GLOBAL EXISTENCE FOR TREE-GRASS COMPETITION MODEL

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Abstract. We present a tree-grass competition model on the basis of the forest kinematic model due to Kuznetsov-Antonovsky-Biktashev-Aponina [6]. The main purpose of the paper is to construct global solutions and to construct a dynamical system generated by the model equations. By numerical computations, we also show that our model actually admits coexisting solutions of trees and grass.

1 Introduction We want to study the kinematics of forest-grassland system from a viewpoint of competitive system between trees and grass.

Our mathematical model is written as the initial-boundary value problem for a parabolic-ordinary system

$$
(1.1)\qquad \begin{cases}
\dfrac{\partial u}{\partial t} = \beta\delta[w-w_*]_+ - (\lambda g + av^2 + c)u - fu & \text{in } \Omega\times(0,\infty),\\
\dfrac{\partial v}{\partial t} = fu - hv & \text{in } \Omega\times(0,\infty),\\
\dfrac{\partial w}{\partial t} = d_w\Delta w - \beta w + \alpha v & \text{in } \Omega\times(0,\infty),\\
\dfrac{\partial g}{\partial t} = d_g\Delta g - \mu v g + \gamma(g-\ell)(1-g)g & \text{in } \Omega\times(0,\infty),\\
\dfrac{\partial w}{\partial n} = \dfrac{\partial g}{\partial n} = 0 & \text{on } \partial\Omega\times(0,\infty),\\
u(x,0)=u_0(x),\ v(x,0)=v_0(x),\ w(x,0)=w_0(x),\ g(x,0)=g_0(x) & \text{in } \Omega,
\end{cases}
$$

in a two-dimensional bounded, $\mathcal{C}^2$ or convex domain Ω. Here, the unknown functions $u(x,t)$ and $v(x,t)$ denote tree densities of young and old age classes, respectively, at a position $x\in\Omega$ and at time $t\in[0,\infty)$. The unknown function $w(x,t)$ denotes a density of seeds in the air at $x\in\Omega$ and $t\in[0,\infty)$. Meanwhile, $g(x,t)$ denotes a density of grass at $x\in\Omega$ and $t\in[0,\infty)$.

The third equation in (1.1) describes the kinetics of seeds; $d_w>0$ is a diffusion constant, and $\alpha>0$ and $\beta>0$ are seed production and seed deposition rates, respectively. The first equation describes growth of young age trees; here, $0<\delta\le 1$ is a seed establishment rate, the term $[w-w_*]_+ = \max\{w-w_*,0\}$ means that a fixed amount w_* of seeds on the ground are consumed (by animals or birds), $\lambda g + av^2 + c$ is a mortality of young age trees which is proportional to the densities g and v^2 with coefficients $\lambda>0$ and $a>0$, $c>0$ being a basic mortality. The second equation describes growth of old age trees; $f>0$ is an aging rate from young age to old age, and $h>0$ is a mortality. Finally, the fourth equation describes growth of grass that is basically given by a reaction-diffusion equation with a diffusion constant $d_g>0$ and with a cubic growth function $\gamma(g-\ell)(1-g)g$, where $0<\ell<1$ is an unstable state and $\gamma>0$ is a reaction rate, the term $-\mu vg$ denotes suppression by the trees

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with a coefficient $\mu > 0$. On w and g, the homogeneous Neumann conditions are imposed on the boundary $\partial\Omega$. Nonnegative initial functions $u_0(x) \geq 0$, $v_0(x) \geq 0$, $w_0(x) \geq 0$ and $g_0(x) \geq 0$ are given in Ω for all unknown functions.

This model is derived by the present authors on the basis of the classical forest kinematic model [6]. The detail of derivation is discussed in Section 2.

First, for suitable initial values (u_0, v_0, w_0, g_0), we construct a unique global solution in the underlying space

$$X = \{(u, v, w, g);\ u \in L_\infty(\Omega),\ v \in L_\infty(\Omega),\ w \in L_2(\Omega),\ g \in L_2(\Omega)\}.$$

As the equations of u and v are an ordinary equation for each $x \in \Omega$, the underlying spaces for u and v must be a Banach algebra. In addition, even if $u_0(x)$ and $v_0(x)$ are continuous functions on $\overline{\Omega}$, $u(x,t)$ and $v(x,t)$ of the global solution can tend to a stationary solution $(\overline{u}, \overline{v}, \overline{w}, \overline{g})$ as $t \to \infty$ in which $\overline{u}$ and $\overline{v}$ are discontinuous functions. By this reason, we set $L_\infty(\Omega)$ for the underlying spaces of u and v. Meanwhile, as w and g satisfy a diffusion equation, $u(t)$ and $g(t)$ belong to $H^2(\Omega) \subset \mathcal{C}(\overline{\Omega})$ for any $t > 0$. In constructing a local solution, we apply the theory of abstract parabolic evolution equations as in [2, 3, 4] (see also [15, Chapter 11]).

Second, after constructing a dynamical system generated by (1.1), we show that there exists a bounded absorbing set (see [14]). This in particular implies that every solution to (1.1) admits a nonempty ω-limit set in a suitable weak topology of X.

Third, by numerical methods, we observe that the model (1.1) includes some solutions showing segregation patterns. Under careful tuning for the parameters in the equations of (1.1), we observe that solutions starting from some class of initial values tend to a stationary solution $(0,0,0,1)$ as $t \to \infty$. Solutions starting another class of initial values tend to a stationary solution of the form $\overline{v}(hf^{-1}, 1, \alpha\beta^{-1}, 0)$, where $\overline{v}$ is a positive solution of the cubic equation

$$ah\overline{v}^3 + [(c+f)h - f\alpha\delta]\overline{v} + f\beta\delta w_* = 0.$$

And solutions starting from the other class of initial values tend to a stationary solution $(\overline{u}, \overline{v}, \overline{w}, \overline{g})$ which is not homogeneous but shows coexistence of trees and grass. As we can see a clear curve which divides Ω into forest and grassland, such a stationary might be called a segregation pattern.

In the forth coming paper [10], the authors will discuss segregation patterns in detail.

2 Kinematics of forests and grasslands Kuznetsov-Antonovsky-Biktashev-Aponina have first presented by their paper [6] a continuous space model describing the kinematics of forests. That is written as

$$(2.1)\qquad \begin{cases} \dfrac{\partial u}{\partial t} = \beta\delta w - \varphi(v)u - fu & \text{in } \quad \Omega \times (0,\infty), \\ \dfrac{\partial v}{\partial t} = fu - hv & \text{in } \quad \Omega \times (0,\infty), \\ \dfrac{\partial w}{\partial t} = d\Delta w - \beta w + \alpha v & \text{in } \quad \Omega \times (0,\infty), \\ \dfrac{\partial w}{\partial n} = 0 & \text{on } \quad \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x),\ v(x,0) = v_0(x),\ w(x,0) = w_0(x), & \text{in } \quad \Omega. \end{cases}$$

As seen, the main difference of (2.1) from (1.1) is that the state variables consist of u, v and w and do not include the density of grass $g(x,t)$. As a consequence, the mortality of young age trees is given by a function of the density of old age trees alone, i.e., $\varphi(v)$. As for a typical form of $\varphi(v)$, they proposed $\varphi(v)$ such that

$$\varphi(v) = a(v-b)^2 + c, \tag{2.2}$$

where a, b, $c > 0$ are positive constants (see [6, p. 220]). This means that the mortality takes its minimum when v is an optimal value b.

It is quite reasonable to assume that the higher the density of old age trees is, the higher the mortality of young age trees, because the dense canopy of old age trees admits only a small amount of light transmission and prevents young age trees under it from growing regularly and because the trees which cease growing die at a higher rate. In the meantime, it is very difficult to understand a reason why the mortality $\varphi(v)$ increases as $v \to 0$ for $0 < v < b$. It may be possible to claim that a canopy of suitable density protects young age trees under it by providing them with a comfortable shelter. On the other hand, according to the articles [1, 5, 8, 9], it is known that trees and grass are always in competition. The old age trees prevent grass's growth and conversely the grass prevents seedling's growth. So, we want to explain by tree-grass competition why the mortality of young age trees increases as old age tree's density decreases less than the critical value b. More precisely, we want to present in this paper a tree-grass competition model for the kinematics of forest together with grassland.

As already shown, our mortality function is given by

$$\gamma(g, v) = \lambda g + av^2 + c. \tag{2.3}$$

It is similar to (2.2) for sufficiently large v. But, for small v, the mortality is governed by the density of grass and is actually proportional to it.

In addition, we need to introduce a growth equation for the grassland. As the basic growth equation, we use the usual reaction-diffusion equation

$$\frac{\partial g}{\partial t} = d_g \Delta g + \gamma(g-\ell)(1-g)g$$

including a cubic growth function $\gamma(g-\ell)(1-g)g$. To this equation, we incorporate the effect of competition with trees that is described by $-\mu v g$.

In this way, our tree-grass competition model (1.1) is derived on the basis of the classical model (2.1) due to Kuznetsov-Antonovsky-Biktashev-Aponina just by incorporating newly competition effects between trees and grass and a growth equation of grassland.

3 Preliminary

I) Some inequality. It is easily verified that

$$(g-\ell)(1-g)g^6 \le \frac{1-\ell}{6}\left(1-g^6\right) \qquad \text{for } 0 \le g < \infty. \tag{3.1}$$

Indeed, we have

$$(g-\ell)g^6 < \frac{1-\ell}{6}\left(1+g+g^2+\cdots+g^5\right) \qquad \text{for } 0 \le g < 1.$$

Meanwhile,

$$(g-\ell)g^6 > \frac{1-\ell}{6}\left(1+g+g^2+\cdots+g^5\right) \qquad \text{for } 1 < g < \infty.$$

II) Function Spaces. Let Ω is a bounded, $\mathcal{C}^2$ or convex domain in $\mathbb{R}^2$. For $0 \le s \le 2$, $H^s(\Omega)$ denotes the complex Sobolev space, its norm being denoted by $\|\cdot\|_{H^s}$ (see [13, Chap. 1]). For $0 \le s_0 \le s \le s_1 \le 2$, $H^s(\Omega)$ coincides with the complex interpolation space $[H^{s_0}(\Omega), H^{s_1}(\Omega)]_\theta$, where $s = (1-\theta)s_0 + \theta s_1$, and among their norms the estimate

$$\|\cdot\|_{H^s} \le C\|\cdot\|_{H^{s_0}}^{1-\theta}\ \|\cdot\|_{H^{s_1}}^{\theta} \tag{3.2}$$

holds true. When $0 \le s < 1$, $H^s(\Omega) \subset L^p(\Omega)$, where $\frac{1}{p} = \frac{1-s}{2}$, with continuous embedding

$$\|\cdot\|_{L^p} \le C_s\|\cdot\|_{H^s}. \tag{3.3}$$

When $s = 1$, $H^1(\Omega) \subset L^q(\Omega)$ for any finite $2 \le q < \infty$ with the estimate

$$\|\cdot\|_{L^q} \le C_{pq}\|\cdot\|_{H^1}^{1-\frac{p}{q}}\|\cdot\|_{L^p}^{\frac{p}{q}}, \tag{3.4}$$

where $1 \le p < q < \infty$. When $s > 1$, $H^s(\Omega) \subset \mathcal{C}(\overline{\Omega})$ with continuous embedding

$$\|\cdot\|_{\mathcal{C}} \le C_s\|\cdot\|_{H^s}. \tag{3.5}$$

III) Linear Operators. Consider a sesquilinear form given by

$$a(u,v) = d\int_\Omega \nabla u\cdot\nabla\overline{v}\,dx + c\int_\Omega u\overline{v}\,dx, \qquad u,\ v \in H^1(\Omega),$$

d and c being positive constants. From this form, one can define a realization Λ of the Laplace operator $-d\Delta + c$ in the space $L_2(\Omega)$ under the homogeneous Neumann conditions on the boundary $\partial\Omega$ (see [12, Chap. VI]).

The realization Λ is a positive definite self-adjoint operator of $L_2(\Omega)$, i.e., $\Lambda \ge c$. When Ω is bounded and, convex or $\mathcal{C}^2$, its domain is characterized by

$$\mathcal{D}(\Lambda) = H^2_N(\Omega) \equiv \{u \in H^2(\Omega);\ \tfrac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega\}. \tag{3.6}$$

For $0 < \theta < 1$, the fractional powers Λ^θ of Λ are defined and also are positive definite self-adjoint in $L_2(\Omega)$. As shown in [15, Sec. 16.4], their domains are characterized by

$$\mathcal{D}(\Lambda^\theta) = \begin{cases} H^{2\theta}(\Omega), & \text{when } 0 \le \theta < \frac{3}{4}, \\ H^{2\theta}_N(\Omega) \equiv \{u \in H^{2\theta}(\Omega);\ \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega\}, & \text{when } \frac{3}{4} < \theta \le 1. \end{cases} \tag{3.7}$$

In addition, the following equivalence estimates

$$C^{-1}\|\Lambda^\theta\cdot\|_{L^2} \le \|\cdot\|_{H^{2\theta}} \le C\|\Lambda^\theta\cdot\|_{L^2} \tag{3.8}$$

hold true with some constant $C > 0$.

Furthermore, let $e^{-t\Lambda}$ $(0 \le t < \infty)$ denote the semigroup generated by $-\Lambda$. Then, the positivity $\Lambda \ge c$ implies that

$$\|e^{-t\Lambda}\|_{\mathcal{L}(L_2)} \le e^{-ct}, \qquad 0 \le t < \infty. \tag{3.9}$$

In addition, it is known for $0 < \theta \le 1$ that

$$\|\Lambda^\theta e^{-t\Lambda}\|_{\mathcal{L}(L_2)} \le Ct^{-\theta}, \qquad 0 < t < \infty, \tag{3.10}$$

with some constant $C > 0$.

IV) Evolution Equations. Consider the Cauchy problem for an evolution equation

$$\begin{cases} \dfrac{du}{dt} + \varLambda u = f(t), \qquad 0 < t \le T, \\ u(0) = u_0 \end{cases}$$

in the space $L_2(\Omega)$, $\varLambda$ being a positive definite self-adjoint operator of $L_2(\Omega)$. Let $f \in \mathcal{C}([0,T]; L_2(\Omega))$ and $u_0 \in L_2(\Omega)$. If $u(t)$ is a strict solution lying in the solution space:

$$u \in \mathcal{C}([0,T]; L_2(\Omega)) \cap \mathcal{C}((0,T]; \mathcal{D}(\varLambda)) \cap \mathcal{C}^1((0,T]; L_2(\Omega)),$$

then $u(t)$ is necessarily represented by the formula

$$u(t) = e^{-t\varLambda} u_0 + \int_0^t e^{-(t-s)\varLambda} f(s)ds, \qquad 0 \le t \le T. \tag{3.11}$$

Next, consider the Cauchy problem of a linear equation

$$\begin{cases} \dfrac{du}{dt} = p(t)u + q(t), \qquad 0 < t \le T, \\ u(0) = u_0 \end{cases}$$

in the space $L_\infty(\Omega)$, $p(t)$ and $q(t)$ being functions such that p, $q \in \mathcal{C}([0,T]; L_\infty(\Omega))$. Then, one can show that, for any initial value $u_0 \in L_\infty(\Omega)$, there exists a unique strict solution $u \in \mathcal{C}^1([0,T]; L_\infty(\Omega))$ and the solution is given by

$$u(t) = e^{\int_0^t p(\tau)d\tau} u_0 + \int_0^t e^{\int_s^t p(\tau)d\tau} q(s)ds, \qquad 0 \le t \le T. \tag{3.12}$$

For the proof, see [15, p. 53].

4 Local solutions In order to construct local solutions to (1.1), we want to apply the theory of abstract semilinear parabolic evolution equations.

As for the first and second equations of (1.1), we handle them in the space $L_\infty(\Omega)$ because they are ordinary differential equations for each $x \in \Omega$. Meanwhile, as for the third and fourth equations which are diffusion equations, we handle them in the space $L_2(\Omega)$. Thereby, we set the following underlying space

$$X \equiv \left\{ \begin{pmatrix} u \\ v \\ w \\ g \end{pmatrix} ;\ u,\, v \in L_\infty(\Omega) \ \text{ and } \ w,\, g \in L_2(\Omega) \right\}. \tag{4.1}$$

In the space X, (1.1) can be formulated as the Cauchy problem

$$\begin{cases} \dfrac{dU}{dt} + AU = F(U), \qquad 0 < t < \infty, \\ U(0) = U_0. \end{cases} \tag{4.2}$$

Here, A denotes a closed linear operator of X of the form

$$A \equiv \begin{pmatrix} f & 0 & 0 & 0 \\ 0 & h & 0 & 0 \\ 0 & 0 & \varLambda_w & 0 \\ 0 & 0 & 0 & \varLambda_g \end{pmatrix} = \operatorname{diag}\{f, h, \varLambda_w, \varLambda_g\}, \tag{4.3}$$

where Λ_w (resp. Λ) is a realization of the Laplace operator $-d_w\Delta+\beta$ (resp. $-d_g\Delta+1$) in $L_2(\Omega)$ under the homogeneous Neumann conditions on $\partial\Omega$. The domain of A is given by

$$\mathcal{D}(A)=L_\infty(\Omega)\times L_\infty(\Omega)\times H^2_N(\Omega)\times H^2_N(\Omega), \tag{4.4}$$

because of (3.6). As A is diagonal, A is easily seen to be a sectorial operator of X with angle 0, namely, its spectrum is contained in the half real line $(0,\infty)$ and its resolvent satisfies the estimate $\|(z-A)^{-1}\|_{\mathcal{L}(X)}\le \frac{M}{|z|+1}$ for $z\notin(0,\infty)$ with some constant $M>0$. Consequently, $-A$ generates an analytic semigroup $e^{-tA}\,(0\le t<\infty)$ on X which is represented by $e^{-tA}=\mathrm{diag}\{e^{-tf},e^{-th},e^{-t\Lambda_w},e^{-t\Lambda_g}\}$.

Similarly, for $0<\theta<1$, the fractional power A^θ of A is represented by

$$A^\theta=\mathrm{diag}\{f^\theta,h^\theta,\Lambda_w^\theta,\Lambda_g^\theta\}\ \text{ with }\ \mathcal{D}(A^\theta)=L_\infty(\Omega)\times L_\infty(\Omega)\times\mathcal{D}(\Lambda_w^\theta)\times\mathcal{D}(\Lambda_g^\theta), \tag{4.5}$$

(as for $\mathcal{D}(\Lambda_w^\theta)$ and $\mathcal{D}(\Lambda_g^\theta)$, see (3.7)).

In the meantime, $F(U)$ denotes a nonlinear operator of X of the form

$$F(U)\equiv\begin{pmatrix}\beta\delta[\mathrm{Re}\,w-w_*]_+-(\lambda g+av^2+c)u\\ fu\\ \alpha v\\ -\mu vg+\gamma(g-\ell)(1-g)g+g\end{pmatrix},\qquad U=\begin{pmatrix}u\\ v\\ w\\ g\end{pmatrix}\in\mathcal{D}(F), \tag{4.6}$$

where $\mathcal{D}(F)=[L_\infty(\Omega)]^4$. In what follows we fix an exponent ϑ arbitrarily so that

$$\tfrac12<\vartheta<\tfrac34. \tag{4.7}$$

Then, on account of (3.5), (3.8) and (4.5), we see that $\mathcal{D}(A^\vartheta)\subset\mathcal{D}(F)$ with continuous embedding. In addition, since the entries of $F(U)$ are a polynomial of u,v,w,g of at most third order except the term $[\mathrm{Re}\,w-w_*]_+$ and since $[\mathrm{Re}\,w-w_*]_+$ is Lipschitz continuous for $w\in\mathbb{C}$, it is directly verified that

$$\|F(U)-F(V)\|_{L_\infty}\le C(\|U\|_{L_\infty}+\|V\|_{L_\infty}+1)^2\|U-V\|_{L_\infty}$$
$$\text{for }\ U={}^t(u_1,v_1,w_1,g_1),\ V={}^t(u_2,v_2,w_2,g_2)\in\mathcal{D}(F),$$

with some constant $C>0$. This then readily implies that

$$\|F(U)-F(V)\|_X\le C(\|A^\vartheta U\|_X+\|A^\vartheta V\|_X+1)^2\|A^\vartheta(U-V)\|_X,\qquad U,V\in\mathcal{D}(A^\vartheta). \tag{4.8}$$

Finally, U_0 denotes an initial value which is taken from $\mathcal{D}(A^\vartheta)$.

We can then conclude the following result.

Theorem 4.1. *Under* (4.7) *let $U_0={}^t(u_0,v_0,w_0,g_0)$ be in $\mathcal{D}(A^\vartheta)$, i.e., $u_0,v_0\in L_\infty(\Omega)$ and $w_0,g_0\in H^{2\vartheta}(\Omega)$. Then,* (4.2) *(and hence* (1.1)*) possesses a unique local solution in the function space:*

$$\begin{cases}u,\ v\in\mathcal{C}([0,T_{U_0}];L_\infty(\Omega))\cap\mathcal{C}^1((0,T_{U_0}];L_\infty(\Omega)),\\ w,\ g\in\mathcal{C}([0,T_{U_0}];H^{2\vartheta}(\Omega))\cap\mathcal{C}^1((0,T_{U_0}];L_2(\Omega))\cap\mathcal{C}((0,T_{U_0}];H^2_N(\Omega)),\\ t^{1-\vartheta}w,\ t^{1-\vartheta}g\in\mathcal{B}((0,T_0];H^2_N(\Omega)).\end{cases} \tag{4.9}$$

Here, $T_{U_0}>0$ is determined only by the norm

$$\|u_0\|_{L_\infty}+\|v_0\|_{L_\infty}+\|w_0\|_{H^{2\vartheta}}+\|g_0\|_{H^{2\vartheta}} \tag{4.10}$$

of the initial value U_0.

Proof. The fundamental existence theorem [15, Theorem 4.1] (presented first in [7]) is available. Indeed, (4.8) shows that the structural assumption [15, (4.2)] is fulfilled with $\beta = \eta = \vartheta$. Therefore, it is concluded that (4.2) possesses a unique local solution in the function space:

$$\begin{cases} U \in \mathcal{C}([0, T_{U_0}]; \mathcal{D}(A^\vartheta)) \cap \mathcal{C}^1((0, T_{U_0}]; X) \cap \mathcal{C}((0, T_{U_0}]; \mathcal{D}(A)), \\ t^{1-\vartheta} U \in \mathcal{B}((0, T_{U_0}]; \mathcal{D}(A)), \end{cases}$$

$T_{U_0} > 0$ being determined by the norm $\|A^\vartheta U_0\|_X$ alone.

Hence, in view of (4.4), each entry of the solution $U(t) = {}^t(u(t), v(t), w(t), g(t))$ belongs to the function space (4.9). From (3.8) and (4.5) it seen that

$$\begin{aligned} C^{-1}(\|u\|_{L_\infty} + \|v\|_{L_\infty} + \|w\|_{H^{2\vartheta}} + \|g\|_{H^{2\vartheta}}) &\le \|A^\vartheta U\|_X \\ &\le C(\|u\|_{L_\infty} + \|v\|_{L_\infty} + \|w\|_{H^{2\vartheta}} + \|g\|_{H^{2\vartheta}}), \qquad U = {}^t(u, v, w, g) \in \mathcal{D}(A^\vartheta). \end{aligned}$$

Hence, T_{U_0} is determined by the norm of (4.10). $\square$

5 Nonnegativity of solutions We shall next verify that nonnegativity of initial functions implies that of the local solution obtained in Theorem 4.1.

Theorem 5.1. *Under* (4.7) *let* $U_0 = {}^t(u_0, v_0, w_0, g_0) \in \mathcal{D}(A^\vartheta)$ *satisfy* $u_0 \ge 0$, $v_0 \ge 0$, $w_0 \ge 0$ *and* $g_0 \ge 0$ *in* Ω. *Then, the local solution* $U(t) = {}^t(u(t), v(t), w(t), g(t))$ *constructed in Theorem 4.1 is also nonnegative, i.e.,* $u(t) \ge 0$, $v(t) \ge 0$, $w(t) \ge 0$ *and* $g(t) \ge 0$ *in* Ω *for every* $0 < t \le T_{U_0}$.

Proof. We want to introduce an auxiliary problem

$$(5.1) \quad \begin{cases} \dfrac{\partial u}{\partial t} = \beta\delta[\operatorname{Re} w - w_*]_+ - [\lambda\chi(\operatorname{Re} g) + av^2 + c]u - fu & \text{in } \Omega \times (0, \infty), \\ \dfrac{\partial v}{\partial t} = fu - hv & \text{in } \Omega \times (0, \infty), \\ \dfrac{\partial w}{\partial t} = d_w \Delta w - \beta w + \alpha v & \text{in } \Omega \times (0, \infty), \\ \dfrac{\partial g}{\partial t} = d_g \Delta g - \mu v g + \gamma(g - \ell)(1 - g)g & \text{in } \Omega \times (0, \infty), \\ \dfrac{\partial w}{\partial n} = \dfrac{\partial g}{\partial n} = 0 & \text{on } \partial\Omega \times (0, \infty), \\ u(x, 0) = u_0(x),\ v(x, 0) = v_0(x),\ w(x, 0) = w_0(x),\ g(x, 0) = g_0(x) & \text{in } \Omega. \end{cases}$$

Here, $\chi(g')$ is a cutoff function for $-\infty < g' < \infty$ given by

$$\chi(g') = \begin{cases} g' & \text{if } g' \ge 0, \\ 0 & \text{if } g' < 0. \end{cases}$$

Since $\chi(\operatorname{Re} g)$ is a uniformly Lipschitz continuous function of $g \in \mathbb{C}$, it is possible to construct a local solution to (5.1) on an interval $[0, \tilde{T}_{U_0}]$ which lies in the same function space as (4.9) and is unique in the function space. Furthermore, the arguments as in the proof of [2, Theorem 4.1] (cf. also [15, Subsec. 11.2.3]) are available to conclude that the local solution satisfies that $u(t) \ge 0$, $v(t) \ge 0$ and $w(t) \ge 0$ in Ω for every $0 < t \le \tilde{T}_{U_0}$.

So, let us here verify that $g(t) \geq 0$ in Ω for every $0 < t \leq \tilde{T}_{U_0}$. First, we notice that, since the function $(u(t), v(t), w(t), \overline{g}(t))$ is also a local solution of (5.1), uniqueness of solution implies that $(u(t), v(t), w(t), \overline{g}(t)) = (u(t), v(t), w(t), g(t))$ for every $0 < t \leq \tilde{T}_{U_0}$. In particular, $g(t)$ is a real valued function of $x \in \Omega$ for each t. Second, in view of this fact, we shall use another cutoff function. Let $H(g)$ be a $\mathcal{C}^{1,1}$ function such that $H(g) = \frac{g^2}{2}$ for $-\infty < g < 0$ and $H(g) = 0$ for $0 \leq g < \infty$. We consider the function

$$\psi(t) = \int_\Omega H(g(x,t))dx, \qquad 0 \leq t \leq \tilde{T}_{U_0}.$$

Clearly, $\psi(t)$ is a nonnegative $\mathcal{C}^1$ function with the derivative

$$\begin{aligned}\psi'(t) = \int_\Omega H'(g(t))\frac{dg}{dt}(t)dx &= \int_\Omega H'(g(t))d_g\Delta g(t)dx \\ &\quad + \int_\Omega H'(g(t))\left[-\mu v(t) + \gamma(g(t)-\ell)(1-g(t))\right]g(t)dx.\end{aligned}$$

Consequently, there is a constant $C_U > 0$ depending on $U(t)$ such that

$$\psi'(t) \leq -d_g\int_\Omega H''(g(t))|\nabla g(t)|^2dx + C_U\int_\Omega H'(g(t))g(t)dx, \qquad 0 < t \leq \tilde{T}_{U_0}.$$

Since $H''(g) \geq 0$ and $H'(g)g = 2H(g)$ for $g \in \mathbb{R}$, it follows that $\psi'(t) \leq 2C_U\psi(t)$. Hence, $0 \leq \psi(t) \leq e^{2C_U t}\psi(0)$ for every $0 < t \leq \tilde{T}_{U_0}$. Finally, $g_0 \geq 0$ implies $\psi(0) = 0$ and hence $\psi(t) = 0$ for every t, i.e., $g(t) \geq 0$ in Ω.

We have thus seen that the local solution to the auxiliary problem (5.1) is nonnegative. This in turn shows that the local solution is as well a local solution of (1.1) (because of $\chi(\operatorname{Re} g(t)) = g(t)$). Uniqueness of local solution for (1.1) then yields that the local solution for (1.1) obtained by Theorem 4.1 coincides with that of (5.1) on the interval $[0, \tilde{T}_{U_0}]$. This means that the assertion of theorem is verified at least on the time interval $[0, \tilde{T}_{U_0}]$.

Consider the time $t_1 = \sup\{0 < t \leq T_{U_0};\ U(s)$ is nonnegative for any $s \in [0,t]\}$. And suppose that $t_1 < T_{U_0}$. Then we can repeat the similar arguments with initial time t_1 and initial value $U_1 = U(t_1)$ to conclude that $U(t)$ is nonnegative for all $t > t_1$ which are sufficiently close to t_1. But this contradicts the definition of the time t_1. Hence, $t_1 = T_{U_0}$. □

6 Global solutions Let us first build up a priori estimates for the local solutions of (1.1).

Proposition 6.1. *Under* (4.7) *let* $0 \leq u_0,\ v_0 \in L_\infty(\Omega)$ *and* $0 \leq w_0,\ g_0 \in H^{2\vartheta}(\Omega)$. *Let* $U = (u, v, w, g)$ *denote any local solution of* (1.1) *on an interval* $[0, T_U]$ *such that*

$$\begin{cases} 0 \leq u,\ v \in \mathcal{C}([0,T_U]; L_\infty(\Omega)) \cap \mathcal{C}^1((0,T_U]; L_\infty(\Omega)), \\ 0 \leq w,\ g \in \mathcal{C}([0,T_U]; H^{2\vartheta}(\Omega)) \cap \mathcal{C}^1((0,T_U]; L_2(\Omega)) \cap \mathcal{C}((0,T_U]; H^2_N(\Omega)). \end{cases} \tag{6.1}$$

Then, the estimate

$$\begin{aligned}&\|u(t)\|_{L_\infty} + \|v(t)\|_{L_\infty} + \|w(t)\|_{H^{2\vartheta}} + \|g(t)\|_{H^{2\vartheta}} \\ &\qquad \leq C\left[\|u_0\|_{L_\infty} + \|v_0\|_{L_\infty} + \|w_0\|_{H^{2\vartheta}} + \|g_0\|^2_{H^{2\vartheta}} + 1\right], \qquad 0 \leq t \leq T_U,\end{aligned} \tag{6.2}$$

holds with some constant C *independent of* T_U.

Proof. Throughout the proof, we shall use a universal notation C to denote positive constants which are determined by the constants d_w, d_g, a, c, f, h, α, β, γ, δ, λ, μ, w_* and ℓ and by Ω. So, C may change from occurrence to occurrence.

Step 1. Let us first estimate the norms $\|u(t)\|_{L_2}$, $\|v(t)\|_{L_2}$ and $\|w(t)\|_{L^2}$. Multiply the first equation of (1.1) by u and integrate the product in Ω. Then, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega u^2dx + f\int_\Omega u^2dx &= \beta\delta\int_\Omega [w-w_*]_+u\,dx - \int_\Omega(\lambda g + av^2 + c)u^2dx\\ &\le \frac{f}{2}\int_\Omega u^2dx + \frac{(\beta\delta)^2}{2f}\int_\Omega (w^2+w_*^2)dx - a\int_\Omega u^2v^2dx.\end{aligned}$$

Therefore,

$$\frac{d}{dt}\int_\Omega u^2dx + f\int_\Omega u^2dx \le (\beta\delta)^2f^{-1}\int_\Omega (w^2+w_*^2)dx - 2a\int_\Omega u^2v^2dx. \tag{6.3}$$

Multiply the second equation of (1.1) by v and integrate the product in Ω. Then,

$$\frac{1}{2}\frac{d}{dt}\int_\Omega v^2dx + h\int_\Omega v^2dx = f\int_\Omega uv\,dx,$$

or

$$\frac{d}{dt}\int_\Omega v^2dx + 2h\int_\Omega v^2dx = 2f\int_\Omega uv\,dx. \tag{6.4}$$

Finally, multiply the third equation of (1.1) by w and integrate the product in Ω. Then,

$$\frac{1}{2}\frac{d}{dt}\int_\Omega w^2dx + \beta\int_\Omega w^2dx = -d_w\int_\Omega |\nabla w|^2dx + \alpha\int_\Omega vw\,dx \le \frac{\beta}{2}\int_\Omega w^2dx + \frac{\alpha^2}{2\beta}\int_\Omega v^2dx.$$

Therefore,

$$\frac{d}{dt}\int_\Omega w^2dx + \beta\int_\Omega w^2dx \le \alpha^2\beta^{-1}\int_\Omega v^2dx. \tag{6.5}$$

Introduce here two positive parameters ρ and η; and, multiply the inequalities (6.3) and (6.5) by ρ and η, respectively. Then, summing up the resulting inequalities and the equation (6.4), we obtain that

$$\begin{aligned}&\frac{d}{dt}\int_\Omega [\rho u^2 + v^2 + \eta w^2]dx + \int_\Omega [f\rho u^2 + 2hv^2 + \beta\eta w^2]dx\\ &\quad\le \int_\Omega [\alpha^2\beta^{-1}\eta v^2 + (\beta\delta)^2f^{-1}\rho w^2]dx + 2\int_\Omega [f(uv) - a\rho(uv)^2]dx + (\beta\delta w_*)^2f^{-1}|\Omega|\rho.\end{aligned} \tag{6.6}$$

Furthermore, fix $\eta > 0$ sufficiently small so that $\alpha^2\beta^{-1}\eta < 2h$ and then fix $\rho > 0$ sufficiently small so that $(\beta\delta)^2f^{-1}\rho < \beta\eta$. Then, as $f(uv) - a\rho(uv)^2 \le f^2(4a\rho)^{-1}$ for u, $v \ge 0$, it follows that

$$\frac{d}{dt}\int_\Omega [\rho u^2 + v^2 + \eta w^2]dx + \varepsilon\int_\Omega [\rho u^2 + v^2 + \eta w^2]dx \le [(\beta\delta w_*)^2f^{-1}\rho + f^2(2a\rho)^{-1}]|\Omega|$$

with some constant $\varepsilon > 0$. Solving this differential inequality, we conclude that

$$\rho\|u(t)\|_{L_2}^2 + \|v(t)\|_{L_2}^2 + \eta\|w(t)\|_{L_2}^2 \le C\left[e^{-\varepsilon t}(\rho\|u_0\|_{L_2}^2 + \|v_0\|_{L_2}^2 + \eta\|w_0\|_{L_2}^2) + 1\right],$$

or

$$\|u(t)\|_{L_2}^2+\|v(t)\|_{L_2}^2+\|w(t)\|_{L_2}^2 \le C_1\left[e^{-\varepsilon t}(\|u_0\|_{L_2}^2+\|v_0\|_{L_2}^2+\|w_0\|_{L_2}^2)+1\right], \qquad 0\le t\le T_U. \tag{6.7}$$

Step 2. The estimate (6.7) directly implies the estimate of $\|w(t)\|_{H^{2\vartheta}}$. In fact, it is known by (3.11) that $w(t)$ is represented by

$$w(t)=e^{-t\Lambda_w}w_0+\int_0^t e^{-(t-s)\Lambda_w}\alpha v(s)ds,$$

where Λ_w is a realization of $-d_w\Delta+\beta$ in $L_2(\Omega)$ under the homogeneous Neumann conditions on $\partial\Omega$ and where $e^{-t\Lambda_w}$ is the semigroup on $L_2(\Omega)$ generated by $-\Lambda_w$. Operating Λ_w^ϑ, we have

$$\Lambda_w^\vartheta w(t)=e^{-t\Lambda_w}[\Lambda_w^\vartheta w_0]+\int_0^t \Lambda_w^\vartheta e^{-\frac{t-s}{2}\Lambda_w}e^{-\frac{t-s}{2}\Lambda_w}\alpha v(s)ds.$$

Therefore, by (3.9) and (3.10),

$$\|\Lambda_w^\vartheta w(t)\|_{L_2}\le Ce^{-\beta t}\|\Lambda_w^\vartheta w_0\|_{L_2}+C\int_0^t (t-s)^{-\vartheta}e^{-\frac{\beta}{2}(t-s)}ds\max_{0\le s\le t}\|v(s)\|_{L_2}.$$

Since

$$\int_0^t (t-s)^{-\vartheta}e^{-\frac{\beta}{2}(t-s)}ds=\int_0^t \sigma^{-\vartheta}e^{-\frac{\beta}{2}\sigma}d\sigma<\int_0^\infty \sigma^{-\vartheta}e^{-\frac{\beta}{2}\sigma}d\sigma=\left(\frac{2}{\beta}\right)^{1-\vartheta}\Gamma(1-\vartheta),$$

we obtain by (6.7) that

$$\|\Lambda_w^\vartheta w(t)\|_{L_2}\le C\left[e^{-\frac{\beta}{2}t}\|\Lambda_w^\vartheta w_0\|_{L_2}+\|u_0\|_{L_2}+\|v_0\|_{L_2}+\|w_0\|_{L_2}+1\right].$$

Hence, in view of (3.8),

$$\|w(t)\|_{H^{2\vartheta}}\le C_2\left[e^{-\frac{\beta}{2}t}\|w_0\|_{H^{2\vartheta}}+\|u_0\|_{L_2}+\|v_0\|_{L_2}+\|w_0\|_{L_2}+1\right], \qquad 0\le t\le T_U. \tag{6.8}$$

Step 3. In view of (3.5), we see from (6.8) that

$$\|w(t)\|_{L_\infty}\le C[\|u_0\|_{L_2}+\|v_0\|_{L_2}+\|w_0\|_{H^{2\vartheta}}+1], \qquad 0\le t\le T_U.$$

By use of this, let us estimate the norms $\|u(t)\|_{L_\infty}$ and $\|v(t)\|_{L_\infty}$.

First, apply the formula (3.12) to the first equation of (1.1). Then, we have

$$u(t)=e^{-\int_0^t[\lambda g(s)+av(s)^2+c+f]ds}u_0+\int_0^t e^{-\int_s^t[\lambda g(\tau)+av(\tau)^2+c+f]d\tau}\beta\delta[w(s)-w_*]_+ds$$

in the space $L_\infty(\Omega)$. Therefore,

$$\|u(t)\|_{L_\infty}\le e^{-(c+f)t}\|u_0\|_{L_\infty}+\beta\delta\int_0^t e^{-(c+f)(t-s)}[\|w(s)\|_{L_\infty}+w_*]ds.$$

Hence,

$$\|u(t)\|_{L_\infty} \le C_3\left[e^{-(c+f)t}\|u_0\|_{L_\infty}+\|u_0\|_{L_2}+\|v_0\|_{L_2}+\|w_0\|_{H^{2\vartheta}}+1\right], \quad 0\le t\le T_U. \tag{6.9}$$

Second, the similar arguments yield from the second equation of (1.1) the estimate

$$\|v(t)\|_{L_\infty} \le C_4\left[e^{-ht}\|v_0\|_{L_\infty}+\|u_0\|_{L_\infty}+\|v_0\|_{L_2}+\|w_0\|_{H^{2\vartheta}}+1\right], \qquad 0\le t\le T_U. \tag{6.10}$$

Of course, (6.9) is used for estimating the integral $\int_0^t e^{-h(t-s)}u(s)ds$ in $L_\infty(\Omega)$.

Step 4. The estimates for the norms of $g(t)$ are carried out quite analogously. Let us first estimate the norm $\|g(t)\|_{L_6}$.

Multiply the fourth equation of (1.1) by $g(t)^5$ and integrate the product in Ω. Then, after some calculations, we have

$$\frac{1}{6}\frac{d}{dt}\int_\Omega g^6dx=-5d_g\int_\Omega g^4|\nabla g|^2dx-\mu\int_\Omega vg^6dx+\gamma\int_\Omega(g-\ell)(1-g)g^6dx.$$

In view of (3.1),

$$\frac{1}{6}\frac{d}{dt}\int_\Omega g^6dx+\frac{\gamma(1-\ell)}{6}\int_\Omega g^6dx\le\frac{\gamma(1-\ell)}{6}\int_\Omega dx=\frac{\gamma(1-\ell)}{6}|\Omega|.$$

Solving this differential inequality, we obtain that

$$\|g(t)\|_{L_6}^6\le e^{-\gamma(1-\ell)t}\|g_0\|_{L_6}^6+|\Omega|, \qquad 0\le t\le T_U. \tag{6.11}$$

Step 5. Regarding $g(t)$ as the solution to a linear evolution equation (i.e., the fourth equation of (1.1)), we describe $g(t)$ by the integral

$$g(t)=e^{-tA_g}g_0+\int_0^t e^{-(t-s)A_g}\left[-\mu v(s)+\gamma(g(s)-\ell)(1-g(s))+1\right]g(s)ds$$

in $L_2(\Omega)$ (due to (3.11)), where A_g is a realization of $-d_g\Delta+1$ in $L_2(\Omega)$ under the homogeneous Neumann conditions on $\partial\Omega$. Then, the similar arguments as in Step 2 yield that

$$\|A_g^\vartheta g(t)\|_{L_2}\le C\left[e^{-t}\|A_g^\vartheta g_0\|_{L_2}+\max_{0\le s\le t}\left\{\|v(s)g(s)\|_{L_2}+\|(1+g(s))^2g(s)\|_{L_2}\right\}\right].$$

Hence we obtain by (6.10) and (6.11) that

$$\|A_g^\vartheta g(t)\|_{L_2}\le C\left[e^{-t}\|A_g^\vartheta g_0\|_{L_2}+\|u_0\|_{L_\infty}+\|v_0\|_{L_\infty}+\|w_0\|_{H^{2\vartheta}}+\|g_0\|_{L_6}^3+1\right],$$

or due to (3.8),

$$\begin{aligned}\|g(t)\|_{H^{2\vartheta}}\le C_5\big[e^{-t}\|g_0\|_{H^{2\vartheta}}+\|u_0\|_{L_\infty}+\|v_0\|_{L_\infty}&\\ +\|w_0\|_{H^{2\vartheta}}+\|g_0\|_{L_6}^3+1\big],&\qquad 0\le t\le T_U.\end{aligned} \tag{6.12}$$

Combing (6.8), (6.9), (6.10) and (6.12), we conclude the desired estimate (6.2). □

As an immediate consequence of the a priori estimates above, we can prove existence and uniqueness of global solution for the problem (1.1).

Theorem 6.1. *Let ϑ be as in* (4.7)*, and let $0 \le u_0,\, v_0 \in L_\infty(\Omega)$ and $0 \le w_0,\, g_0 \in H^{2\vartheta}(\Omega)$. Then,* (1.1) *possesses a unique global solution in the function space:*

$$\begin{cases} 0 \le u,\, v \in \mathcal{C}([0,\infty); L_\infty(\Omega)) \cap \mathcal{C}^1((0,\infty); L_\infty(\Omega)), \\ 0 \le w,\, g \in \mathcal{C}([0,\infty); H^{2\vartheta}(\Omega)) \cap \mathcal{C}^1((0,\infty); L_2(\Omega)) \cap \mathcal{C}((0,\infty); H^2_N(\Omega)). \end{cases}$$

Of course, the global solution satisfies all the estimates (6.7)$\sim$(6.12) *on the interval* $[0,\infty)$.

Proof. First, by Theorems 4.1 and 5.1, there exists a unique local solution (u,v,w,g) to (1.1) on an interval $[0,T_{U_0}]$ which is nonnegative.

Second, consider any local solution of (1.1) on an interval $[0,T_U]$ in the function space (6.1). Then, Proposition 6.1 provides that the norm $\|A^\vartheta U(T_U)\|_X$ is estimated by $\|A^\vartheta U_0\|_X$ alone (independently of the end time T_U). We can then apply Theorems 4.1 and 5.1 with initial time T_U and initial value $U(T_U)$ and know that the solution (u,v,w,g) can be uniquely extended as a nonnegative local solution on an interval $[0,T_U+\tau]$, where $\tau > 0$ depends on the norm $\|A^\vartheta U(T_U)\|_X$ and hence depends only on the norm $\|A^\vartheta U_0\|_X$.

Thus, we have verified that any local solution on $[0,T_U]$ in the function space (6.1) can always be extended as a nonnegative local solution on a longer interval $[0,T_U+\tau]$ with a fixed length $\tau > 0$. This evidently means that the assertion of theorem is true. $\square$

Let us finally observe Lipschitz continuity of solutions $U(t)$ in the initial values U_0. Let B be a bounded set of $\mathcal{D}(A^\vartheta)$ such that

$$B_R = \{U_0 \in \mathcal{D}(A^\vartheta); \|A^\vartheta U_0\|_X \le R \text{ and } U_0 \ge 0\} \tag{6.13}$$

with radius $R > 0$. Then, there exists a unique global solution to (1.1) for each $U_0 \in B$. As a direct consequence of [15, Theorem 4.3], we observe the following result.

Proposition 6.2. *Let $U(t)$ (resp. $V(t)$) denote the global solution to* (1.1) *for initial value $U_0 \in B_R$ (resp. $V_0 \in B_R$). Then, for each fixed time $T > 0$, there exists some constants $C_{R,T} > 0$ depending on R and T alone such that*

$$\|A^\vartheta[U(t) - V(t)]\|_X \le C_{R,T}\|A^\vartheta[U_0 - V_0]\|_X \quad \textit{for any } 0 \le t \le T. \tag{6.14}$$

7 Dynamical system

This section is devoted to constructing a dynamical system generated by the problem (1.1). As for the phase space we set

$$K = \{U_0 \in \mathcal{D}(A^\vartheta);\ U_0 \ge 0\} \subset \mathcal{D}(A^\vartheta),$$

K being a metric space equipped with the distance induced by the norm $\|A^\vartheta \cdot\|_X$.

As shown by Theorem 6.1, for each $U_0 \in K$, there exists a unique global solution $U(t;U_0)$ of (1.1) with values in K. Therefore, we can define a nonlinear semigroup $\{S(t)\}_{0\le t<\infty}$ acting on K by the formula $S(t)U_0 = U(t;U_0)$. As shown by Proposition 6.2, $U_0 \mapsto U(t;U_0)$ is locally Lipschitz continuous from K into itself. Furthermore, according to (6.14), the Lipschitz constant is uniform on any bounded set B_R of K and on any finite interval $[0,T]$. It then easily follows that the mapping $(t,U_0) \mapsto S(t)U_0$ is continuous from $[0,\infty) \times K$ into K, namely, $S(t)$ is a continuous semigroup on K. Hence, (1.1) generates a dynamical system $(S(t), K, \mathcal{D}(A^\vartheta))$.

The a priori estimates (6.7)$\sim$(6.12) we have established in the proof of Proposition 6.2 provide existence of a bounded absorbing set of K.

Theorem 7.1. *The dynamical system $(S(t), K, \mathcal{D}(A^\vartheta))$ possesses a bounded, invariant and absorbing subset $\widetilde{B}$ of K.*

Proof. Let $R > 0$ and let B_R be a bounded subset of the form (6.13). Let $U_0 \in B_R$ be any initial value and put $S(t)U_0 = {}^t(u(t), v(t), w(t), g(t))$.

From (6.7) we see that there is a time $t_1 > 0$ depending only on R such that

$$\|u(t)\|_{L_2}^2 + \|v(t)\|_{L_2}^2 + \|w(t)\|_{L_2}^2 \le 2C_1, \qquad t_1 \le \forall t < \infty.$$

Apply (6.8) to $w(t)$ but with initial time t_1 and initial value $S(t_1)U_0$. Then,

$$\|w(t)\|_{H^{2\vartheta}} \le C_2\Big[e^{-\frac{\beta}{2}(t-t_1)}\|w(t_1)\|_{H^{2\vartheta}} \\ + \|u(t_1)\|_{L_2} + \|v(t_1)\|_{L_2} + \|w(t_1)\|_{L_2} + 1\Big], \qquad t_1 \le \forall t < \infty.$$

From this we see that there is a time $t_2 > t_1$ depending only on R such that

$$\|w(t)\|_{H^{2\vartheta}} \le C_2\left[\sqrt{3}\sqrt{2C_1} + 2\right], \qquad t_2 \le \forall t < \infty.$$

We repeat the similar arguments by using (6.9)∼(6.12) to see ultimately that there is a time $T_R > 0$ depending only on R such that

$$\|A^\vartheta S(t)U_0\|_X \le \widetilde{C}, \qquad t_R \le \forall t < \infty,$$

here $\widetilde{C} > 0$ is a suitable universal constant determined by C_1, C_2, C_3, C_4, C_5 alone.

Set $B \equiv B_{\widetilde{C}} = \{U \in K;\ \|A^\vartheta U\|_X \le \widetilde{C}\}$. Then, as shown above, B is an absorbing set of $(S(t), K, \mathcal{D}(A^\vartheta))$. Since B itself is a bounded subset of K, there is a time $t_{\widetilde{C}}$ such that $S(t)B \subset B$ for all $t \ge t_{\widetilde{C}}$. We then set

$$\widetilde{B} = \bigcup_{0 \le t < \infty} S(t)B = \bigcup_{0 \le t \le t_{\widetilde{C}}} S(t)B.$$

It is clear that $\widetilde{B}$ is an invariant set. Since $B \subset \widetilde{B}$, $\widetilde{B}$ is also an absorbing set. Proposition 6.1 means that $\widetilde{B}$ is a bounded subset. Hence, $\widetilde{B}$ is a subset to be constructed. □

Let us now consider the ω-limit set. For each global solution $S(t)U_0$, its ω-limit set is usually defined by

$$\omega(U_0) = \bigcap_{0 \le t < \infty} \overline{\{S(\tau)U_0;\ t \le \tau < \infty\}} \qquad \text{(closure in the topology of } K).$$

In the present case, however, the trajectory $\{S(t)U_0;\ 0 \le t < \infty\}$ is not necessarily a relatively compact set of K. So, $\omega(U_0)$ may be an empty set in general. So, we will introduce another ω-limit set with respect to some weak topology of K.

We introduce the weak* topology of K: a sequence $\{(u_n, v_n, w_n, g_n)\}$ in K is said to be weak* convergent to $(\overline{u}, \overline{v}, \overline{w}, \overline{g})$ as $n \to \infty$ if

$$\begin{cases} u_n \to \overline{u} \ \text{ and } \ v_n \to \overline{v} & \text{weak* in } \ L_\infty(\Omega), \\ w_n \to \overline{w} \ \text{ and } \ g_n \to \overline{g} & \text{weakly in } \ H^{2\vartheta}(\Omega). \end{cases}$$

The weak* ω-limit set of $S(t)U_0$ is then defined by

$$\text{w}^*\text{-}\omega(U_0) = \left\{\overline{U} \in K;\ \exists t_n \nearrow \infty \ \text{ such that } \ S(t_n)U_0 \to \overline{U} \ \text{ in weak* topology}\right\}. \tag{7.1}$$

Theorem 7.2. *For each $U_0 \in K$, w^*-$\omega(U_0)$ is not an empty set.*

Proof. Put $S(t)U_0 = {}^t(u(t), v(t), w(t), g(t))$. Since $\{(u(t), v(t)); 0 \leq t < \infty\}$ is a bounded subset of $L_\infty(\Omega) \times L_\infty(\Omega)$, the Banach-Alaoglu theorem [11, p. 65] guarantees the trajectory $S(t)U_0$ to contain a sequence $(u(t_n), v(t_n))$, where $t_n \nearrow \infty$, which converges to $(\overline{u}, \overline{v})$ in the weak* topology of $L_\infty(\Omega) \times L_\infty(\Omega)$. It is easy to see that $\overline{u} \geq 0$ and $\overline{v} \geq 0$ in Ω. On the other hand, as $H^{2\vartheta}(\Omega)$ is a Hilbert space, $(w(t_n), g(t_n))$ contains a subsequence $(w(t_{n'}), g(t_{n'}))$ which is convergent to $(\overline{w}, \overline{g})$ in the weak topology of $H^{2\vartheta}(\Omega) \times H^{2\vartheta}(\Omega)$. It is clear that $\overline{w} \geq 0$ and $\overline{g} \geq 0$ in Ω. Hence, as $n' \to \infty$, $S(t_{n'})U_0$ is weak* convergent to $\overline{U} = {}^t(\overline{u}, \overline{v}, \overline{w}, \overline{g}) \in K$. Then, by the definition (7.1), we conclude that $\overline{U}$ belongs to w*-$\omega(U_0)$. □

8 Numerical Examples We conclude this paper by presenting some numerical results. These results show that our problem (1.1) can actually admit some coexisting solutions of trees and grass together with the boundary which divides forest and grassland.

Throughout the numerical computations, the domain is set as $\Omega = (0, 1) \times (0, 1)$. The constants in (1.1) are fixed as $d_w = 0.1$, $d_g = 1 \times 10^{-6}$, $a = 1$, $c = 0$, $f = 1$, $h = 0.5$, $\alpha = \beta = 1$, $\gamma = 40$, $\delta = 1$, $\lambda = 9$, $\mu = 50$, $w_* = 0.1$ and $\ell = 0.1$.

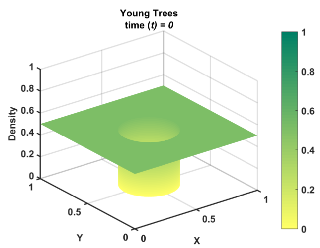
As in Figure 1, the initial functions $u_0(x)$, $v_0(x)$, $w_0(x)$ and $g_0(x)$ are taken as

$$u_0(x),\ v_0(x) \ \text{and}\ w_0(x) \equiv \begin{cases} 0 & \text{for } x \in B(x_0; r), \\ 0.5 & \text{for } x \in \Omega - B(x_0; r), \end{cases}$$

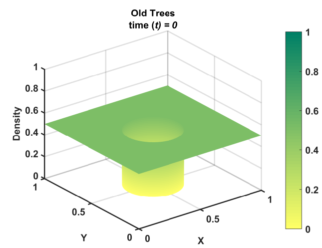
$$g_0(x) \equiv \begin{cases} 0.1 & \text{for } x \in B(x_0; r), \\ 0 & \text{for } x \in \Omega - B(x_0; r), \end{cases}$$

where x_0 denotes the central point $(0.5, 0.5)$ of Ω and $0 < r < 0.5$ denotes a radius of disk to be adjusted in our simulations. Starting from such initial functions, computations are continued until the approximate solution is stabilized numerically (almost $T = 1000$).

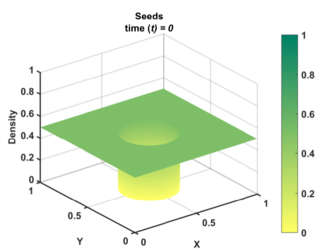
When $r = 0.1$, the solution tends to a state of homogeneous forest and no grass, see Figure 2. When $r = 0.2$ the solution tends to a coexisting state of trees and grass, see Figure 3. Finally, when $r = 0.3$, the solution tends to a state of homogeneous grass and no trees, see Figure 4.



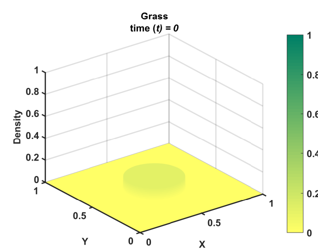
(a) Graph of young trees



(b) Graph of old trees

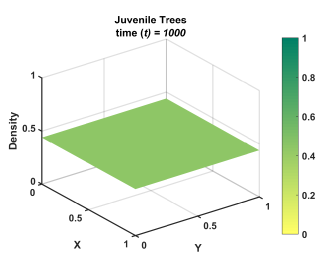


(c) Graph of seeds

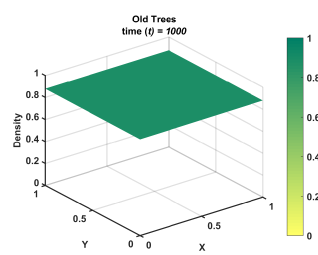


(d) Graph of grass

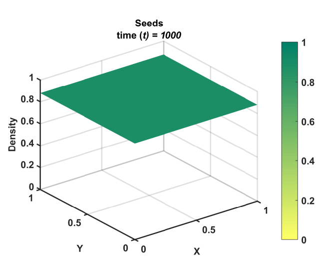
Fig. 1: Initial function.



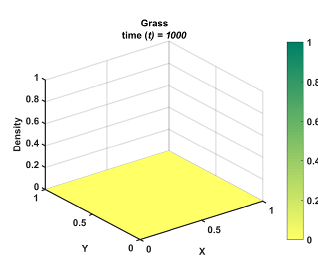
(a) Graph of young trees



(b) Graph of old trees

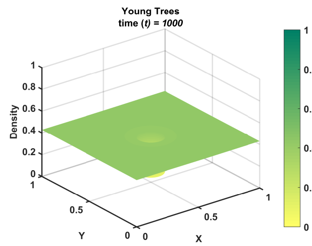


(c) Graph of seeds

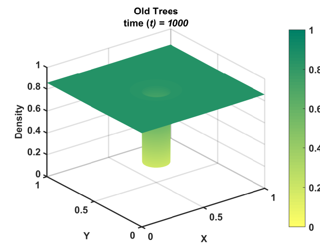


(d) Graph of grass

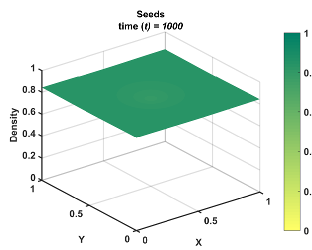
Fig. 2: When $r = 0.1$, the solution tends to a state of homogeneous forest and no grass.



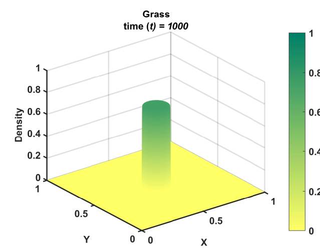
(a) Graph of young trees



(b) Graph of old trees



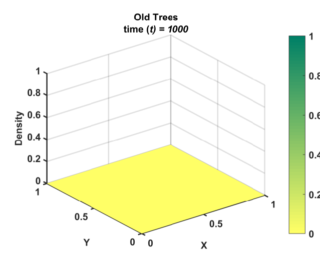
(c) Graph of seeds



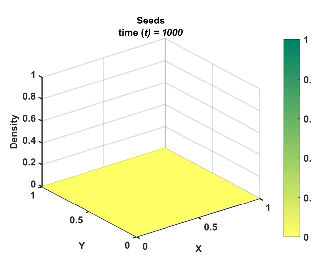
(d) Graph of grass

Fig. 3: When $r = 0.2$ the solution tends to a coexisting state of trees and grass.

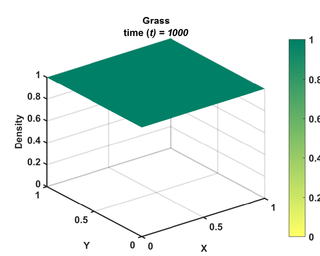
(a) Graph of young trees



(b) Graph of old trees



(c) Graph of seeds



(d) Graph of grass

Fig. 4: When $r = 0.3$, the solution tends to a state of homogeneous grass and no trees.

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