). Let $j, j' \geq 1$, $u, u' \in \mathbb{Z}_n^*$, and $t, t' \in \mathbb{Z}_n$ with $t \neq w_j$ and $t' \neq w_{j'}$. It suffices to show the following statements:

(*a*) $j = j'$ whenever $\sigma^j(w) = \sigma^{j'}(w)$.
(*b*) $u = u'$ whenever $uw = u'w$.
(*c*) $u = u'$, $t = t'$, and $j = j'$ whenever $ut\sigma^j(w) = u't'\sigma^{j'}(w)$.
(*d*) $\sigma^j(w) \neq uw$.

(e) $\sigma^j(w) \neq ut'\sigma^{j'}(w)$.
(f) $uw \neq u't\sigma^j(w)$.

The statements (b) and (d) directly follow from Lemma 1.

Suppose that $ut\sigma^j(w) = u't'\sigma^{j'}(w)$ and $j \leq j'$. Since $\sigma^j(w) = w_{j+1} \dots w_{j'}\sigma^{j'}(w)$, by Lemma 1, we have $utw_{j+1} \dots w_{j'} = u't'$. Since $t' \neq w_{j'}$, we see $j = j'$, thus $u = u'$ and $t = t'$, which proves (c). Similarly, we can show (a).

If $j \geq j'$, by Lemma 1, $ut'\sigma^{j'}(w) = ut'w_{j'+1} \dots w_j\sigma^j(w) \neq \sigma^j(w)$. Suppose that $j \leq j'$ and $\sigma^j(w) = ut'\sigma^{j'}(w)$. Since $\sigma^j(w) = w_{j+1} \dots w_{j'}\sigma^{j'}(w)$, $w_{j+1} \dots w_{j'}\sigma^{j'}(w) = ut'\sigma^{j'}(w)$. Hence by Lemma 1 $w_{j+1} \dots w_{j'} = ut'$. Thus $w_{j'} = t'$, a contradiction, and (e) is proved.

Since $w_j \neq t$, $uw_1 \dots w_j \neq u't$. By Lemma 1, $uw = uw_1 \dots w_j\sigma^j(w) \neq u't\sigma^j(w)$, which proves (f). □

Lemma 9. *Let $n \geq 2$ and $x \in \mathbb{R}$ represented by $w \in \mathbb{Z}_n^\omega$. Then,* $\mathrm{Orb}_{G_n}(x) = \bigsqcup_{v \in V_w} D_v$.

Proof. By Lemmas 5,6 and 8, $\bigcup_{v \in V_w} D_v = \bigsqcup_{v \in V_w} D_v$. Thus it suffices to show that $\mathrm{Orb}_{G_n}(x) = \bigcup_{v \in V_w} D_v$. Since $x \in D_w \subset \bigcup_{v \in V_w} D_v$, by Lemma 7,

$$\mathrm{Orb}_{G_n}(x) \subset \bigcup_{g \in G_n} \bigcup_{v \in V_w} g(D_v) = \bigcup_{v \in V_w} D_v.$$

Let $j \geq 0$ and $u \in \mathbb{Z}_n^*$. It suffices to show that $D_{u\sigma^j(w)} \subset \mathrm{Orb}_{G_n}(x)$. We have

$$\begin{aligned}
D_{u\sigma^j(w)} &= \mathbb{Z} + \sum_{i \geq 1} (u\sigma^j(w))_i / n^i \\
&= \mathbb{Z} + \sum_{i=1}^{|u|} u_i/n^i + \sum_{l \geq j+1} w_l / n^{l-j+|u|} \\
&= \mathbb{Z} + \sum_{i=1}^{|u|} u_i/n^i + n^{j-|u|}\Big(\sum_{l \geq 1} w_l/n^l - \sum_{l=1}^{j} w_l/n^l\Big) \\
&= \mathbb{Z} + n^{-|u|}\Big(\sum_{i=1}^{|u|} n^{|u|-i}u_i - \sum_{i=1}^{j} n^{j-i}w_i + n^j(x - \lfloor x \rfloor)\Big) \\
&= \{b^k a^{-|u|} b^{(\sum_{i=1}^{|u|} n^{|u|-i}u_i - \sum_{i=1}^{j} n^{j-i}w_i)} a^j b^{-\lfloor x \rfloor}(x) \mid k \in \mathbb{Z}\} \subset \mathrm{Orb}_{G_n}(x).
\end{aligned}$$

□

Theorem 3. *Let $n \geq 2$ and x be a real number represented by $w \in \mathbb{Z}_n^\omega$. Then, there exists a homomorphism $h = (f, \psi, \gamma) : (\mathrm{Orb}_{G_n}(x), S_n) \to \Gamma_w$ such that for every*

$v \in V_w$, the subgraph $h^{-1}(v) = (D_v, D_v \times \{b\}^{\pm}, S_n, \alpha|, \beta|, l|)$ is isomorphic to $\Gamma_{\mathbb{Z}}$, where $h^{-1}(v) = (f^{-1}(v), \psi^{-1}(v), S_n, \alpha|, \beta|, l|)$.

Proof. It suffices to find a homomorphism $h = (f, \psi, \gamma) : (\mathrm{Orb}_{G_n}(x), S_n) \to \Gamma_w$ such that for every $v \in V_w$, the subgraph $h^{-1}(v)$ is isomorphic to $\Gamma_{\mathbb{Z}}$. By Lemmas 8 and 9, for any $y \in \mathrm{Orb}_{G_n}(x)$, there exists a unique $v_y \in V_w$ and $k \in \mathbb{Z}_n$ such that $y \in D^k_{v_y} \subset D_{v_y}$. Thus, we can define $f : \mathrm{Orb}_{G_n}(x) \to V_w$, $\psi : \mathrm{Orb}_{G_n}(x) \times S_n \to E_w \sqcup V_w$ and $\gamma : S_n \to L_w$ by $f(y) = v_y$, $\psi(y, a) = (f(y), a)$, $\psi(y, a^{-1}) = (f(y), k)$, $\psi(y, b) = f(y)$, $\psi(y, b^{-1}) = f(y)$, $\gamma(a) = a$, $\gamma(a^{-1}) = a^{-1}$, $\gamma(b) = a$, and $\gamma(b^{-1}) = a^{-1}$. $\square$

4. Classification of Schreier graphs

In this section we classify Schreier graphs described in the previous section.

Lemma 10. *Let $v \in \widetilde{\mathbb{Z}_n}$. For $i \geq 1$ set $W_i = b^{-(v^\infty)_i} a$ and $Z_i = b^{(v^\infty)_i} a$. Then, for every $k \geq 1$, $W_k \cdots W_1$ and $Z_k \cdots Z_1$ are nontrivial affine maps with the slopes n^k such that*

$$(W_k \cdots W_1)(\sum_{j\geq 1} (v^\infty)_j / n^j) = \sum_{j \geq 1} (v^\infty)_{k+j} / n^j \text{ and}$$

$$(Z_k \cdots Z_1)(-\sum_{j\geq 1} (v^\infty)_j / n^j) = -\sum_{j \geq 1} (v^\infty)_{k+j} / n^j.$$

Proof. The proof is by induction on k. The affine map W_1 has the slope n such that

$$\begin{aligned} W_1(\sum_{j\geq 1} (v^\infty)_j / n^j) = b^{-(v^\infty)_1}\, a\,(\sum_{j\geq 1} (v^\infty)_j / n^j) &= b^{-(v^\infty)_1}\,((v^\infty)_1 + \sum_{j\geq 2} (v^\infty)_j / n^{j-1}) \\ &= \sum_{j\geq 1} (v^\infty)_{1+j} / n^j. \end{aligned}$$

Assume the formula holds for $k-1$, we have

$$\begin{aligned} (W_k W_{k-1} \cdots W_1)\,(\sum_{j\geq 1} (v^\infty)_j / n^j) &= W_k(\sum_{j\geq 1} (v^\infty)_{k-1+j} / n^j) \\ &= b^{-(v^\infty)_k}\, a\,(\sum_{j\geq 1} (v^\infty)_{k-1+j} / n^j) \\ &= b^{-(v^\infty)_k}\,((v^\infty)_k + \sum_{j\geq 2} (v^\infty)_{k-1+j} / n^{j-1}) \\ &= \sum_{j\geq 1} (v^\infty)_{k+j} / n^j \end{aligned}$$

and the affine map $W_k \cdots W_1$ has the slope n^k. Similarly, we can prove it for $Z_k \cdots Z_1$. □

REMARK 3. Let $x, y \in \mathbb{R}$. Then, by Remark 1, Schreier graphs $(\mathrm{Orb}_{G_n}(x), S_n)$ and $(\mathrm{Orb}_{G_n}(y), S_n)$ are isomorphic if and only if there exist two bijections $f : \mathrm{Orb}_{G_n}(x) \to \mathrm{Orb}_{G_n}(y)$ and $\gamma : S_n \to S_n$ such that $\gamma(s)(f(z)) = f(s(z))$ for each $z \in \mathrm{Orb}_{G_n}(x)$ and each $s \in S_n$.

LEMMA 11. *Let $x, y \in \mathbb{R}$. Suppose that the Schreier graph $(\mathrm{Orb}_{G_n}(x), S_n)$ is isomorphic to the Schreier graph $(\mathrm{Orb}_{G_n}(y), S_n)$ by a bijection $\gamma : S_n \to S_n$. Then*

$$\gamma(a)\gamma(b)\gamma(a^{-1})\gamma(b^{-1})^n = 1_{\mathbb{R}} \text{ in } G_n$$

if and only if

$$\gamma = 1_{S_n} \quad \text{or} \quad \gamma(a) = a, \gamma(a^{-1}) = a^{-1}, \gamma(b) = b^{-1}, \text{ and } \gamma(b^{-1}) = b.$$

PROOF. Let $f : \mathrm{Orb}_{G_n}(x) \to \mathrm{Orb}_{G_n}(y)$ be a bijection as in Remark 3. For any $s \in S$ and any $x_0 \in \mathrm{Orb}_{G_n}(x)$, $\gamma(s)\gamma(s^{-1})(f(x_0)) = f(ss^{-1}(x_0)) = f(x_0)$ by Remark 3. Since f is a bijection, $\gamma(s)\gamma(s^{-1}) = 1_{\mathrm{Orb}_{G_n}(y)}$. Since $\gamma(s)\gamma(s^{-1})$ is an affine map, $\gamma(s)\gamma(s^{-1}) = 1_{\mathbb{R}}$, thus $\gamma(s)^{-1} = \gamma(s^{-1}) \in \mathrm{Aff}(\mathbb{R})$.

Suppose that $\gamma(a)\gamma(b)\gamma(a^{-1})\gamma(b^{-1})^n = 1_{\mathbb{R}}$ and $\gamma \neq 1_{S_n}$. Since $a(x) = nx$ and $\gamma(b^{-1})$ has the n-th power, $\gamma(b^{-1}) \in \{b\}^{\pm}$.

Suppose that $\gamma(b^{-1}) = b^{-1}$. Then $\gamma(b) = b$. Since $\gamma \neq 1_{S_n}$, we have $\gamma(a) = a^{-1}$. Then $\gamma(a)\gamma(b)\gamma(a^{-1})\gamma(b^{-1})^n = a^{-1}bab^{-n} \neq 1_{\mathbb{R}}$, a contradiction. Thus $\gamma(b^{-1}) = b$ and $\gamma(b) = b^{-1}$.

If $\gamma(a) = a^{-1}$, then $\gamma(a^{-1}) = a$ and $\gamma(a)\gamma(b)\gamma(a^{-1})\gamma(b^{-1})^n = a^{-1}b^{-1}ab^n \neq 1_{\mathbb{R}}$, a contradiction. Hence $\gamma(a) = a$ and $\gamma(a^{-1}) = a^{-1}$. □

THEOREM 4. *Let $m, n \geq 2$ with $m \neq n$.*

(1) *For any $x, y \in \mathbb{R}$, the Schreier graph $(\mathrm{Orb}_{G_m}(x), S_m)$ is not isomorphic to the Schreier graph $(\mathrm{Orb}_{G_n}(y), S_n)$ as labelled directed graphs.*
(2) *For any $\alpha_1, \alpha_2 \in \mathbb{R} \setminus \mathbb{Q}$, the Schreier graph $(\mathrm{Orb}_{G_n}(\alpha_1), S_n, \alpha_1)$ is S_n-isomorphic to the Schreier graph $(\mathrm{Orb}_{G_n}(\alpha_2), S_n, \alpha_2)$ as marked labelled directed graphs.*
(3) *For any $q \in \mathbb{Q}$ and any $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, the Schreier graph $(\mathrm{Orb}_{G_n}(q), S_n)$ is not isomorphic to the Schreier graph $(\mathrm{Orb}_{G_n}(\alpha), S_n)$ as labelled directed graphs.*
(4) *Let $q_1, q_2 \in \mathbb{Q}$. Then, the following statements are equivalent.*
    (a) *The Schreier graph $(\mathrm{Orb}_{G_n}(q_1), S_n)$ is isomorphic to the Schreier graph $(\mathrm{Orb}_{G_n}(q_2), S_n)$ as labelled directed graphs.*
    (b) $\mathrm{Orb}_{G_n}(q_1) = \mathrm{Orb}_{G_n}(q_2)$ *or* $\mathrm{Orb}_{G_n}(-q_1) = \mathrm{Orb}_{G_n}(q_2)$.

PROOF. On the contrary, suppose that the Schreier graphs $(\mathrm{Orb}_{G_m}(x), S_m)$ and $(\mathrm{Orb}_{G_n}(y), S_n)$ are isomorphic by bijections $f : \mathrm{Orb}_{G_m}(x) \to \mathrm{Orb}_{G_n}(y)$ and $\gamma : S_m \to$

S_n as in Remark 1. We check at once that $\gamma(a)\gamma(b)\gamma(a^{-1})\gamma(b^{-1})^m \neq 1_{\mathbb{R}} \in G_n$. By Remark 1, $\gamma(a)\gamma(b)\gamma(a^{-1})\gamma(b^{-1})^m(f(z)) = f(aba^{-1}b^{-m}(z)) = f(z)$ for each $z \in \mathrm{Orb}_{G_m}(x)$, contradiction, which proves (1). Since $\mathrm{Stab}_{G_n}(\alpha) = 1$ for any $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, by Proposition 1, the statement (2) is proved.

Let q be a rational number represented by uv^∞ and $x \in \mathbb{R}$ such that the Schreier graph $(\mathrm{Orb}_{G_n}(q), S_n)$ is isomorphic to the Schreier graph $(\mathrm{Orb}_{G_n}(x), S_n)$ as labelled directed graphs by bijections $f : \mathrm{Orb}_{G_n}(q) \to \mathrm{Orb}_{G_n}(x)$ and $\gamma : S_n \to S_n$ as in Remark 3. Let $q_0 = \sum_{j\geq 1}(v^\infty)_j/n^j \in \mathrm{Orb}_{G_n}(q)$. Since $aba^{-1}b^{-n}(q') = q'$ for each $q' \in \mathrm{Orb}_{G_n}(q)$, by Remark 3, we have $\gamma(a)\gamma(b)\gamma(a^{-1})\gamma(b^{-1})^n(f(q')) = f(aba^{-1}b^{-n}(q')) = f(q')$. Hence, $\gamma(a)\gamma(b)\gamma(a^{-1})\gamma(b^{-1})^n = 1_{\mathbb{R}}$. By Lemma 11,

$$\gamma = 1_{S_n} \quad \text{or} \quad \gamma(a) = a, \gamma(a^{-1}) = a^{-1}, \gamma(b) = b^{-1}, \text{ and } \gamma(b^{-1}) = b. \tag{E}$$

On the other hand, by Lemma 10, there exists a nontrivial affine map $W_{|v|}\cdots W_1 = c_k \cdots c_1$ such that $c_k \cdots c_1(q_0) = q_0$, where $c_i \in \{a, b^{-1}\}$. By Remark 3, we have $\gamma(c_k)\cdots\gamma(c_1)(f(q_0)) = f(c_k \cdots c_1(q_0)) = f(q_0)$.

(i) If $\gamma = 1_{S_n}$, then the nontrivial affine map $c_k \cdots c_1$ fixes both q_0 and $f(q_0)$. Hence, $f(q_0) = q_0$.

(ii) If $\gamma(a) = a, \gamma(a^{-1}) = a^{-1}, \gamma(b) = b^{-1}$, and $\gamma(b^{-1}) = b$, then by Lemma 10, $\gamma(c_k)\cdots\gamma(c_1)(-q_0) = Z_{|v|}\cdots Z_1(-q_0) = -q_0$. Since the nontrivial affine map $\gamma(c_k)\cdots\gamma(c_1)$ fixes both $-q_0$ and $f(q_0)$, we have $-q_0 = f(q_0)$.

We start to prove (3). On the contrary, if $x = \alpha \in \mathbb{R} \setminus \mathbb{Q}$, by the above, we see $f(q_0) \in \mathbb{Q}$, a contradiction, which proves (3).

Next we prove (4). Suppose that the statement (a) holds, i.e.,$q = q_1$, $x = q_2 \in \mathbb{Q}$ above. If $\gamma = 1_{S_n}$, by (i) above, $\mathrm{Orb}_{G_n}(q_1) = \mathrm{Orb}_{G_n}(q_0) = \mathrm{Orb}_{G_n}(q_2)$. If $\gamma \neq 1_{S_n}$, by (ii) above, $\mathrm{Orb}_{G_n}(-q_1) = \mathrm{Orb}_{G_n}(-q_0) = \mathrm{Orb}_{G_n}(q_2)$, which proves (b).

Suppose that the statement (b) holds. We show that $(\mathrm{Orb}_{G_n}(q_1), S_n)$ and $(\mathrm{Orb}_{G_n}(q_2), S_n)$ are isomorphic. Without loss of generality, we can assume that $\mathrm{Orb}_{G_n}(-q_1) = \mathrm{Orb}_{G_n}(q_2)$. Define $\gamma : S_n \to S_n$ by $\gamma(a) = a$, $\gamma(a^{-1}) = a^{-1}$, $\gamma(b) = b^{-1}$, and $\gamma(b^{-1}) = b$. In addition define $f : \mathrm{Orb}_{G_n}(q_1) \to \mathrm{Orb}_{G_n}(q_2)$ by $f(c_k \cdots c_1(q_1)) = \gamma(c_k)\cdots\gamma(c_1)(-q_1)$, where $c_i \in S_n$. By induction on k, we can show that $(c_k \cdots c_1)(q_1) + (\gamma(c_k)\cdots\gamma(c_1))(-q_1) = 0$ for each $k \geq 1$ and each $c_i \in S_n$. Hence, f is well-defined and an injection. By definition, f is a surjection satisfying that $f(s(z)) = \gamma(s)(f(z))$ for each $z \in \mathrm{Orb}_{G_n}(q_1)$ and each $s \in S_n$. By Remark 3, the Schreier graphs $(\mathrm{Orb}_{G_n}(q_1), S_n)$ and $(\mathrm{Orb}_{G_n}(q_2), S_n)$ are isomorphic by f and γ. $\square$

Corollary 1. *Let q_1, q_2 be rational numbers. Then, the following statements are equivalent.*

- (a) *The Schreier graph $(\mathrm{Orb}_{G_n}(q_1), S_n, q_1)$ is isomorphic to the Schreier graph $(\mathrm{Orb}_{G_n}(q_2), S_n, q_2)$ as marked labelled directed graphs.*
- (b) $|q_1| = |q_2|$.

PROOF. From the latter part of the proof of Theorem 4, we can show that (b) implies (a). Suppose that $(\mathrm{Orb}_{G_n}(q_1), S_n, q_1)$ is isomorphic to $(\mathrm{Orb}_{G_n}(q_2), S_n, q_2)$ by bijections $f : \mathrm{Orb}_{G_n}(q_1) \to \mathrm{Orb}_{G_n}(q_2)$ with $f(q_1) = q_2$ and $\gamma : S_n \to S_n$ as in Remark 3. It suffices to show that $|q_1| = |q_2|$. Let us represent by $uv^\infty \in \mathbb{Z}_n^\omega$ $q_1 \in \mathbb{Q}$. Set $q_0 = \sum_{j\geq 1}(v^\infty)_j/n^j \in \mathrm{Orb}_{G_n}(q_1)$. Then, there exist $d_1, \dots, d_j \in S_n$ such that $(d_j \cdots d_1)(q_1) = q_0$. From the proof of Theorem 4, the map γ satisfies (E) in the proof of Theorem 4, and the map f satisfies

$$f(q_0) = \begin{cases} q_0 & \text{if } \gamma = 1_{S_n} \\ -q_0 & \text{if } \gamma \neq 1_{S_n}. \end{cases}$$

Moreover, there exist $c_1, \dots, c_k \in S_n$ such that $(c_k \cdots c_1)(q_0) = q_0$ and $\gamma(c_k)\cdots\gamma(c_1)(f(q_0)) = f(q_0)$. Then

$$(d_j \cdots d_1)^{-1}(c_k \cdots c_1)(d_j \cdots d_1)(q_1) = q_1.$$

By Remark 3

$$\gamma(d_1)^{-1} \cdots \gamma(d_j)^{-1}\gamma(c_k)\cdots\gamma(c_1)\gamma(d_j)\cdots\gamma(d_1)(q_2) = q_2.$$

Thus $\gamma(c_k)\cdots\gamma(c_1)(\gamma(d_j)\cdots\gamma(d_1)(q_2)) = \gamma(d_j)\cdots\gamma(d_1)(q_2)$.

Suppose that $\gamma = 1_{S_n}$. Then, $(c_k \cdots c_1)((d_j \cdots d_1)(q_2)) = (d_j \cdots d_1)(q_2)$. Since the nontrivial affine map $c_k \cdots c_1$ fixes both $q_0 = (d_j \cdots d_1)(q_1)$ and $(d_j \cdots d_1)(q_2)$, $(d_j \cdots d_1)(q_1) = (d_j \cdots d_1)(q_2)$. We conclude that $q_1 = q_2$.

Suppose that $\gamma \neq 1_{S_n}$. By Remark 3, $\gamma(d_j)\cdots\gamma(d_1)(q_2) = (\gamma(d_j)\cdots\gamma(d_1))(f(q_1)) = f((d_j \cdots d_1)(q_1)) = f(q_0) = -q_0 = -(d_j \cdots d_1)(q_1)$. Since the map γ satisfies (E) in the proof of Theorem 4, by induction on j, we can show $q_1 = -q_2$. □

5. Applications

First we determine the group structure of stabilizers for all rational numbers by using the Schreier graphs described in the previous section. The proof of next proposition allows us to understand a word stood for a generator as well as the group structure. We note that the the stabilizer $\mathrm{Stab}_{G_n}(q)$ is an infinite index subgroup of G_n since the orbit $\mathrm{Orb}_{G_n}(q)$ is an infinite set.

PROPOSITION 2. *Let $n \geq 2$ and q be a rational number represented by $uv^\infty \in \mathbb{Z}_n^\omega$. Then, there exists $f \in \mathrm{Aff}(\mathbb{R})$ such that $f(x) = n^{|v|}(x - q) + q$ for each $x \in \mathbb{R}$, and* $\mathrm{Stab}_{G_n}(q) = \langle f \rangle \cong \mathbb{Z}$.

Proof. For $i \geq 1$ set $\widetilde{W_i} = b^{-(uv^\infty)_i} a$. By Lemma 10 we have

$$\begin{aligned}
\widetilde{W}_{|u|+|v|} \cdots \widetilde{W}_{|u|+1} \widetilde{W}_{|u|} \cdots \widetilde{W}_1 (b^{-\lfloor q \rfloor}(q)) &= \widetilde{W}_{|u|+|v|} \cdots \widetilde{W}_{|u|+1} (\sum_{i \geq 1} (v^\infty)_i / n^i) \\
&= W_{|v|} \cdots W_1 (\sum_{i \geq 1} (v^\infty)_i / n^i) \\
&= \sum_{i \geq 1} (v^\infty)_i / n^i \\
&= \widetilde{W}_{|u|} \cdots \widetilde{W}_1 \, (b^{-\lfloor q \rfloor}(q)).
\end{aligned}$$

Set $f = b^{\lfloor q \rfloor} \widetilde{W}_1^{-1} \cdots \widetilde{W}_{|u|}^{-1} \widetilde{W}_{|u|+|v|} \cdots \widetilde{W}_{|u|+1} \widetilde{W}_{|u|} \cdots \widetilde{W}_1 b^{-\lfloor q \rfloor}$. Then, f is an affine map with the slope $n^{|v|}$ such that $f(q) = q$. Hence $\langle f \rangle < \mathrm{Stab}_{G_n}(q)$.

Let $g \in \mathrm{Stab}_{G_n}(q)$. By $(*)_n$, there exists $i \in \mathbb{Z}$ such that $g(x) = n^i(x - q) + q$ for any $x \in \mathbb{R}$. If $|v| = 1$, f has the slope n, thus $g = f^i$. Hence, we may assume that $|v| \geq 2$. On the contrary, suppose that there exist $h \in \mathrm{Stab}_{G_n}(q) \setminus \langle f \rangle$, $0 < r < |v|$, $j \in \mathbb{Z}$, and $k \geq 0$ such that $h(x) = n^r x + j/n^k$ and $h(q) = q$. Then, we have

$$q = \frac{-j}{n^k(n^r - 1)}.$$

There exist $m \geq 0$ and $z = z_1 z_2 \ldots z_r \in \widetilde{\mathbb{Z}_n}$ with $z \neq (n-1)^r$ such that

$$|j| = \left(\sum_{i=0}^{r-1} (n-1) n^i \right) m + \sum_{i=0}^{r-1} z_{r-i} n^i = n^r \left(m \sum_{i=1}^{r} \frac{n-1}{n^i} + \sum_{i=1}^{r} \frac{z_i}{n^i} \right).$$

Since

$$\frac{n^r}{n^r - 1} = \sum_{j \geq 0} \left(\frac{1}{n^r} \right)^j,$$

we have

$$qn^k = m + \sum_{i \geq 1} \frac{(z^\infty)_i}{n^i} \quad \text{or} \quad qn^k = -(m+1) + \sum_{i \geq 1} \frac{(\overline{z}^\infty)_i}{n^i},$$

where $\overline{z} = (n - 1 - z_1) \ldots (n - 1 - z_r) \in \widetilde{\mathbb{Z}_n}$. Thus, qn^k has a repeating part whose length is the period of z^∞. However,

$$qn^k = \left(\lfloor q \rfloor + \sum_{i \geq 1} \frac{(uv^\infty)_i}{n^i} \right) n^k = \left(\lfloor q \rfloor n^k + \sum_{i=1}^{k} (uv^\infty)_i n^{k-i} \right) + \sum_{i \geq 1} \frac{(uv^\infty)_{i+k}}{n^i},$$

which contradicts (2) in Definition 2. $\square$

Next we introduce the definition of being isomorphic in presentations for subgroups in order to translate the graphical expression of the Schreier graphs into the algebraic expression of subgroups. Consequently, we get a relevance to presentations for the stabilizers from the previous result about the classification of the Schreier graphs (see Theorem 5).

For $i \in \{1,2\}$, let G_i be a group with a generating set T_i. Let $T_i^{-1} = \{t^{-1} \mid t \in T_i\}$ and $T_i^{\pm} = T_i \cup T_i^{-1}$. We assume that

$$(*) \quad t \in T_i \cap T_i^{-1} \text{ if and only if } t \in T_i, \quad t^2 = 1.$$

For $i \in \{1,2\}$ let $X_i = \{ x_t \mid t \in T_i \}$. Set $X_i^{-1} = \{x_t^{-1} \mid t \in T_i\}$, where x_t^{-1} denotes a new symbol corresponding to the element x_t. We assume that $X_i \cap X_i^{-1} = \emptyset$ and that the expression $(x_t^{-1})^{-1}$ denotes the element x_t. For $i \in \{1,2\}$ the free group with the basis X_i is denoted by $F(X_i)$, and for a subset R_i of $F(X_i)$ the normal closure of the set R_i in $F(X_i)$ is denoted by $\langle\langle R_i \rangle\rangle$. Let G_i be the group with the presentation $\langle X_i \mid R_i \rangle$ with respect to the epimorphism $\psi_i : F(X_i) \to G_i$ given by $\psi_i(x_t) = t$.

Definition 4. For $i \in \{1,2\}$, let H_i be a subgroup of G_i. H_1 and H_2 are *isomorphic in presentations $\langle X_1|R_1\rangle$ and $\langle X_2|R_2\rangle$ respectively* if there exists a bijection $\gamma : X_1^{\pm} \to X_2^{\pm}$ with $\gamma(x_t^{-1}) = \gamma(x_t)^{-1}$ such that $\widetilde{\gamma}\,(\psi_1^{-1}(H_1)) = \psi_2^{-1}(H_2)$ and $\widetilde{\gamma}(\langle\langle R_1\rangle\rangle) = \langle\langle R_2\rangle\rangle$, where $\widetilde{\gamma} : F(X_1) \to F(X_2)$ is defined by $\widetilde{\gamma}(x_{t_1}^{\varepsilon_1} \cdots x_{t_k}^{\varepsilon_k}) = \gamma(x_{t_1})^{\varepsilon_1} \cdots \gamma(x_{t_k})^{\varepsilon_k}$ for $\varepsilon_i \in \{\pm 1\}$. Then, $\widetilde{\gamma}$ is an isomorphism and $H_1 \cong H_2$. Conversely, if there exists an isomorphism $\widetilde{\gamma} : F(X_1) \to F(X_2)$ such that $\widetilde{\gamma}(K_1) = K_2$ for each $K_i \in \{\psi_i^{-1}(H_i), \mathrm{Ker}\psi_i, X_i^{\pm}\}$, then $\gamma = \widetilde{\gamma}|_{X_1^{\pm}}$ satisfies the above condition.

Proposition 3. *Let $\Gamma_i = (G_i/H_i, T_i^{\pm}, H_i)$ and $\Gamma_i' = (F(X_i)/\psi_i^{-1}(H_i), X_i^{\pm}, \psi_i^{-1}(H_i))$ be Schreier coset graphs for $i \in \{1,2\}$. Then, the following statements are equivalent.*

- (a) *Γ_1 is isomorphic to Γ_2 as marked labelled directed graphs by a bijection $\gamma : T_1^{\pm} \to T_2^{\pm}$ such that $\gamma(t^{-1}) = \gamma(t)^{-1}$ for every $t \in T_1$.*
- (b) *Γ_1' is isomorphic to Γ_2' as marked labelled directed graphs by a bijection $\gamma' : X_1^{\pm} \to X_2^{\pm}$ with $\gamma'(x_t^{-1}) = \gamma'(x_t)^{-1}$ for every $x_t \in X_1$ satisfying the condition*

$$(B) \quad \psi_1(x_t)^2 = 1_{G_1} \quad \textit{if and only if} \quad \psi_2(\gamma'(x_t))^2 = 1_{G_2}.$$

Proof. Let $\varphi_i : G_i \to \mathrm{Aut}(G_i/H_i)$ and $\varphi_i' : F(X_i) \to \mathrm{Aut}(F(X_i)/\psi_i^{-1}(H_i))$ be the usual left actions for $i \in \{1,2\}$. We define $\Psi_i : F(X_i)/\psi_i^{-1}(H_i) \to G_i/H_i$ by $\Psi_i(y\,\psi_i^{-1}(H_i)) = \psi_i(y)H_i$. Since $y^{-1}y' \in \psi_i^{-1}(H_i)$ is equivalent to $\psi_i(y)^{-1}\psi_i(y') \in H_i$, Ψ_i is well-defined and an injection. Since ψ_i is a surjection, Ψ_i is also a surjection.

Suppose that the statement (a) holds. Let $f : G_1/H_1 \to G_2/H_2$ be a bijection between vertices such that $f(H_1) = H_2$ and $f\varphi_1(t) = \varphi_2(\gamma(t))f$ for every $t \in T_1$.

Set $f' = \Psi_2^{-1} f \Psi_1 : F(X_1)/\psi_1^{-1}(H_1) \to F(X_2)/\psi_2^{-1}(H_2)$. Clearly f' is bijective with $f'(\psi_1^{-1}(H_1)) = \psi_2^{-1}(H_2)$.

Define $\gamma' : X_1^{\pm} \to X_2^{\pm}$ by

$$\gamma'(x_t^{\varepsilon}) = \begin{cases} x_{\gamma(t)}^{\varepsilon} & \text{if } \gamma(t) \in T_2 \text{ and } \varepsilon \in \{\pm 1\}, \\ x_{\gamma(t)^{-1}}^{-\varepsilon} & \text{if } \gamma(t) \notin T_2 \text{ and } \varepsilon \in \{\pm 1\}. \end{cases}$$

Then we have $\gamma'(x_t^{-1}) = \gamma'(x_t)^{-1}$. To show that γ' is bijective, we define $\sigma : X_2^{\pm} \to X_1^{\pm}$ by

$$\sigma(x_t^{\varepsilon}) = \begin{cases} x_{\gamma^{-1}(t)}^{\varepsilon} & \text{if } \gamma^{-1}(t) \in T_1 \text{ and } \varepsilon \in \{\pm 1\}, \\ x_{\gamma^{-1}(t)^{-1}}^{-\varepsilon} & \text{if } \gamma^{-1}(t) \notin T_1 \text{ and } \varepsilon \in \{\pm 1\}. \end{cases}$$

Then

$$\sigma\, \gamma'(x_t^{\varepsilon}) = \begin{cases} \sigma(x_{\gamma(t)}^{\varepsilon}) & \text{if } \gamma(t) \in T_2 \text{ and } \varepsilon \in \{\pm 1\}, \\ \sigma(x_{\gamma(t)^{-1}}^{-\varepsilon}) & \text{if } \gamma(t) \notin T_2 \text{ and } \varepsilon \in \{\pm 1\}. \end{cases}$$

If $\gamma(t) \in T_2$, $\gamma^{-1}(\gamma(t)) = t \in T_1$. If $\gamma(t) \notin T_2$, $\gamma^{-1}(\gamma(t)^{-1}) = \gamma^{-1}(\gamma(t^{-1})) = t^{-1} \notin T_1$ by $(*)$. Since $\gamma(t^{-1}) = \gamma(t)^{-1}$, we have $\gamma^{-1}(s^{-1}) = \gamma^{-1}(s)^{-1}$. Hence we have

$$\sigma\, \gamma'(x_t^{\varepsilon}) = \begin{cases} x_t^{\varepsilon} & \text{if } \gamma(t) \in T_2 \text{ and } \varepsilon \in \{\pm 1\}, \\ x_t^{\varepsilon} & \text{if } \gamma(t) \notin T_2 \text{ and } \varepsilon \in \{\pm 1\}, \end{cases}$$

thus $\sigma\, \gamma' = 1_{X_1^{\pm}}$. The similar argument gives $\gamma' \sigma = 1_{X_2^{\pm}}$. Thus γ' is a bijection.

Since $\psi_2(\gamma'(x_t)) = \gamma(t)$ and $t^2 = 1_{G_1}$ if and only if $\gamma(t)^2 = 1_{G_2}$, we have $\psi_1(x_t)^2 = 1_{G_1}$ if and only if $\psi_2(\gamma'(x_t))^2 = 1_{G_2}$, which establishes (B).

Since $\Psi_1 \varphi_1'(x_t) = \varphi_1(t) \Psi_1$ and $\Psi_2 \varphi_2'(\gamma'(x_t)) = \varphi_2(\gamma(t)) \Psi_2$, we have

$$\begin{aligned} \varphi_2'(\gamma'(x_t)) f' \varphi_1'(x_t)^{-1} = \varphi_2'(\gamma'(x_t)) \Psi_2^{-1} f \Psi_1 \varphi_1'(x_t)^{-1} &= \Psi_2^{-1} \varphi_2(\gamma(t)) f \varphi_1(t)^{-1} \Psi_1 \\ &= \Psi_2^{-1} f \Psi_1 \\ &= f'. \end{aligned}$$

By Remark 1 we obtain (b).

Suppose that the statement (b) holds. Let $f' : F(X_1)/\psi_1^{-1}(H_1) \to F(X_2)/\psi_2^{-1}(H_2)$ be a bijection between vertices such that $f'(\psi_1^{-1}(H_1)) = \psi_2^{-1}(H_2)$ and $f' \varphi_1'(x_t) = \varphi_2'(\gamma'(x_t)) f'$ for every $x_t \in X_1$. Set $f = \Psi_2 f' \Psi_1^{-1} : G_1/H_1 \to G_2/H_2$. Clearly f is bijective with $f(H_1) = H_2$.

Define $\gamma : T_1^{\pm} \to T_2^{\pm}$ by $\gamma(t^{\varepsilon}) = \psi_2(\gamma'(x_t^{\varepsilon}))$ for each $t \in T_1$ and $\varepsilon \in \{\pm 1\}$. First we show that γ is well-defined. Suppose that $t_1^{\varepsilon_1} = t_2^{\varepsilon_2}$. If $\varepsilon_1 = \varepsilon_2$ and $t_1 = t_2$, then $\psi_2(\gamma'(x_{t_1}^{\varepsilon_1})) = \psi_2(\gamma'(x_{t_2}^{\varepsilon_2}))$. If $\varepsilon_1 \neq \varepsilon_2$, then $t_1 = t_2$. Since $\psi_2(\gamma'(x_{t_j}))^2 = 1_{G_2}$

by (B), $\psi_2(\gamma'(x_{t_1}^{\varepsilon_1})) = \psi_2(\gamma'(x_{t_1}^{-\varepsilon_1})) = \psi_2(\gamma'(x_{t_2}^{\varepsilon_2}))$. Thus γ is well-defined. Then we have $\gamma(t^{-1}) = \gamma(t)^{-1}$. Next we show that γ is bijective. We define $\rho : T_2^{\pm} \to T_1^{\pm}$ by $\rho(t^{\varepsilon}) = \psi_1(\gamma'^{-1}(x_t^{\varepsilon}))$ for each $t \in T_2$ and $\varepsilon \in \{\pm 1\}$. Since γ' satisfies the condition (B), $\psi_2(x_t)^2 = 1_{G_2}$ if and only if $\psi_1(\gamma'^{-1}(x_t))^2 = 1_{G_1}$. Hence ρ is well-defined. We can easily see that $\gamma\rho = 1_{T_2^{\pm}}$ and $\rho\gamma = 1_{T_1^{\pm}}$. Hence γ is a bijection.

Since $\Psi_1\varphi_1'(x_t) = \varphi_1(t)\Psi_1$ and $\Psi_2\varphi_2'(\gamma'(x_t)) = \varphi_2(\gamma(t))\Psi_2$,

$$\begin{aligned}\varphi_2(\gamma(t))f\varphi_1(t)^{-1} = \varphi_2(\gamma(t))\Psi_2 f'\Psi_1^{-1}\varphi_1(t)^{-1} &= \Psi_2\varphi_2'(\gamma'(x_t))f'\varphi_1'(x_t)^{-1}\Psi_1^{-1}\\ &= \Psi_2 f'\Psi_1^{-1}\\ &= f.\end{aligned}$$

By Remark 1 we obtain (a). □

Lemma 12. *Let $\Gamma_i = (G_i/H_i, T_i^{\pm}, H_i)$ be Schreier coset graphs for $i \in \{1,2\}$. Then the following statements are equivalent.*

(a) *Γ_1 is isomorphic to Γ_2 as marked labelled directed graphs by a bijection $\gamma : T_1^{\pm} \to T_2^{\pm}$ satisfying the following condition: for any $t_1, \dots, t_k \in T_1$ and any $\varepsilon_1, \dots, \varepsilon_k \in \{\pm 1\}$,*

$$(C) \qquad t_1^{\varepsilon_1} \cdots t_k^{\varepsilon_k} = 1_{G_1} \text{ if and only if } \gamma(t_1^{\varepsilon_1}) \cdots \gamma(t_k^{\varepsilon_k}) = 1_{G_2}.$$

(b) *H_1 and H_2 are isomorphic in presentations $\langle X_1|R_1\rangle$ and $\langle X_2|R_2\rangle$ respectively.*

Proof. By Proposition 3, (a) is equivalent to the following statement.

(a') Γ_1' is isomorphic to Γ_2' as marked labelled directed graphs by a bijection $\gamma' : X_1^{\pm} \to X_2^{\pm}$ such that $\gamma'(x_t^{-1}) = \gamma'(x_t)^{-1}$ for every $x_t \in X_1$ and

$$(C') \quad \psi_1(x_{t_1}^{\varepsilon_1}) \cdots \psi_1(x_{t_k}^{\varepsilon_k}) = 1_{G_1} \text{ if and only if } \psi_2(\gamma'(x_{t_1}^{\varepsilon_1})) \cdots \psi_2(\gamma'(x_{t_k}^{\varepsilon_k})) = 1_{G_2}.$$

In addition we note that the following statements are equivalent.

(1) There exists a bijection $\gamma' : X_1^{\pm} \to X_2^{\pm}$ with $\gamma'(x_t^{-1}) = \gamma'(x_t)^{-1}$ satisfying the condition (C').
(2) There exists a group isomorphism $\delta : F(X_1) \to F(X_2)$ such that $\delta(X_1^{\pm}) = X_2^{\pm}$ and $\delta(\langle\langle R_1\rangle\rangle) = \langle\langle R_2\rangle\rangle$.

Suppose that the statement (a) holds. By the above, we may suppose that the statement (a') holds, and can take $\widetilde{\gamma'}$ as δ in (2), where $\widetilde{\gamma'} : F(X_1) \to F(X_2)$ given by $\widetilde{\gamma'}(x_{t_1}^{\varepsilon_1} \cdots x_{t_k}^{\varepsilon_k}) = \gamma'(x_{t_1})^{\varepsilon_1} \cdots \gamma'(x_{t_k})^{\varepsilon_k}$. It suffices to prove that $\widetilde{\gamma'}(\psi_1^{-1}(H_1))$ $= \psi_2^{-1}(H_2)$. Let $f' : F(X_1)/\psi_1^{-1}(H_1) \to F(X_2)/\psi_2^{-1}(H_2)$ be a bijection between vertices which preserves marked vertices. Now, we note that for $i \in \{1,2\}$, $\psi_i^{-1}(H_i) =$

$\{\, l(P) \,|\, P$ is an edge path in Γ_i' from $\psi_i^{-1}(H_i)$ to itself $\}$, where $l(P) = l(e_n) \ldots l(e_1)$ whenever $P = e_1 \ldots e_n$.

Let $l(P) \in \psi_1^{-1}(H_1)$, where $e_j = (x_{t_{j-1}}^{\varepsilon_{j-1}} \cdots x_{t_1}^{\varepsilon_1} \psi_1^{-1}(H_1),\, x_{t_j}^{\varepsilon_j})$ and $P = e_1 \ldots e_n$. Since $x_{t_n}^{\varepsilon_n} \cdots x_{t_1}^{\varepsilon_1} \psi_1^{-1}(H_1) = \beta(e_n) = \psi_1^{-1}(H_1)$, by Remark 1,

$$\begin{aligned}
\widetilde{\gamma'}(l(P))\psi_2^{-1}(H_2) = \gamma'(x_{t_n}^{\varepsilon_n}) \cdots \gamma'(x_{t_1}^{\varepsilon_1}) f'(\psi_1^{-1}(H_1)) &= f'(x_{t_n}^{\varepsilon_n} \cdots x_{t_1}^{\varepsilon_1} \psi_1^{-1}(H_1)) \\
&= f'(\psi_1^{-1}(H_1)) \\
&= \psi_2^{-1}(H_2).
\end{aligned}$$

Thus we have $\widetilde{\gamma'}(\psi_1^{-1}(H_1)) \subset \psi_2^{-1}(H_2)$. Similarly $\widetilde{\gamma'}^{-1}(\psi_2^{-1}(H_2)) \subset \psi_1^{-1}(H_1)$, which proves $\widetilde{\gamma'}\,(\psi_1^{-1}(H_1)) = \psi_2^{-1}(H_2)$.

Suppose that the statement (b) holds. There exists a bijection $\gamma' : X_1^{\pm} \to X_2^{\pm}$ with $\gamma'(x_t^{-1}) = \gamma'(x_t)^{-1}$ such that $\widetilde{\gamma'}\,(\psi_1^{-1}(H_1)) = \psi_2^{-1}(H_2)$ and $\widetilde{\gamma'}(\langle\langle R_1 \rangle\rangle) = \langle\langle R_2 \rangle\rangle$, which establishes (2). Define $f' : F(X_1)/\psi_1^{-1}(H_1) \to F(X_2)/\psi_2^{-1}(H_2)$ by $f'(g\psi_1^{-1}(H_1)) = \widetilde{\gamma'}(g)\psi_2^{-1}(H_2)$. Since $g_2^{-1}g_1 \in \psi_1^{-1}(H_1)$ is equivalent to $\widetilde{\gamma'}(g_2^{-1}g_1) \in \widetilde{\gamma'}(\psi_1^{-1}(H_1)) = \psi_2^{-1}(H_2)$, f' is well-defined and an injection. Since $\widetilde{\gamma'}$ is a surjection, f' is also a surjection. Since

$$\begin{aligned}
f'\varphi_1'(x_t)(g\psi_1^{-1}(H_1)) = f'(x_t g\psi_1^{-1}(H_1)) &= \widetilde{\gamma'}(x_t g)\psi_2^{-1}(H_2) \\
&= \widetilde{\gamma'}(x_t)\widetilde{\gamma'}(g)\psi_2^{-1}(H_2) \\
&= \varphi_2'(\gamma'(x_t)) f'(g\psi_1^{-1}(H_1)),
\end{aligned}$$

we have $f'\varphi_1'(x_t) = \varphi_2'(\gamma'(x_t))f'$ for every $x_t \in X_1$. Thus Γ_1' is isomorphic to Γ_2' as marked labelled directed graphs by a bijection $\gamma' : X_1^{\pm} \to X_2^{\pm}$, which establishes (a'), i.e., (a). $\square$

By Lemmas 11 and 12, Corollary 1, (1) in Theorem 4 and the isomorphism h_n, we obtain the following theorem.

Theorem 5. *Let $m, n \geq 2$ and q_1, $q_2 \in \mathbb{Q}$. Then the following statements are equivalent.*

- (a) $\mathrm{Stab}_{BS(1,m)}(q_1)$ *and* $\mathrm{Stab}_{BS(1,n)}(q_2)$ *are isomorphic in presentations $BS(1,m)$ and $BS(1,n)$ respectively.*
- (b) $m = n$ *and* $|q_1| = |q_2|$.

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