PATTERN FORMATION FOR SELF-REGUIATING HOMEOSTASIS MODEL IN A RECTANGLE

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Abstract. We continue the study on two-dimensional self-regulating homeostasis models. In the previous paper [4], after introducing a homeostasis model on a sphere, we showed global existence of solutions and constructed exponential attractors for the dynamical system generated by the model. We furthermore showed by numerical computations that white daisy and black daisy perform very clear segregation patterns on the sphere.

This paper is then devoted to investigating more on this pattern formation in a rectangular domain. We show that the competition of white and black daisies and the interaction with temperature create several types of segregation patterns and bring homeostasis of the global temperature to the planet.

1 Introduction We continue the study on two-dimensional self-regulating homeostasis models. In the previous paper [4], after introducing a homeostasis model on a sphere on the basis of the classical work Watson-Lovelock [6], we showed global existence of solutions and constructed exponential attractors for the dynamical system generated by the model. We furthermore showed by numerical computations that white daisy and black daisy perform very clear segregation patterns on the sphere. This paper is then devoted to investigating more on this pattern formation.

We consider the following reaction diffusion system

$$
\begin{cases}
\dfrac{\partial u}{\partial t} = d\Delta u + [(1-u-v)\Phi(u,v,w) - f]\, u & \text{in} \quad \Omega \times (0,\infty), \\
\dfrac{\partial v}{\partial t} = d\Delta v + [(1-u-v)\Psi(u,v,w) - f]\, v & \text{in} \quad \Omega \times (0,\infty), \\
\dfrac{\partial w}{\partial t} = D\Delta w + [1-g(u,v)]R - \sigma w^4 & \text{in} \quad \Omega \times (0,\infty),
\end{cases}
\tag{1.1}
$$

in a rectangular domain $\Omega = (-\ell_x, \ell_x) \times (0, \ell_y)$, where $0 < \ell_x,\ \ell_y < \infty$. As in [4], the variables $u = u(x,y,t)$ and $v = v(x,y,t)$ denote the coverage rate of white and black daisy, respectively, at position $(x,y) \in \Omega$ and time t. Therefore, $u \geq 0$, $v \geq 0$ and $u + v \leq 1$ at any (x,y,t), and $1 - u - v$ denotes a rate of uncovered ground. The third state variable $w = w(x,y,t)$ denotes a surface temperature. We assume that u and v satisfy a diffusion equation on Ω with diffusion rate $d > 0$. It is the same for w with diffusion rate $D > 0$. The function $g(u,v)$ stands for an averaged albedo of the surface that is given at each point as a function of u, v in the form

$$
g(u,v) = a_w u + a_b v + a_g(1-u-v) = (a_w - a_g)u + (a_b - a_g)v + a_g,
\tag{1.2}
$$

where a_w, a_b and a_g denote the proper albedo of white daisy, black daisy and bare ground, respectively. In general, we have $0 < a_b < a_g < a_w < 1$; as a consequence, it is always the case that

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$a_b \le g(u,v) \le a_w$. Furthermore, $\Phi(u,v,w)$ and $\Psi(u,v,w)$ denote a growth rate of white and black daisy, respectively. According as [6], we set

$$\Phi(u,v,w) = \{1 - \delta(\overline{w} - w - q[g(u,v) - a_w])^2\}_+,$$
$$\Psi(u,v,w) = \{1 - \delta(\overline{w} - w - q[g(u,v) - a_b])^2\}_+.$$

Here, $\overline{w}$ is a fixed optimal temperature for growing for both white daisy and black daisy. The term $q[g(u,v) - a_w]$ (resp. $q[g(u,v) - a_b]$) means some suitable adjustment on a local temperature to the global one w at any position where white daisy (resp. black daisy) grows, $q > 0$ being some coefficient. Since $g(u,v) \le a_w$ (resp. $g(u,v) \ge a_b$), we see that w is always adjusted negatively (resp. positively) where white daisy (resp. black daisy) grows. The notation $\{w\}_+ = \max\{w, 0\}$ denotes a positive cutoff of the function w for $-\infty < w < \infty$; consequently, $\{1 - \delta(\overline{w} - w)^2\}_+$ is a positive cutoff of the square function $1 - \delta(\overline{w} - w)^2$ for $-\infty < w < \infty$, $\delta > 0$ being some coefficient. Both white daisy and black daisy die at a rate $f > 0$. Finally, the term $[1 - g(u,v)]R$ denotes an increasing rate of the global temperature which is determined by the averaged albedo $g(u,v)$ mentioned above and the incoming energy R from the sun which is assumed to be constant in Ω. And, the term $-\sigma w^4$ denotes a decaying rate of the temperature due to the Stefan-Boltzmann law, $\sigma > 0$ being the Stefan-Boltzmann constant of the surface.

We impose, as boundary conditions, the periodic conditions in x-variable and the homogeneous Neumann conditions in y-variable for all of u, v and w. That is,

$$\begin{cases} \zeta(-\ell_x, y, t) = \zeta(\ell_x, y, t) \quad \text{and} \quad \zeta_x(-\ell_x, y, t) = \zeta_x(\ell_x, y, t) \\ \qquad\qquad\qquad\qquad \text{on } \{-\ell_x, \ell_x\} \times (0, \ell_y) \times (0, \infty), \\ \zeta_y(x, 0, t) = \zeta_y(x, \ell_y, t) = 0, \qquad \text{on } (-\ell_x, \ell_x) \times \{0, \ell_y\} \times (0, \infty), \end{cases} \tag{1.3}$$

where ζ stands for u, v and w. Finally, the initial conditions are set as

$$u(x,y,0) = u_0(x,y), \ \ v(x,y,0) = v_0(x,y) \ \text{and} \ w(x,y,0) = w_0(x,y) \quad \text{in} \quad \Omega. \tag{1.4}$$

The main interest of the present paper is as mentioned above to investigate when homogeneous distribution of white and black daisies becomes unstable and how segregation patterns are created by the competition of two daisies and the interaction with global temperature. For this purpose we want to consider the case where (1.1) has a stationary solution which is homogeneous in the spatial variables (x,y). This is reason why we assume that the incoming energy R is constant with respect to the variables (x,y). (In [4], R depends on the latitude.) In addition, for simplicity, we want to consider (1.1) on the cylindrical surface instead of on the sphere. This is reason why we handle (1.1) in $(-\ell_x, \ell_x) \times (0, \ell_y)$ under the periodic-Neumann boundary conditions (1.3) on u, v and w. If R is constant, then similar results will be obtained for the problem (1.1) and (1.4) on the sphere.

Global solutions are constructed as in [4], although we have to prepare and use the Proposition 2.1 which may not be so standard. Construction of the dynamical system and its exponential attractors can be carried out in a quite analogous way as in [4]. In order to investigate stability and instability of the homogeneous stationary solutions, we will restrict our interest only to a typical case where the parameters in (1.1) are fixed as

$$a_b = \frac{1}{4}, \ a_g = \frac{1}{2}, \ a_w = \frac{3}{4}, \ q = 20, \ \delta = 3.265 \times 10^{-3},$$
$$f = 0.3, \ \overline{w} = 295.5 \ \text{and} \ \sigma = 5.67 \times 10^{-8},$$

except R that is treated as a tuning parameter. Such a setting is suggested by [6]. Then, it is proved that there is an interval (R_*, R^*) for R such that if $R \notin [R_*, R^*]$ there is no positive homogeneous stationary solution, meanwhile if $R \in (R_*, R^*)$ there is a unique one $U_* = {}^t(u_*, v_*, w_*)$. Furthermore, for

$$\varphi(u, v, w) = [(1 - u - v)\Phi(u, v, w) - f]\, u,$$
$$\psi(u, v, w) = [(1 - u - v)\Psi(u, v, w) - f]\, v,$$

it is proved that, if (u_*, v_*, w_*) satisfies

$$\varphi_u(u_*, v_*, w_*)\psi_v(u_*, v_*, w_*) \geq \varphi_v(u_*, v_*, w_*)\psi_u(u_*, v_*, w_*),$$

then U_* is stable, meanwhile if (u_*, v_*, w_*) satisfies

$$\varphi_u(u_*, v_*, w_*)\psi_v(u_*, v_*, w_*) < \varphi_v(u_*, v_*, w_*)\psi_u(u_*, v_*, w_*),$$

and if the diffusion coefficient D is sufficiently large with respect to the other d, then U_* becomes unstable. Roughly speaking, if the intra-species competition is stronger than the inter-species one at U_*, then U_* is stable. Meanwhile, if the intra-species competition is weaker than the inter-species one at U_* and if global temperature diffuses much faster than daisies, U_* loses its stability, that is, the diffusion driven instability takes place.

As the dynamical system possesses a finite-dimensional attractor, when U_* is unstable, the trajectories are attracted to some states of a finite number of freedoms which does not include the homogeneous state. This fact then suggests that some pattern might be created spontaneously by the white and black daisies. As a matter of fact, we find by numerical computations under suitably fixed diffusion coefficients d and D that some segregation patterns emerge and they change their types from homogeneous, spot, island and to labyrinth as R changes. On the other hand, the mean of the global temperature, i.e.,

$$W(\infty) = \frac{1}{|\Omega|} \iint_\Omega w(x, y, \infty)dxdy,$$

is observed to be stable during R changes in this range. In this way, the competition between two daisies and the interaction with global temperature create several types of segregation patterns of daisies, and simultaneously they bring the homeostasis of global temperature to the planet.

The mechanism of self-regulating homeostasis has already been studied by using zero and one-dimensional Daisyworld models. For a survey, we refer the reader to [4, Introduction].

2 Local Solutions

2.1 Laplacian under periodic-Neumann boundary conditions In order to formulate (1.1)-(1.3) in the space $L_2(\Omega)$, we have to define Δ as a linear operator of $L_2(\Omega)$ under the boundary conditions stated in (1.3).

For this purpose, we consider the sesquilinear form

$$a(u, v) = a \int_\Omega \nabla u \cdot \nabla \overline{v}\, dx + c \int_\Omega u\overline{v}\, dx, \qquad u,\, v \in V, \tag{2.1}$$

where a and c are positive constants, on the space

$$H^1_{\rm per}(\Omega) = \{u \in H^1(\Omega);\ u(-\ell_x, y) = u(\ell_x, y) \ \text{ in the interval } \ (0, \ell_y)\}. \tag{2.2}$$

As $u \in H^1(\Omega)$ implies $u_{|\partial\Omega} \in H^{\frac{1}{2}}(\partial\Omega) \subset L_2(\partial\Omega)$, the coincidence $u(-\ell_x, y) = u(\ell_x, y)$ is meaningful as a function of $L_2(0, \ell_y)$. Thereby, $H^1_{\rm per}(\Omega)$ is a closed subspace of $H^1(\Omega)$ and becomes a Hilbert space with the H^1-inner product. Of course, $H^1_{\rm per}(\Omega)$ is dense in $L_2(\Omega)$. Therefore,

$$H^1_{\rm per}(\Omega) \subset L_2(\Omega) \subset H^1_{\rm per}(\Omega)^*$$

defines a triplet of spaces. In the meantime, $a(u, v)$ given by (2.1) is continuous and coercive on $H^1_{\rm per}(\Omega)$. By the theory of variation (see Dautray-Lions [2]), $a(u, v)$ then determines a linear operator $\mathcal{A}$ by the formula $a(u, v) = \langle \mathcal{A}u, v\rangle_{H^1_{\rm per}{}^* \times H^1_{\rm per}}$ for all $u, v \in H^1_{\rm per}(\Omega)$. The operator $\mathcal{A}$ is seen to be a sectorial operator of $H^1_{\rm per}(\Omega)^*$ with the domain $\mathcal{D}(\mathcal{A}) = H^1_{\rm per}(\Omega)$ and is therefore regarded as a realization of $-a\Delta + c$ in the space $H^1_{\rm per}(\Omega)^*$.

The part of $\mathcal{A}$ in the space $L_2(\Omega)$ is defined by

$$\begin{cases} \mathcal{D}(A) = \{u \in H^1_{\rm per}(\Omega);\ \mathcal{A}u \in L_2(\Omega)\}, \\ Au = \mathcal{A}u. \end{cases}$$

In other words, $u \in \mathcal{D}(A)$ if and only if $a(u, v) = (f, v)$ for all $v \in H^1_{\rm per}(\Omega)$ with some $f \in L_2(\Omega)$. By the theory of variation, again, A is a densely defined linear operator of $L_2(\Omega)$. As $a(u, v)$ is symmetric, A is a positive definite self-adjoint operator of $L_2(\Omega)$. In the present case, we can characterize the domain $\mathcal{D}(A)$ as follows.

Proposition 2.1. *The domain $\mathcal{D}(A)$ is given by*

$$\mathcal{D}(A) = \{u \in H^2(\Omega);\ u \textit{ satisfies the conditions on } \partial\Omega \textit{ stated in } (1.3)\}. \tag{2.3}$$

Moreover, it holds true that

$$\|u\|_{H^2} \le C\|Au\|_{L_2}, \qquad u \in \mathcal{D}(A). \tag{2.4}$$

Proof. Let $u \in H^2(\Omega)$ satisfy (1.3) and let $v \in H^1_{\rm per}(\Omega)$ be any function. By integration by parts,

$$\iint_\Omega u_x \overline{v}_x \, dxdy = \int_0^{\ell_y} dy \int_{-\ell_x}^{\ell_x} u_x \overline{v}_x \, dx = \int_0^{\ell_y} dy \left\{ [u_x \overline{v}]_{x=-\ell_x}^{x=\ell_x} - \int_{-\ell_x}^{\ell_x} u_{xx} \overline{v} \, dx \right\}.$$

Here, the periodic conditions on u yield that

$$[u_x \overline{v}]_{x=-\ell_x}^{x=\ell_x} = u_x(\ell_x, y)\overline{v}(\ell_x, y) - u_x(-\ell_x, y)\overline{v}(-\ell_x, y) = 0 \quad \text{for a.e. } y \in (0, \ell_y).$$

Therefore, $\iint_\Omega u_x \overline{v}_x \, dxdy = -\iint_\Omega u_{xx} \overline{v} \, dxdy$. By the similar arguments, we have

$$\iint_\Omega u_y \overline{v}_y \, dxdy = \int_{-\ell_x}^{\ell_x} dx \left\{ [u_y \overline{v}]_{y=0}^{y=\ell_y} - \int_0^{\ell_y} u_{yy} \overline{v} \, dy \right\} = -\iint_\Omega u_{yy} \overline{v} \, dxdy.$$

In this way, we observe that $(\nabla u, \nabla v) = (-\Delta u, v)$. In view of (2.1), this in fact shows that $a(u, v) = (-a\Delta u + cu, v)$, hence $u \in \mathcal{D}(A)$ and $Au = -a\Delta u + cu$.

In order to prove that $u \in \mathcal{D}(A)$ implies $u \in H^2(\Omega)$, we will use a double Fourier expansion for the functions of $L_2(\Omega)$. For the variable $x \in (-\ell_x, \ell_x)$, we use an expansion by the base functions

$\cos \frac{m\pi}{\ell_x} x$ and $\sin \frac{m\pi}{\ell_x} x$ for $m = 0, 1, 2, \ldots$; for the variable $y \in (0, \ell_y)$, an expansion by the base functions $\cos \frac{n\pi}{\ell_y} y$ for $n = 0, 1, 2, \ldots$. Then, u can be expressed by the series

$$u = \sum_{m,n=0}^{\infty} \left[u_{mn} \cos \frac{m\pi}{\ell_x} x + v_{mn} \sin \frac{m\pi}{\ell_x} x \right] \cos \frac{n\pi}{\ell_y} y$$

with Fourier coefficients u_{mn} and v_{mn} determined by the base functions. And they satisfy $\sum_{m,n} |u_{mn}|^2 < \infty$ and $\sum_{m,n} |v_{mn}|^2 < \infty$. In the distribution sense, we observe that

$$-\Delta u = \sum_{m,n=0}^{\infty} \left[\left(\frac{m\pi}{\ell_x} \right)^2 + \left(\frac{n\pi}{\ell_y} \right)^2 \right] \left[u_{mn} \cos \frac{m\pi}{\ell_x} x + v_{mn} \sin \frac{m\pi}{\ell_x} x \right] \cos \frac{n\pi}{\ell_y} y.$$

If $u \in \mathcal{D}(A)$, then, since $\mathcal{C}_0^{\infty}(\Omega) \subset H^1_{\text{per}}(\Omega)$, the condition $\mathcal{A}u \in L_2(\Omega)$ implies that $-\Delta u = f \in L_2(\Omega)$. So, if f_{mn} and g_{mn} are the Fourier coefficients of f, then it follows that

$$u_{mn} = \left[\left(\frac{m\pi}{\ell_x} \right)^2 + \left(\frac{n\pi}{\ell_y} \right)^2 \right]^{-1} f_{mn} \quad \text{and} \quad v_{mn} = \left[\left(\frac{m\pi}{\ell_x} \right)^2 + \left(\frac{n\pi}{\ell_y} \right)^2 \right]^{-1} g_{mn}$$

for every $(m, n) \neq (0, 0)$. Furthermore, it follows that

$$\|u_{xx}\|^2_{L_2} + \|u_{xy}\|^2_{L_2} + \|u_{yy}\|^2_{L_2} \leq C \left(\sum_{m,n} |f_{mn}|^2 + \sum_{m,n} |g_{mn}|^2 \right) \leq C \|f\|^2_{L_2}. \tag{2.5}$$

Hence, u belongs to $H^2(\Omega)$.

Knowing that $u \in \mathcal{D}(A)$ implies $u \in H^2(\Omega)$, we can repeat the arguments above to conclude that the two integrals

$$\int_0^{\ell_y} [u_x \overline{v}]_{x=-\ell_x}^{x=\ell_x} \, dy = \int_0^{\ell_y} [u_x(\ell_x, y)\overline{v}(\ell_x, y) - u_x(-\ell_x, y)\overline{v}(-\ell_x, y)] dy,$$
$$\int_{-\ell_x}^{\ell_x} [u_y \overline{v}]_{y=0}^{y=\ell_y} \, dx = \int_{-\ell_x}^{\ell_x} [u_y(x, \ell_y)\overline{v}(x, \ell_y) - u_y(x, 0)\overline{v}(x, 0)] dx$$

must vanish for all $v \in H^1_{\text{per}}(\Omega)$. Remembering the definition (2.2), we verify that $u_x(\ell_x, y) - u_x(-\ell_x, y) = 0$ for a.e. $y \in (0, \ell_y)$ and $u_y(x, \ell_y) = u_y(x, 0) = 0$ for a.e. $x \in (-\ell_x, \ell_x)$, that is, u satisfies the boundary conditions of (1.3).

Finally, since $\|u\|_{H^1} \leq C\|Au\|_{L_2}$ is already known, (2.4) is immediately verified from (2.5). □

We have thus shown that A is a realization of $-a\Delta + c$ in $L_2(\Omega)$ under the periodic-Neumann boundary conditions stated in (1.3).

2.2 Abstract formulation Let us formulate the problems (1.1)-(1.4) as the Cauchy problem for an abstract evolution equation

$$\begin{cases} \dfrac{dU}{dt} + AU = F(U), & 0 < t < \infty, \\ U(0) = U_0, \end{cases} \tag{2.6}$$

in a Banach space X. As X we set the product L_2-space, i.e.,

$$X = \left\{ U = \begin{pmatrix} u \\ v \\ w \end{pmatrix};\ u \in L_2(\Omega),\ v \in L_2(\Omega),\ w \in L_2(\Omega) \right\}.$$

The operator A denotes an operator matrix acting in X given by $\operatorname{diag}\{A_d, A_d, A_D\}$, where A_d (resp. A_D) is the realization of $-d\Delta + f$ (resp. $-D\Delta + 1$) in $L_2(\Omega)$ under the boundary conditions stated in (1.3). Then, A is a positive definite self-adjoint operator of X. Of course the domain $\mathcal{D}(A)$ is characterized by (2.3).

The nonlinear operator $F(U)$ is defined from the reaction terms including in (1.1). However, in view of our modeling, we expect that the solutions must exist in the ranges of $u \geq 0$, $v \geq 0$, $u+v \leq 1$ and $0 \leq w \leq (R/\sigma)^{\frac{1}{4}}$. On account of this expectation on the ranges, we will define $F(U)$ as follows:

$$F(U) = \begin{pmatrix} H_1(1 - \operatorname{Re} u - \operatorname{Re} v)\, \Phi\big(H_1(\operatorname{Re} u), H_1(\operatorname{Re} v), H_2(\operatorname{Re} w)\big)\, H_1(\operatorname{Re} u) \\ H_1(1 - \operatorname{Re} u - \operatorname{Re} v)\, \Psi\big(H_1(\operatorname{Re} u), H_1(\operatorname{Re} v), H_2(\operatorname{Re} w)\big)\, H_1(\operatorname{Re} v) \\ [1 - g\big(H_1(\operatorname{Re} u), H_1(\operatorname{Re} v)\big)]R - \sigma H_2(\operatorname{Re} w)^4 + H_2(\operatorname{Re} w) \end{pmatrix}.$$

Here, $H_1(\xi)$ and $H_2(\xi)$ are cutoff functions defined by

$$H_1(\xi) = \begin{cases} 0, & -\infty < \xi \leq 0, \\ \xi, & 0 < \xi \leq 1, \\ 1, & 1 < \xi < \infty, \end{cases} \qquad H_2(\xi) = \begin{cases} 0, & -\infty < \xi \leq 0, \\ \xi & 0 < \xi \leq (R/\sigma)^{\frac{1}{4}}, \\ (R/\sigma)^{\frac{1}{4}}, & (R/\sigma)^{\frac{1}{4}} < \xi < \infty, \end{cases}$$

respectively.

Finally, the initial value U_0 is taken from the space

$$K = \left\{ U_0 = \begin{pmatrix} u_0 \\ v_0 \\ w_0 \end{pmatrix} \in X;\ u_0 \geq 0,\ v_0 \geq 0,\ u_0 + v_0 \leq 1, 0 \leq w_0 \leq \left(\frac{R}{\sigma}\right)^{\frac{1}{4}} \right\}, \tag{2.7}$$

K being thus the space of initial values.

2.3 Construction of local solutions Construction of the local solution to (2.6) is easily carried out if we employ the theory of abstract parabolic evolution equations.

In fact, it is clear that $H_1(\xi)$ and $H_2(\xi)$ are uniformly bounded and globally Lipschitz continuous functions for $-\infty < \xi < \infty$. Consequently, $\Phi(H_1(\operatorname{Re} u), H_1(\operatorname{Re} v), H_2(\operatorname{Re} w))$ and $\Psi(H_1(\operatorname{Re} u), H_1(\operatorname{Re} v), H_2(\operatorname{Re} w))$ are uniformly bounded and globally Lipschitz continuous functions for $(u, v, w) \in \mathbb{C}^3$. Therefore, it is easily verified that $F(U)$ is a bounded operator on X and satisfies the Lipschitz condition, i.e.,

$$\|F(U)\|_X \leq C_1, \qquad U \in X,$$
$$\|F(U) - F(V)\|_X \leq C_2 \|U - V\|_X, \qquad U, V \in X,$$

with suitable constants $C_i > 0\,(i = 1, 2)$.

It is then possible to apply [7, Theorem 4.4] to obtain that for any $U_0 \in X$, there exists a unique local solution to (2.6) in the function space:

$$U \in \mathcal{C}([0, T_{U_0}]; X) \cap \mathcal{C}^1((0, T_{U_0}]; X) \cap \mathcal{C}((0, T_{U_0}]; \mathcal{D}(A)).$$

In addition, the solution $U(t)$ satisfies the norm estimate

$$\|U(t)\|_X + t\|AU(t)\|_X \leq C_{U_0}, \qquad 0 < t \leq T_{U_0}. \tag{2.8}$$

Here, the constant C_{U_0} and the time $T_{U_0} > 0$ are determined by the norm $\|U_0\|_X$ alone.

Let us next prove that, if U_0 is in K, then the local solution $U(t)$ also takes values in K for every $0 < t \leq T_{U_0}$.

Proposition 2.2. *If $U_0 \in K$, then $U(t) \in K$ for every $0 < t \leq T_{U_0}$.*

Proof. It is easy to verify that the complex conjugate $\overline{U(t)}$ of $U(t)$ is also a local solution to (2.6). Uniqueness of solution yields that $U(t) = \overline{U(t)}$ for every $0 < t \leq T_{U_0}$, hence $U(t)$ is real valued.

First, let us see that $u(t) \geq 0$. For this purpose, we use the cutoff function given by $H(u) = \frac{1}{2}u^2$ for $-\infty < u < 0$ and $H(u) = 0$ for $0 \leq u < \infty$. Put $g(t) = \iint_\Omega H(u(x,y,t))dxdy$. Then, for $0 < t \leq T_{U_0}$,

$$\begin{aligned}\frac{dg}{dt}(t) = \iint_\Omega H'(u)\frac{\partial u}{\partial t}\,dxdy = d\iint_\Omega H'(u)\Delta u\,dxdy \\ + \iint_\Omega H'(u)\left[H_1(1-u-v)\Phi\big(H_1(u), H_1(v), H_2(w)\big)H_1(u) - fu\right]dxdy.\end{aligned}$$

Here, on account of $H'(u) \in H^1_{\text{per}}(\Omega)$, we observe that

$$\iint_\Omega H'(u)\Delta u\,dxdy = -\iint_\Omega \nabla H'(u)\cdot\nabla u\,dxdy = -\iint_\Omega H''(u)|\nabla u|^2 dxdy \leq 0.$$

Meanwhile, since $H'(u)H_1(u) = 0$ and $-H'(u)u \leq 0$ for all $-\infty < u < \infty$, it follows that $\frac{dg}{dt}(t) \leq 0$, i.e., $g(t) \leq g(0) = 0$. This means that $u(t) \geq 0$ for every $0 < t \leq T_{U_0}$

The same arguments for $v(t)$ conclude that $v(t) \geq 0$ for every $0 < t \leq T_{U_0}$.

Second, in order to see that $u(t) + v(t) \leq 1$, we notice that $z(t) = 1 - u(t) - v(t)$ is regarded as a solution to

$$\frac{\partial z}{\partial t} = d\Delta z - \left[\Phi\big(H_1(u), H_1(v), H_2(w)\big) + \Psi\big(H_1(u), H_1(v), H_2(w)\big)\right]H_1(z) + f[u+v].$$

We can then repeat the same arguments for $z(t)$ to conclude that $z(t) \geq 0$, i.e., $u(t) + v(t) \leq 1$ for every $0 < t \leq T_{U_0}$.

Third, let us observe that $0 \leq w(t) \leq (R/\sigma)^{\frac{1}{4}}$. But observation of the non negativity $w(t) \geq 0$ is the same as for $u(t)$ and $v(t)$. Putting $w_1(t) = (R/\sigma)^{\frac{1}{4}} - w(t)$, we notice that $w_1(t)$ is a solution to

$$\frac{\partial w_1}{\partial t} = D\Delta w_1 - \sigma[R/\sigma - H_2(w)^4] + Rg(u,v) + [w - H_2(w)].$$

Then, consider the function $h(t) = \iint_\Omega H(w_1(x,y,t))dxdy$ and differentiate it. Since $H'((R/\sigma)^{\frac{1}{4}} - w)[R/\sigma - H_2(w)^4] = 0$ and $H'((R/\sigma)^{\frac{1}{4}} - w)[w - H_2(w)] \leq 0$ for all $-\infty < w < \infty$, it follows that $\frac{dh}{dt}(t) \leq 0$, i.e., $h(t) \leq h(0) = 0$. Hence, $(R/\sigma)^{\frac{1}{4}} - w(t) \geq 0$ for every $0 < t \leq T_{U_0}$.

We have thus verified all the conditions in order that $U(t)$ lies in K. □

Once $U(t) \in K$, $U(t)$ actually satisfies that $H_1(u(t)) = u(t)$, $H_1(v(t)) = v(t)$, $H_1(1 - u(t) - v(t)) = 1 - u(t) - v(t)$ and $H_2(w(t)) = w(t)$. This means that the local solution $U(t)$ to (2.6) constructed above can be regard as a local solution to the original problem (1.1), (1.3) and (1.4), too.

3 Global Solutions and Dynamical System This section is devoted to constructing global solutions, a dynamical system generated by (2.6) and its exponential attractors. But the similar techniques used in [4] are available equally to the present problem.

3.1 Construction of global solutions It is immediate to construct a unique global solution to (2.6) for any initial value in K. In fact, let $U_0 \in K$. Then, Proposition 2.2 provides that the norm $\|U(t)\|_X$ remains uniformly bounded on the interval $[0, T_{U_0}]$. This then means that we can extend this local solution over some time interval $[0, T_{U_0} + \tau]$, $\tau > 0$ being determined by the norm $\|U(T_{U_0})\|_X$ alone. It is then possible to repeat such an extension, for any local solution of (2.6) (with this initial value U_0) takes its values in K for every t and the extended time interval $\tau > 0$ is taken uniformly.

Therefore, we obtain the following existence theorem.

Theorem 3.1. *For any $U_0 \in K$, (2.6) possesses a unique global solution lying in*

$$U \in \mathcal{C}([0,\infty); X) \cap \mathcal{C}^1((0,\infty); X) \cap \mathcal{C}((0,\infty); \mathcal{D}(A)).$$

The solution $U(t)$ takes its values in K for every $0 < t < \infty$ and satisfies the estimate

$$\|U(t)\|_X + t(1+t)^{-1}\|AU(t)\|_X \leq C_3, \qquad 0 < t < \infty, \tag{3.1}$$

with some constant $C_3 > 0$ which is uniform for the initial values from K.

Proof. It suffices to prove the estimate (3.1). We already know that (3.1) holds true locally in the interval $(0, \tau]$, where τ is the time interval mentioned above. We then reset an initial value $U_1 = U(\frac{\tau}{2}) \in K$ and apply (2.8) to this local solution. Then,

$$\|U(t)\|_X + (t - \tfrac{\tau}{2})\|AU(t)\|_X \leq C, \qquad \tau \leq t \leq \tfrac{3\tau}{2}.$$

This shows that (3.1) holds true in the extended interval $(0, \frac{3\tau}{2}]$. Repeating this procedure, we obtain (3.1) on the whole interval $(0, \infty)$. □

It is also verified that the global solution is Lipschitz continuous with respect to the initial value in K. But, as the proof is quite analogous to that of [4, Theorem 3.3], we state the following theorem without its proof.

Theorem 3.2. *Let U_0, $V_0 \in K$ and let $U(t)$ and $V(t)$ be the global solutions of (2.6) with initial values U_0 and V_0, respectively. Then,*

$$\|U(t) - V(t)\|_X \leq C_4 e^{\beta t}\|U_0 - V_0\|_X, \qquad 0 \leq t < \infty, \tag{3.2}$$

$$\sqrt{t}\|\nabla[U(t) - V(t)]\|_X \leq C_4 e^{\beta t}\|U_0 - V_0\|_X, \qquad 0 < t < \infty, \tag{3.3}$$

with some exponent $\beta > 0$ and some constant $C_4 > 0$ which are both uniform for the initial values from K.

3.2 Dynamical system By utilizing the theory of dynamical systems for semilinear abstract parabolic evolution equations (see [7, Section 6.5]), it is immediate to construct a dynamical system generated by (2.6) in the space X.

For $U_0 \in K$, let $U(t; U_0)$ denote the global solution of (2.6) and set

$$S(t)U_0 = U(t; U_0), \qquad 0 \leq t < \infty.$$

Then, $S(t)$ is a nonlinear semigroup acting on K, i.e., $S(0) = I$ and $S(t+s) = S(t)S(s)$ for $0 \le s, t < \infty$. Furthermore, $S(t)$ is seen to be continuous in the sense that $(t, U_0) \mapsto S(t)U_0$ is continuous from $[0, \infty) \times K$ into K. Indeed, due to (3.2), we have

$$\begin{aligned} \|S(s)V_0 - S(t)U_0\|_X &\le \|S(s)V_0 - S(s)U_0\|_X + \|S(s)U_0 - S(t)U_0\|_X \\ &\le e^{\beta s}\|V_0 - U_0\|_X + \|S(s)U_0 - S(t)U_0\|_X. \end{aligned}$$

Then, $(s, V_0) \to (t, U_0)$ implies $S(s)V_0 \to S(t)U_0$ in X.

The nonlinear semigroup $S(t)$ thus defines a dynamical system in the space X, which is denoted by $(S(t), K, X)$. The phase space K presented by (2.7) is a bounded, closed subset of X.

As well known (see Babin-Vishik [1] and Temam [5]), the dissipative estimate provides existence of the global attractor. Consider a subset B of K defined by

$$B = K \cap \{U \in \mathcal{D}(A); \|AU\|_X \le C_3 + 1\}.$$

Then, (3.1) means that there is a time T such that $S(t)K \subset B$ for every $t \ge T$, i.e., B is an absorbing set. In addition, B is a compact set of X. Thereby, B is a compact absorbing set of $(S(t), K, X)$. In view of the fact that $S(T)B \subset S(T)K \subset B$, we reset a phase space as

$$\mathcal{K} \equiv \bigcup_{0 \le t \le T} S(t)B \ \subset \ K.$$

It is obvious that $S(t)\mathcal{K} \subset \mathcal{K}$ for every $t > 0$, i.e., $\mathcal{K}$ is an invariant set. Therefore, $\mathcal{K}$ is not only compact and absorbing but also invariant. This means that the asymptotic behavior of trajectories of $(S(t), K, X)$ can be reduced to a sub dynamical system $(S(t), \mathcal{K}, X)$ in which the phase space $\mathcal{K}$ is a compact set of X. By the usual arguments, it is then seen that $\mathcal{B} = \bigcap_{0 \le t < \infty} S(t)\mathcal{K}$ becomes a global attractor of $(S(t), \mathcal{K}, X)$.

Furthermore, thanks to the estimate (3.3), we can construct the exponential attractors. Remember (see Eden-Foias-Nicolaenko-Temam [3]) that a subset $\mathcal{M} \subset \mathcal{K}$ satisfying the following conditions is called the exponential attractor of $(S(t), \mathcal{K}, X)$:

1. $\mathcal{M}$ is a compact subset of X with finite fractal dimension.
2. $\mathcal{M}$ includes the global attractor $\mathcal{B}$.
3. $\mathcal{M}$ is an invariant set, i.e., $S(t)\mathcal{M} \subset \mathcal{M}$ for every $t > 0$.
4. There exists an exponent $k > 0$ such that

   $$h(S(t)\mathcal{K}, \mathcal{M}) \le C_5 e^{-kt}, \qquad 0 < t < \infty,$$

   with a constant $C_5 > 0$.

Here, $h(K_1, K_2) = \sup_{F \in K_1} \inf_{G \in K_2} \|F - G\|_X$ is a semi-distance of two subsets K_1 and K_2 of $\mathcal{K}$.

As explained in [7, Section 6.4], the compact smoothing property

$$\|S(t^*)U_0 - S(t^*)V_0\|_{H^1(\Omega)} \le C_6\|U_0 - V_0\|_X, \qquad U_0,\, V_0 \in \mathcal{K},$$

of $S(t^*)$ with any fixed time $t^* > 0$ provides existence of exponential attractors. But, in the present case, this property is nothing more than the estimate (3.3). In this way, we obtain the following theorem.

Theorem 3.3. *The dynamical system $(S(t), K, X)$ possesses exponential attractors.*

Proof. As noticed above, we already know that there exists an exponential attractor $\mathcal{M}$ for $(S(t), \mathcal{K}, X)$. Then, as $S(T)K \subset B \subset \mathcal{K}$, it is readily verified that $\mathcal{M}$ is an exponential attractor for $(S(t), K, X)$, too. □

4 Homogeneous Stationary Solutions Consider the system of equations for u, v and w:

$$\varphi(u,v,w) \equiv \big[(1-u-v)\{1-\delta(\overline{w}-w-q[g(u,v)-a_w])^2\}-f\big]u = 0, \tag{4.1}$$

$$\psi(u,v,w) \equiv \big[(1-u-v)\{1-\delta(\overline{w}-w-q[g(u,v)-a_b])^2\}-f\big]v = 0, \tag{4.2}$$

$$\chi(u,v,w) \equiv [1-g(u,v)]R - \sigma w^4 = 0, \tag{4.3}$$

where $g(u,v)$ is the function given by (1.2). Here, according to [6], we want to handle a typical case that the parameters are given by

$$a_b = \frac{1}{4},\ a_g = \frac{1}{2},\ a_w = \frac{3}{4},\ q = 20,\ \delta = 3.265 \times 10^{-3}, \qquad f = 0.3,\ \overline{w} = 295.5 \text{ and } \sigma = 5.67 \times 10^{-8}, \tag{4.4}$$

except R that is treated as a tuning parameter.

4.1 Positive solutions We are concerned with the solutions such that $0 < u < 1$ and $0 < v < 1$. Then, since $1 - u - v \neq 0$, it follows from (4.1) and (4.2) that

$$1-\delta(\overline{w}-w-q[g(u,v)-a_w])^2 = 1-\delta(\overline{w}-w-q[g(u,v)-a_b])^2.$$

Therefore, $2(\overline{w}-w) - q[2g(u,v) - a_w - a_b] = 0$. In view of $a_b + a_w = 1$, we have

$$g(u,v) = \frac{1}{q}(\overline{w}-w) + \frac{1}{2}. \tag{4.5}$$

It then follows from (1.2) that

$$u - v = \frac{4}{q}(\overline{w}-w). \tag{4.6}$$

Meanwhile, (4.5) together with (4.3) yields the 4-th order equation

$$w^4 - \rho(w - w_0) = 0 \tag{4.7}$$

for w, where $\rho = \frac{R}{q\sigma}$ and $w_0 \equiv \overline{w} - \frac{q}{2} > 0$. On the other hand, (4.5) together with (4.1) yields the equation

$$u + v = 1 - \frac{f}{1-(q/4)^2\delta}. \tag{4.8}$$

In this way, we have observed that the equations (4.1)-(4.3) reduced to (4.6)-(4.8).

Let us next solve the equations (4.6)-(4.8). We first observe that when $\rho = \frac{4^4}{3^3}w_0^3$, i.e., $R = R_0 \equiv \frac{4^4}{3^3}q\sigma w_0^3$, (4.7) has a unique solution $w = \frac{4}{3}w_0$. Consequently, when $R > R_0$, (4.7) has two solutions $w_* < w^*$ such that $w_0 < w_* < \frac{4}{3}w_0 < w^*$. But, here, we easily see for w^* that the equations (4.6) and (4.8) cannot have positive solutions. Meanwhile, there is a range for w_* in which (4.6) and (4.8) admit a unique positive solution. As $R > R_0$ increases, w_* monotonously decreases in the range $\frac{4}{3}w_0 > w_* > w_0$. Therefore, we verify the following result.

Proposition 4.1. *There is a range* (R_*, R^*) *of* R *for which* (4.6)-(4.8) *have a unique positive solution* (u_*, v_*, w_*).

Moreover, under (4.4) it is easy to see that

$$1 - \delta(\overline{w} - w_* - q[g(u_*, v_*) - a_i])^2 \geq 0$$

for $i = w, b$. This shows that for $R_* < R < R^*$, $U_* = (u_*, v_*, w_*)$ gives a unique positive homogeneous stationary solution of (2.6).

4.2 Stability and instability of U_* We investigate stability and instability of the homogeneous positive stationary solution U_* when $R_* < R < R^*$.

For this purpose we use the linearization principle. Linearizing (2.6) in a neighborhood of U^*, let us consider the linear problem

$$\begin{cases} \dfrac{dU}{dt} + AU = F'(U_*)U, \qquad 0 < t < \infty, \\ U(0) = U_0. \end{cases} \tag{4.9}$$

Here, $F'(U_*)$ is a multiplicative operator of X by the matrix

$$F'(U_*) = \begin{pmatrix} \varphi_u^* & \varphi_v^* & \varphi_w^* \\ \psi_u^* & \psi_v^* & \psi_w^* \\ \chi_u^* & \chi_v^* & \chi_w^* \end{pmatrix} \equiv \begin{pmatrix} \varphi_u(u_*, v_*, w_*) & \varphi_v(u_*, v_*, w_*) & \varphi_w(u_*, v_*, w_*) \\ \psi_u(u_*, v_*, w_*) & \psi_v(u_*, v_*, w_*) & \psi_w(u_*, v_*, w_*) \\ \chi_u(u_*, v_*, w_*) & \chi_v(u_*, v_*, w_*) & \chi_w(u_*, v_*, w_*) \end{pmatrix}.$$

By elementary calculations, we observe that

$$\varphi_u^* = \left[\tfrac{q^2\delta}{16} + \tfrac{2fq^2\delta}{16-q^2\delta} - 1\right] u_*, \quad \varphi_v^* = \left[\tfrac{q^2\delta}{16} - \tfrac{2fq^2\delta}{16-q^2\delta} - 1\right] u_*, \quad \varphi_w^* = \tfrac{8fq\delta}{16-q^2\delta} u_*, \tag{4.10}$$

$$\psi_u^* = \left[\tfrac{q^2\delta}{16} - \tfrac{2fq^2\delta}{16-q^2\delta} - 1\right] v_*, \quad \psi_v^* = \left[\tfrac{q^2\delta}{16} + \tfrac{2fq^2\delta}{16-q^2\delta} - 1\right] v_*, \quad \psi_w^* = -\tfrac{8fq\delta}{16-q^2\delta} v_*, \tag{4.11}$$

$$\chi_u^* = -\tfrac{R}{4}, \qquad \chi_v^* = \tfrac{R}{4}, \qquad \chi_w^* = -4\sigma w_*^3. \tag{4.12}$$

We utilize again the base functions

$$\begin{Bmatrix} \cos \dfrac{m\pi}{\ell_x} x \\ \sin \dfrac{m\pi}{\ell_x} x \end{Bmatrix} \times \cos \frac{n\pi}{\ell_y} y, \qquad m, n = 0, 1, 2, \ldots,$$

which have been introduced in the proof of Proposition 2.1. They compose an orthogonal basis of $L_2(\Omega)$ and are an eigenfunction of $-\Delta$ under the periodic-Neumann boundary conditions with the eigenvalue

$$\mu_{mn} = \left(\frac{m\pi}{\ell_x}\right)^2 + \left(\frac{n\pi}{\ell_y}\right)^2, \qquad m, n = 0, 1, 2, \ldots, \quad \text{respectively.}$$

Consider the subspaces of X which are defined by

$$X_{mn}^c = \text{Span}\,\{e_1 \cos \tfrac{m\pi}{\ell_x} x \cdot \cos \tfrac{n\pi}{\ell_y} y,\ e_2 \cos \tfrac{m\pi}{\ell_x} x \cdot \cos \tfrac{n\pi}{\ell_y} y,\ e_3 \cos \tfrac{m\pi}{\ell_x} x \cdot \cos \tfrac{n\pi}{\ell_y} y\},$$

$$X_{mn}^s = \text{Span}\,\{e_1 \sin \tfrac{m\pi}{\ell_x} x \cdot \cos \tfrac{n\pi}{\ell_y} y,\ e_2 \sin \tfrac{m\pi}{\ell_x} x \cdot \cos \tfrac{n\pi}{\ell_y} y,\ e_3 \sin \tfrac{m\pi}{\ell_x} x \cdot \cos \tfrac{n\pi}{\ell_y} y\},$$

where $e_1 = {}^t(1,0,0)$, $e_2 = {}^t(0,1,0)$, $e_3 = {}^t(0,0,1)$. Then, it is easily verified that they are all a three-dimensional subspace of X, are mutually orthogonal in X and their Hilbert sum coincides with the space X, i.e.,

$$X = \sum_{0\le m,n<\infty} X^c_{mn} + \sum_{1\le m<\infty, 0\le n<\infty} X^s_{mn}.$$

Furthermore, it is verified that they are all an invariant subspace of the operator $-A + F'(U_*)$. Hence, the problem (4.9) can be decomposed into the infinite number of subproblems of (4.9) in the three-dimensional subspaces X^c_{mn} and X^s_{mn}.

By the way, the transformation matrices of $-A + F'(U_*)$ both in X^c_{mn} and X^s_{mn} are given by $M_{\mu_{mn}}$, where we put

$$M_\mu = \begin{pmatrix} -d\mu + \varphi^*_u & \varphi^*_v & \varphi^*_w \\ \psi^*_u & -d\mu + \psi^*_v & \psi^*_w \\ \chi^*_u & \chi^*_v & -D\mu + \chi^*_w \end{pmatrix} \qquad \text{for } 0 \le \mu < \infty.$$

If for all $M_{\mu_{mn}}$, their eigenvalues have negative real parts, then U_* is concluded to be a stable stationary solution. To the contrary, if there exists at least one $M_{\mu_{mn}}$ such that one of its eigenvalues has a positive real part, then U_* is concluded to be an unstable one. The characteristic polynomial of M_μ is given by

$$P_\mu(\lambda) \equiv \det(\lambda I - M_\mu) = \lambda^3 + p_1\lambda^2 + p_2\lambda + p_3$$

with the following coefficients:

$$\begin{aligned} p_1 &= (2d + D)\mu - (\varphi^*_u + \psi^*_v + \chi^*_w), \qquad p_3 = -\det M_\mu, \\ p_2 &= (d^2 + 2dD)\mu^2 - [(\varphi^*_u + \psi^*_v)D + (\psi^*_v + \chi^*_w)d + (\chi^*_w + \varphi^*_u)d]\mu \\ &\qquad + (\varphi^*_u\psi^*_v + \psi^*_v\chi^*_w + \chi^*_w\varphi^*_u) - (\varphi^*_v\psi^*_u + \varphi^*_w\chi^*_u + \psi^*_w\chi^*_v). \end{aligned}$$

Furthermore, p_3 is described as a third order polynomial of μ by

$$\begin{aligned} p_3 &= d^2D\mu^3 - [(\varphi^*_u + \psi^*_v)dD + \chi^*_w d^2]\mu^2 \\ &\quad + [(\varphi^*_u\psi^*_v - \varphi^*_v\psi^*_u)D + (\varphi^*_u\chi^*_w + \psi^*_v\chi^*_w - \psi^*_w\chi^*_v - \varphi^*_w\chi^*_u)d]\mu - \det M_0. \end{aligned}$$

Here, it is verified from (4.10)-(4.12) that $p_1 > 0$ and $p_1p_2 - p_3 > 0$. The Routh-Hurwitz theorem then provides that $P_\mu(\lambda)$ has a root of positive real part if and only if $p_3 < 0$. But we notice that

$$\begin{aligned} \varphi^*_u\psi^*_v - \varphi^*_v\psi^*_u &= \left(\left[\frac{q^2\delta}{16} + \frac{2fq^2\delta}{16 - q^2\delta} - 1 \right]^2 - \left[\frac{q^2\delta}{16} - \frac{2fq^2\delta}{16 - q^2\delta} - 1 \right]^2 \right) u_*v_* \\ &= \frac{4fq^2\delta}{16 - q^2\delta} \left(\frac{q^2\delta}{8} - 2 \right) u_*v_* < 0. \end{aligned}$$

This shows that, if the diffusion coefficient D is sufficiently large with respect to the other d, then $p_3 < 0$ for μ varying in some interval (μ_*, μ^*). Consequently, for $\mu \in (\mu_*, \mu^*)$, the polynomial $P_\mu(\lambda)$ has at least one positive root. As explained above, if there is some eigenvalue μ_{mn} that is included in this interval, then U^* is unstable.

For example, set $R = 917$ in addition to (4.4). Then,

$$p_3 \approx k(d\mu)^3 + (5.813 + 0.582k)(d\mu)^2 + (5.032 - 0.021k)(d\mu) + 0.885,$$

where we put $D = kd$. Thereby, if

$$k > 5118.845, \tag{4.13}$$

then there exists the interval (μ_*, μ^*) of μ in which p_3 takes negative values.

5 Numerical Results This section is devoted to showing numerical results for (2.6).

Set $\Omega = (0, 2\pi) \times (0, \pi)$, and set the parameters appearing in (1.1) as (4.4). The parameter R is tuned as a control parameter. In view of (4.13), the diffusion coefficients are fixed by

$$D = 1 \quad \text{and} \quad d = 10^{-5}.$$

According to the thermal physics, the incoming energy R is more precisely described by $R = S \times L$, where S is a radiation energy of sunlight and L is intensity of sunlight. Setting $S = 917$, we actually tune L in a range

$$R = 917 \times L \qquad \text{for} \quad 0.75 \le L \le 1.35.$$

(Consequently, R varies in $[687.75, 1237.95]$.) By the results obtained in Section 4, we know for each L of this range that (2.6) has a unique positive homogeneous stationary solution U_*. The initial value U_0 is then set by a random small perturbation of this homogeneous stationary solution.

All the numerical computations are performed by using the two-dimensional ADI methods.

5.1 Segregation patterns We vary L from 0.75 to 1.35 with step size $\mathit{\Delta} L = 0.05$.

For $0.75 \le L \le 1.30$, the stationary solution U_* is unstable. So, in these cases, the perturbation added to U_* increases and the trajectory $S(t)U_0$ leaves from U_* and goes far away. About $t = 6,000$, the numerical solution is almost stabilized. The trajectory $S(t)U_0$ might have been attracted by the exponential attractors. The profiles of the graphs of $u(t)$ and $v(t)$ at $t = 6,000$ are illustrated by means of the color graduation by Fig. 1 ($L = 0.75$), Fig. 2 ($L = 0.80$), Fig. 3 ($L = 0.85$), Fig. 4 ($L = 0.90$), Fig. 5 ($L = 0.95$), Fig. 6 ($L = 1.00$), Fig. 7 ($L = 1.05$), Fig. 8 ($L = 1.10$), Fig. 9 ($L = 1.15$), Fig. 10 ($L = 1.20$), Fig. 11 ($L = 1.25$) and Fig. 12 ($L = 1.30$), respectively. On the contrary, for $L = 1.35$, the stationary solution U_* is stable. So, the trajectory $S(t)U_0$ goes back to U_*, see Fig. 13 . But, as the stability is very weak, it takes longtime ($t = 6,000$) until $S(t)U_0$ is numerically stabilized.

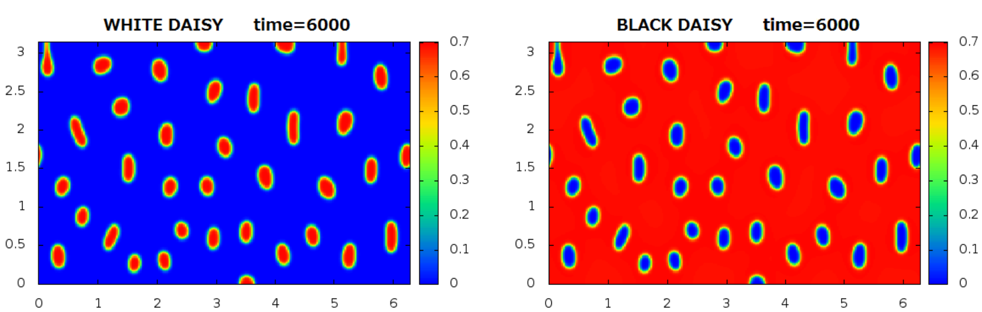

Fig. 1: $L = 0.75$.

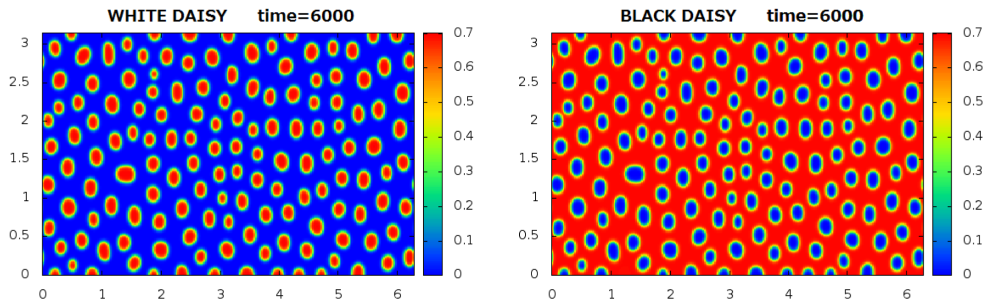

Fig. 2: $L = 0.80$.

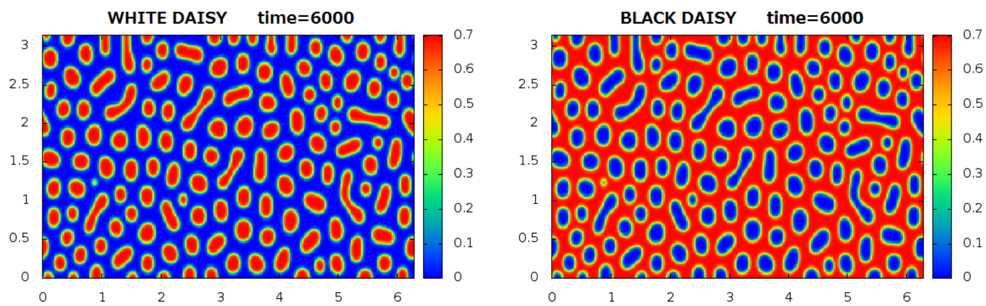

Fig. 3: $L = 0.85$.

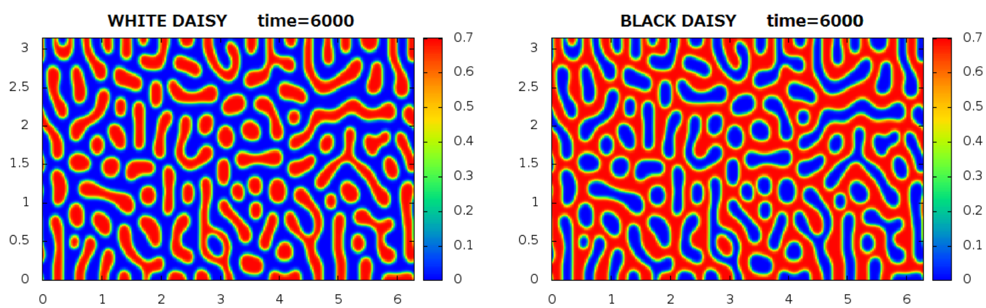

Fig. 4: $L = 0.90$.

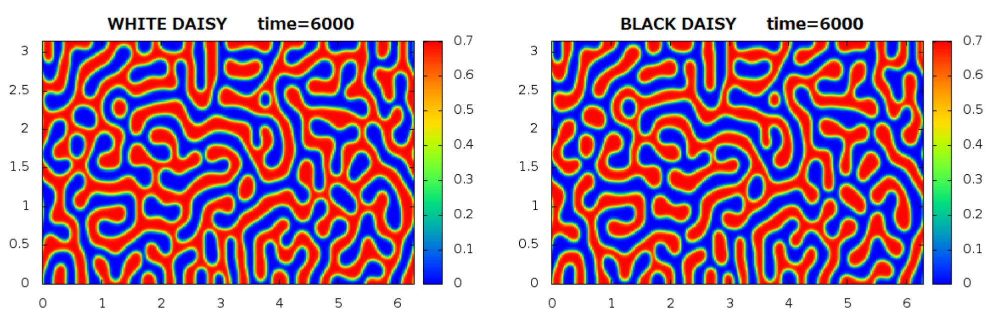

Fig. 5: $L = 0.95$.

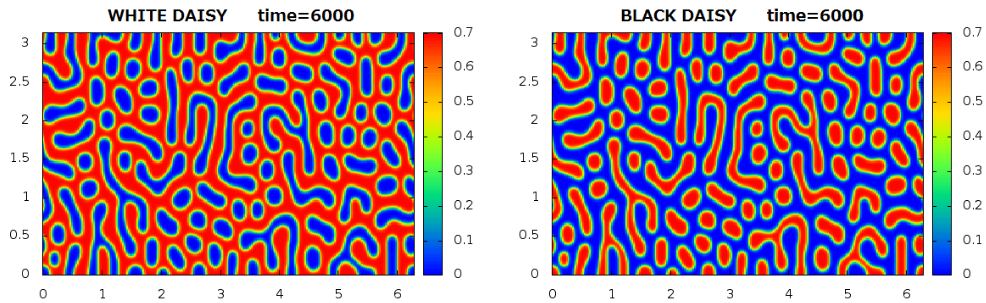

Fig. 6: $L = 1.00$.

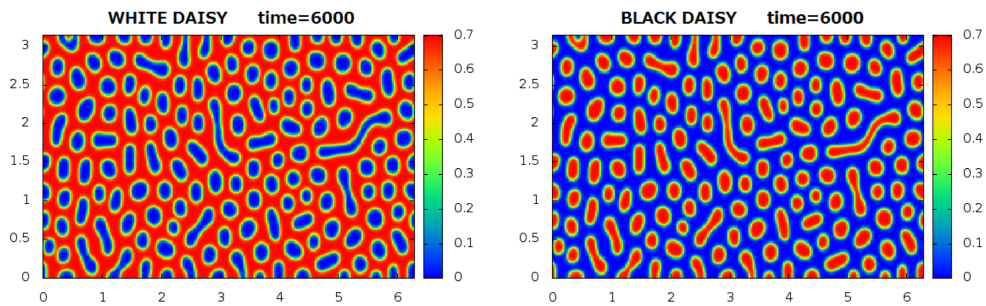

Fig. 7: $L = 1.05$.

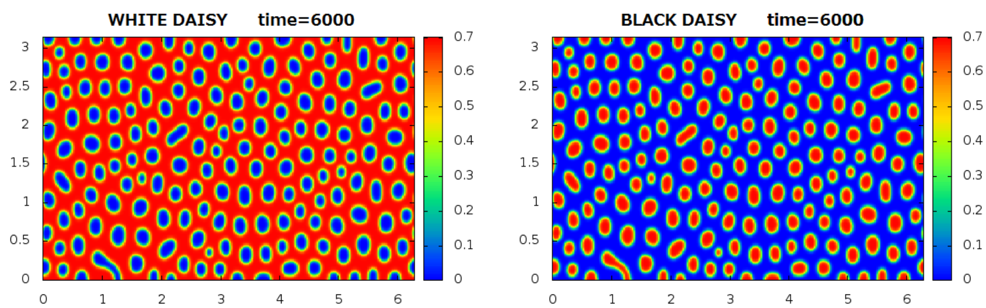

Fig. 8: $L = 1.10$.

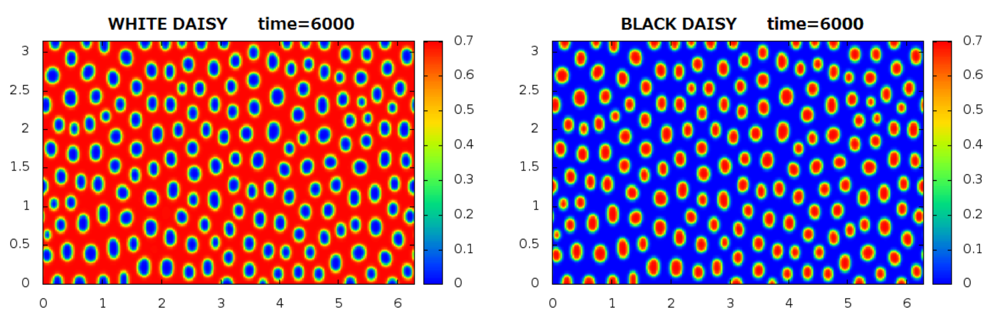

Fig. 9: $L = 1.15$.

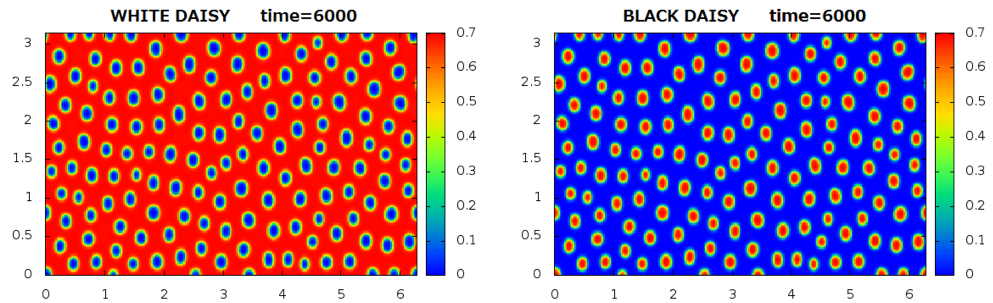

Fig. 10: $L = 1.20$.

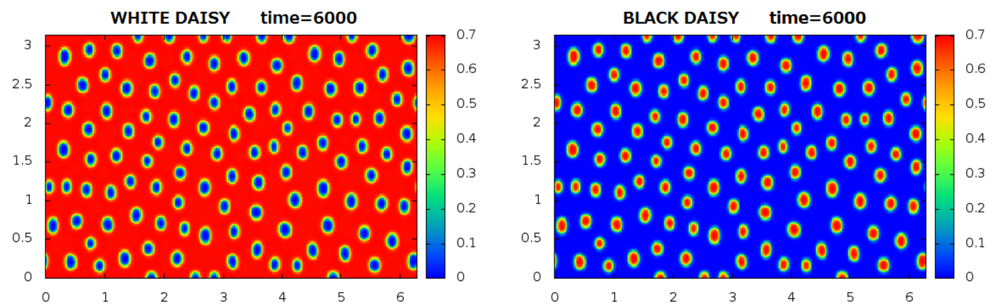

Fig. 11: $L = 1.25$.

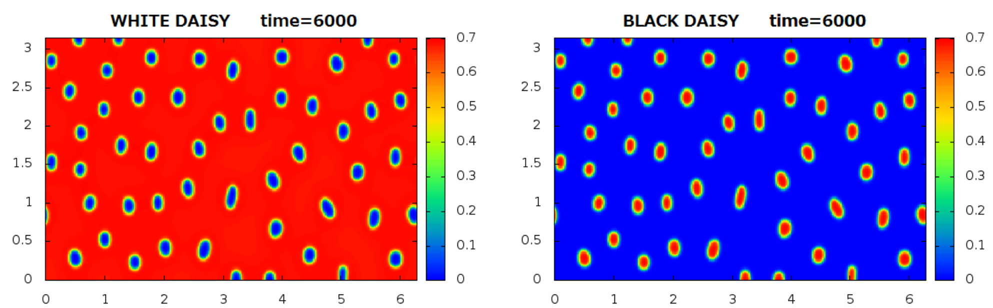

Fig. 12: $L = 1.30$.

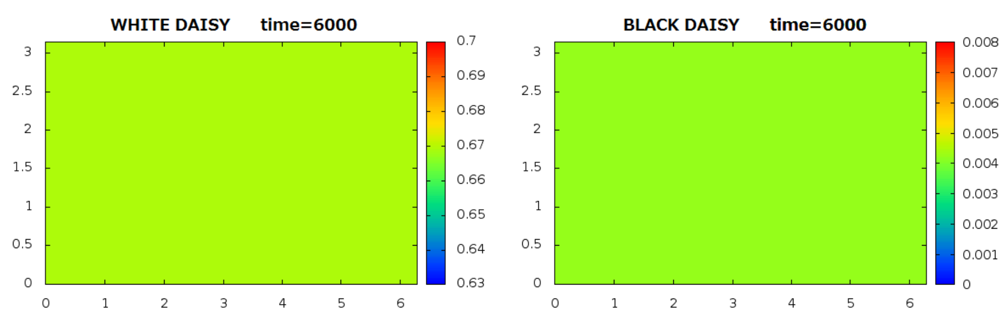

Fig. 13: $L = 1.35$.

For $0.75 \le L \le 1.30$, we find clear segregation patterns formed by the white and black daisies. At $L = 0.75$, black daisy is dominant in Ω and white daisy occurs only in a small number of spots. At $L = 0.80$, the number of spots generated by white daisy increases; but, at $L = 0.85$ and 0.90, some of these spots are jointed to make a long island of white daisy. At $L = 0.95$, the growth of two daisies seems to balance in Ω and both of them form a labyrinth pattern. For $1.00 \le L \le 1.30$, white daisy in turn becomes dominant. As L increases, the very reversed patterns of white daisy and black daisy are successively performed. At $L = 1.35$, white and black daisies coexist but two daisies are distributed homogeneously in Ω.

5.2 Mean of global temperature For $0.75 \le L \le 1.35$, the numerical values of $w(t)$ are as well stabilized about $t = 6,000$. The profiles of the graphs of $w(t)$ at $t = 6,000$ are illustrated by means of the color graduation by Figs. 14-26. Of course, the distribution of the global temperature depends closely on those of white and black daisies. So, we want to consider the spatial mean of $w(x, y, t)$, i.e.,

$$W(t) = \frac{1}{|\Omega|} \iint_\Omega w(x, y, t) dxdy, \qquad 0 \le t < \infty.$$

For each L, an approximate value of $W(6,000)$ is computed by a numerical integration. Its graph is drawn by Fig. 27. (However, the temperature is expressed in degrees Celsius.) We find that during the interval $[0.75, 1.35]$ of L, the mean of the global temperature is completely stabilized.

We thus observe that the homeostasis in the global temperature is maintained in Ω with respect to a change of intensity of sunlight, although the segregation pattern of white and black daisies clearly changes its types from homogeneous, spot, island and to labyrinth.

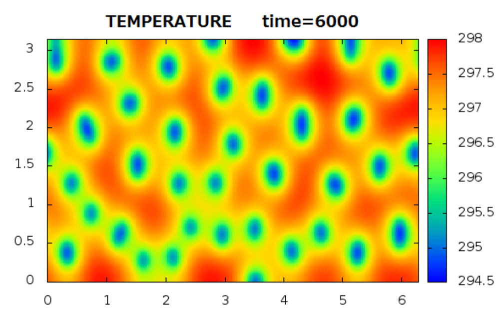

Fig. 14: $L = 0.75$.

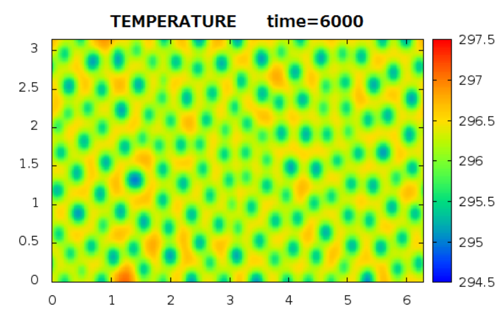

Fig. 15: $L = 0.80$.

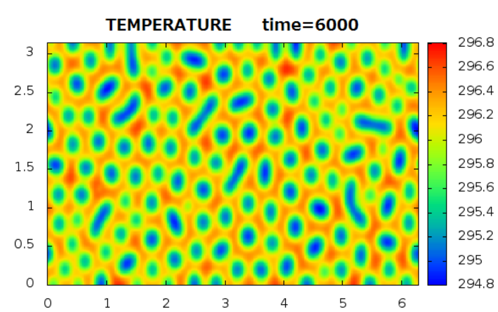

Fig. 16: $L = 0.85$.

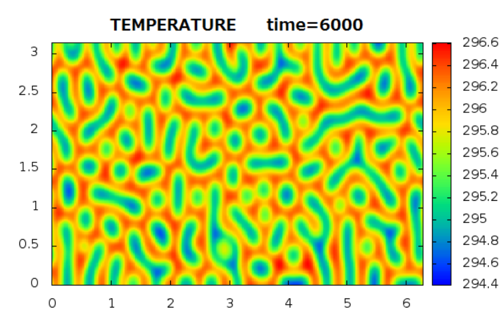

Fig. 17: $L = 0.90$.

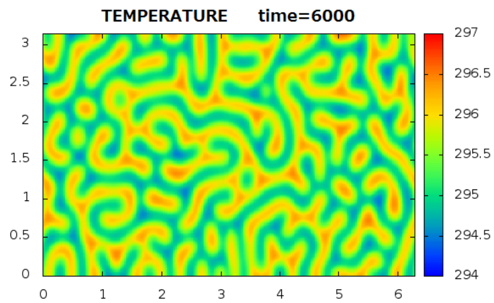

Fig. 18: $L = 0.95$.

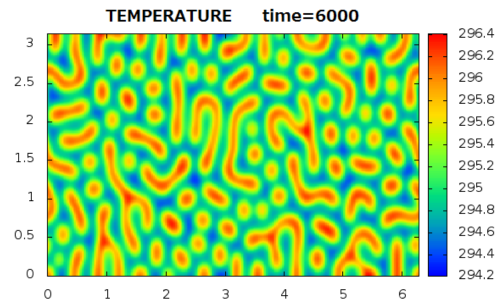

Fig. 19: $L = 1.00$.

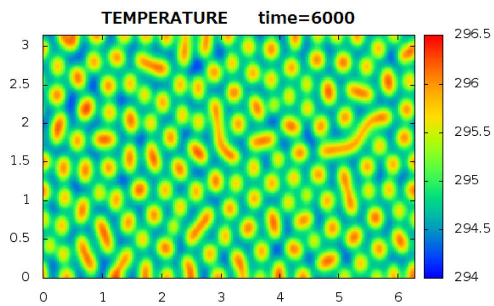

Fig. 20: $L = 1.05$.

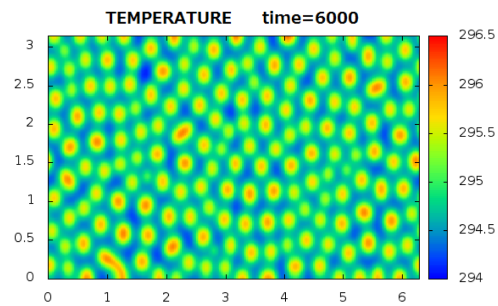

Fig. 21: $L = 1.10$.

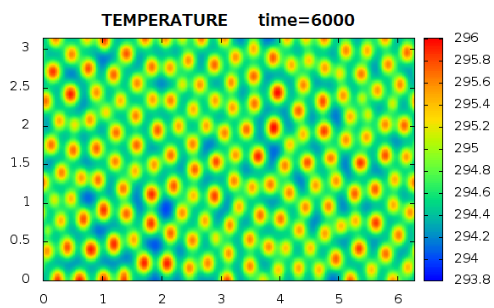

Fig. 22: $L = 1.15$.

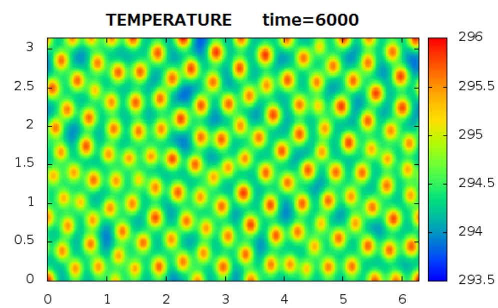

Fig. 23: $L = 1.20$.

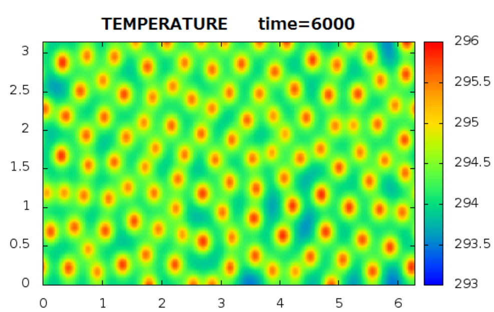

Fig. 24: $L = 1.25$.

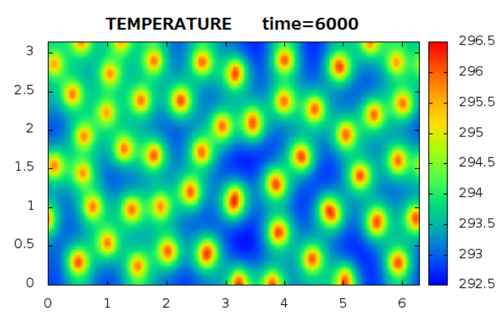

Fig. 25: $L = 1.30$.

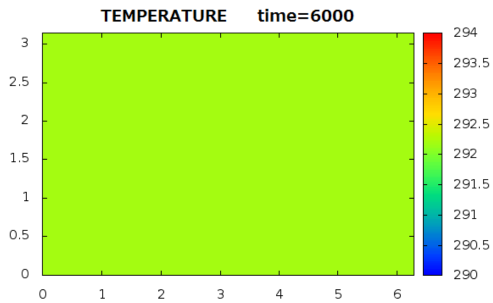

Fig. 26: $L = 1.35$.

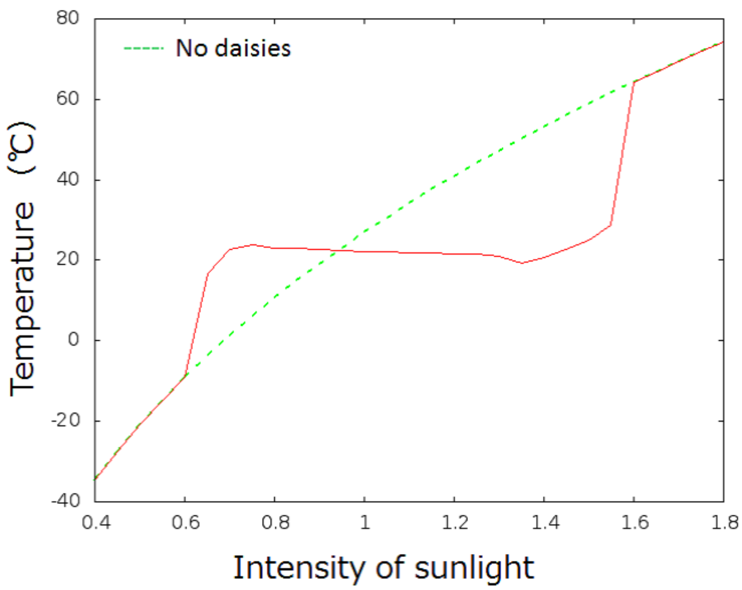

Fig. 27: The spatial mean of temperature.

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