COMMON INVARIANT SUBSPACES OF A FAMILY OF TOEPLITZ OPERATORS

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Abstract. Let Φ be a subset of L^∞ containing H^∞ and T_Φ the family of Toeplitz operators $\{T_\varphi\}_{\varphi\in\Phi}$. In this paper, we study invariant subspaces of T_Φ and their properties. Moreover, we provide a concrete description of nontrivial invariant subspaces of T_Φ for some Φ.

1 Introduction Let Γ be the unit circle centered at the origin in the complex plane, and $H^2(\Gamma^n)$ be the Hardy space on Γ^n. In [5], the second author showed that $H^2(\Gamma)$ has a certain rigidity (see Theorem 2.1 stated below), and pointed out that $H^2(\Gamma^2)$ does not have this property. The purpose of this paper is to study this phenomenon with examples.

We introduce notions in this paper. Let $L^2(\Gamma^n)$ be the usual L^2 space with respect to the normalized Lebesgue measure on Γ^n. Let P be the orthogonal projection from $L^2(\Gamma^n)$ onto $H^2(\Gamma^n)$. For $\varphi \in L^\infty(\Gamma^n)$, we define

$$T_\varphi f = P(\varphi f) \quad (f \in H^2).$$

Then T_φ is called the Toeplitz operator with symbol φ. For a subset Φ in $L^\infty(\Gamma^n)$, T_Φ denotes the set of Toeplitz operators whose symbols are in Φ, that is, we set

$$T_\Phi = \{T_\varphi : \varphi \in \Phi\}.$$

The collection of all closed subspaces of $H^2(\Gamma^n)$ invariant under every $T_\varphi \in T_\Phi$ is denoted by $\operatorname{Lat} T_\Phi$. Throughout this paper, we assume that $H^\infty \subseteq \Phi \subseteq L^\infty$.

This paper consists of five sections. In Section 2, we consider one variable Hardy space and recall results in [5]. In Section 3, we introduce some classes of functions in order to study $\operatorname{Lat} T_\Phi$. In Section 4, we study $\operatorname{Lat} T_\Phi$ for some Φ's. In Section 5, we show that $\operatorname{Lat} T_\Phi$ is nontrivial for some Φ, and present examples of invariant subspaces of T_z and T_w.

2 A certain rigidity of $H^2(\Gamma)$ The following theorem was given in [5], which shows that $H^2(\Gamma)$ has a certain rigidity.

Theorem 2.1 ([5]). *If $\Phi = H^\infty(\Gamma) \cup \{\varphi\}$ for $\varphi \in L^\infty(\Gamma) \setminus H^\infty(\Gamma)$, then* $\operatorname{Lat} T_\Phi = \{\langle 0\rangle, H^2(\Gamma)\}$.

The original proof is based on the theory of uniform algebras. We shall give another proof to this theorem.

Proof. In this proof, we will write $H^2 = H^2(\Gamma)$, $H^\infty = H^\infty(\Gamma)$ and so on. Suppose that $\mathcal{M} \in \operatorname{Lat} T_\Phi$ and $\mathcal{M}$ is nontrivial. Then, $\mathcal{M}$ is an invariant subspace of H^2. Hence, there exists a non-constant inner function q such that $\mathcal{M} = qH^2$ by Beurling's theorem. We note that $T_\varphi \mathcal{M} \subset \mathcal{M}$ is equivalent to that

$$P_{H^2}\varphi q H^2 \subset qH^2.$$

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Hence, for any function $h \in H^2$, there exists a function $g_h \in H^2$ such that $P_{H^2}(\varphi q h) = q g_h$. Then we have that $P_{H^2}(\varphi q h - q g_h) = 0$, and which is equivalent to that $\varphi q h - q g_h \in \overline{H_0^2}$, where $\overline{H_0^2} = L^2 \ominus H^2$. Therefore we have that

$$\varphi q h \in \mathcal{M} \oplus \overline{H_0^2} \quad (h \in H^2). \tag{2.1.1}$$

In particular, for $h = 1$, there exist $g_1 \in H^2$ and $k \in H_0^2$ such that

$$\varphi q = q g_1 + \overline{k}. \tag{2.1.2}$$

Put $\mathcal{N} = H^2 \ominus \mathcal{M}$. Multiplying both sides of (2.1.2) by $h \in H^\infty$, we obtain

$$\begin{aligned} \varphi q h &= \{P_{\mathcal{M}} + P_{\mathcal{N}} + (I_{L^2} - P_{H^2})\}(q g_1 h + \overline{k} h) \\ &= (q g_1 h + P_{\mathcal{M}} \overline{k} h) \oplus P_{\mathcal{N}} \overline{k} h \oplus (I_{L^2} - P_{H^2}) \overline{k} h. \end{aligned}$$

Then, by (2.1.1), we note that

$$P_{\mathcal{N}} \overline{k} h = P_{\mathcal{N}} \varphi q h = 0.$$

Let $\mathbb{D}$ be the open unit disc in the complex plane. Now, setting

$$k = \sum_{j=1}^{\infty} c_j z^j, \quad k_n = \sum_{j=1}^{n} c_j z^j \quad \text{and} \quad s_\lambda = \frac{1}{1 - \overline{\lambda} z} \quad (\lambda \in \mathbb{D}),$$

we have that

$$\begin{aligned} \|P_{\mathcal{N}} \overline{k_n} s_\lambda\| &= \|P_{\mathcal{N}} \overline{k_n} s_\lambda - P_{\mathcal{N}} \overline{k} s_\lambda\| \\ &\leq \|\overline{k_n} s_\lambda - \overline{k} s_\lambda\| \\ &\leq \|s_\lambda\|_\infty \|k_n - k\| \\ &\to 0 \end{aligned}$$

as $n \to \infty$. On the other hand,

$$\begin{aligned} P_{\mathcal{N}} \overline{k_n} s_\lambda &= P_{\mathcal{N}} T_{k_n}^* s_\lambda \\ &= P_{\mathcal{N}} \overline{k_n(\lambda)} s_\lambda \\ &\to P_{\mathcal{N}} \overline{k(\lambda)} s_\lambda \end{aligned}$$

as $n \to \infty$. Therefore $P_{\mathcal{N}} \overline{k(\lambda)} s_\lambda = 0$ for any $\lambda \in \mathbb{D}$. If $k(\lambda) \neq 0$ for some λ, then $P_{\mathcal{N}} s_\lambda = 0$. However,

$$P_{\mathcal{N}} s_\lambda = \frac{1 - \overline{q(\lambda)} q}{1 - \overline{\lambda} z} \neq 0.$$

Hence $k(\lambda) = 0$ for all $\lambda \in \mathbb{D}$. Then we see that $\varphi q = q g_1$ in (2.1.2), and which implies $\varphi = g_1 \in H^2$. This contradicts that $\varphi \in L^\infty \setminus H^\infty$. □

From Theorem 2.1, in $H^2(\Gamma)$, $\operatorname{Lat} T_\Phi$ has only trivial invariant subspaces if Φ contains $H^\infty(\Gamma)$ properly. On the other hand, in the case of $H^2(\Gamma^2)$, $\operatorname{Lat} T_\Phi$ may not be $\{\langle 0 \rangle, H^2(\Gamma^2)\}$ even if Φ properly contains $H^\infty(\Gamma^2)$. The following is an example.

Example 2.2. We set $\mathcal{M} = zH^2(\Gamma^2) + wH^2(\Gamma^2)$. Then $\mathcal{M} \in \operatorname{Lat} T_\Phi$ for $\Phi = H^\infty(\Gamma^2) \cup \{\overline{z} w\}$.

We will see more examples in Section 5.

3 $\mathcal{M}_\Phi$, $\mathcal{M}^\Phi$ and $K^\Phi_{\mathcal{M}}$ We focus on the structure of $H^2(\Gamma^2)$, so that we will write $L^2 = L^2(\Gamma^2)$, $H^2 = H^2(\Gamma^2)$ and so on, if no confusion occurs. In this section, some classes of functions which play important roles in this paper are introduced.

Definition 3.1. *Let φ be a function in L^∞. For $\mathcal{M} \in \operatorname{Lat} T_\varphi$, we put*

$$\mathcal{M}_\varphi = \{f \in \mathcal{M} : \varphi f \in \mathcal{M}\} \quad \text{and} \quad \mathcal{M}^\varphi = \mathcal{M} \ominus \mathcal{M}_\varphi.$$

Moreover, let Φ be a subset of L^∞. For $\mathcal{M} \in \operatorname{Lat} T_\Phi$, we put

$$\mathcal{M}_\Phi = \bigcap_{\varphi \in \Phi} \mathcal{M}_\varphi \quad \text{and} \quad \mathcal{M}^\Phi = \mathcal{M} \ominus \mathcal{M}_\Phi.$$

Example 3.1. $\mathcal{M}_{\overline{z}} = z\mathcal{M}$ and $\mathcal{M}^{\overline{z}} = \mathcal{M} \ominus z\mathcal{M}$. Further, if $\Phi = H^\infty \cup \{\overline{z}, \overline{w}\}$, then $\mathcal{M}_\Phi = zw\mathcal{M}$ and $\mathcal{M}^\Phi = \mathcal{M} \ominus zw\mathcal{M}$.

We are mainly interested in the case where Φ is a subset of L^∞ which contains H^∞ properly. We shall give some general facts on $\mathcal{M}_\Phi$ and $\mathcal{M}^\Phi$.

Proposition 3.2. *Let Φ be a subset of L^∞ which contains H^∞ properly. Then $\mathcal{M}_\Phi$ is an invariant subspace in H^2.*

Proof. It suffices to show that $\mathcal{M}_\varphi$ is an invariant subspace for any $\varphi \in \Phi$. If $f \in \mathcal{M}_\varphi$ then $\varphi f \in \mathcal{M}$. It follows from this that $z\varphi f \in \mathcal{M}$, that is, $zf \in \mathcal{M}_\varphi$. Hence $\mathcal{M}_\varphi$ is invariant under multiplication by z. Moreover, if $f_n \in \mathcal{M}_\varphi$ and $f_n \to f$ $(n \to \infty)$, then $f \in \mathcal{M}$ and $\varphi f_n \to \varphi f$ $(n \to \infty)$ in $\mathcal{M}$. Hence we have that $f \in \mathcal{M}_\varphi$, that is, $\mathcal{M}_\varphi$ is closed. These conclude that $\mathcal{M}$ is an invariant subspace in H^2. □

In order to give the next theorem on $\mathcal{M}^\Phi$, we need a lemma.

Lemma 3.3. *Let Φ be a subset of L^∞ which contains H^∞ properly. Suppose that $\mathcal{M} \in \operatorname{Lat} T_\Phi$. For any $f \in H^\infty$, we define $Q_f = P_{\mathcal{M}^\Phi} T_f|_{\mathcal{M}^\Phi}$. Then*

$$Q_{fg} = Q_f Q_g \quad (f \text{ and } g \in H^\infty).$$

Proof. It follows from Proposition 3.2 that

$$\begin{aligned} Q_{fg} - Q_f Q_g &= P_{\mathcal{M}^\Phi} T_{fg} P_{\mathcal{M}^\Phi} - P_{\mathcal{M}^\Phi} T_f P_{\mathcal{M}^\Phi} T_g P_{\mathcal{M}^\Phi} \\ &= P_{\mathcal{M}^\Phi} T_f (P_{\mathcal{M}} - P_{\mathcal{M}^\Phi}) T_g P_{\mathcal{M}^\Phi} \\ &= P_{\mathcal{M}^\Phi} T_f P_{\mathcal{M}_\Phi} T_g P_{\mathcal{M}^\Phi} \\ &= 0. \end{aligned}$$

□

Theorem 3.4. *Let Φ be a subset of L^∞ which contains H^∞ properly. If $\mathcal{M} \in \operatorname{Lat} T_\Phi$ then $\dim \mathcal{M}^\Phi = \infty$.*

Proof. Suppose $\dim \mathcal{M}^\Phi = n < \infty$. Then, by Lemma 3.3, there exists a finite Blaschke product $b_1(z)$ such that $Q_{b_1(z)} = 0$. Hence we have $b_1(z)\mathcal{M}^\Phi \subset \mathcal{M}_\Phi$. Further, it follows from Proposition 3.2 that $b_1(z)\mathcal{M}_\Phi \subset \mathcal{M}_\Phi$, that is,

$$b_1(z)\varphi\mathcal{M} \subset \mathcal{M} \quad (\varphi \in \Phi).$$

Similarly, there exists a finite Blaschke product $b_2(w)$ such that

$$b_2(w)\varphi\mathcal{M} \subset \mathcal{M} \quad (\varphi \in \Phi).$$

Hence $b_1(z)\varphi$ and $b_2(w)\varphi$ belong to H^2 for all $\varphi \in \Phi$. Therefore we have

$$\varphi \in \overline{b_1(z)}H^2 \cap \overline{b_2(w)}H^2 \subset H^2.$$

However, this is a contradiction. □

Next, we introduce a kind of complement of $\mathcal{M}$ in our problem.

Definition 3.2. *For $\mathcal{M} \in \operatorname{Lat} T_\Phi$ and $\varphi \in \Phi$, put*

$$K = \{\overline{f} : f \in L^2 \ominus H^2\}$$

and

$$K_{\mathcal{M}}^{\varphi} = \{k \in K : \overline{k} = \varphi f - g \text{ for some } f \text{ and } g \in \mathcal{M}\},$$

where $\overline{f}$ denotes the complex conjugate of f. Moreover, we set

$$K_{\mathcal{M}}^{\Phi} = \bigcup_{\varphi\in\Phi} K_{\mathcal{M}}^{\varphi}.$$

If $\varphi \in H^\infty$ and $k \in K_{\mathcal{M}}^{\varphi}$, then there exist f and $g \in \mathcal{M}$ such that $\overline{k} = \varphi f - g$. However, it follows from $\overline{K} \cap \mathcal{M} = \langle 0\rangle$ that $k = 0$, that is, $K_{\mathcal{M}}^{\varphi} = \langle 0\rangle$ for $\varphi \in H^\infty$, so that we may define

$$K_{\mathcal{M}}^{\Phi} = \bigcup_{\varphi\in\Phi\setminus H^\infty} K_{\mathcal{M}}^{\varphi}.$$

Remark 3.5. In $H^2(\Gamma)$,

$$K = \{\overline{f} : f \in L^2(\Gamma) \ominus H^2(\Gamma)\} = H_0^2(\Gamma)$$

and we have already dealt with $K_{\mathcal{M}}^{\varphi}$ in the proof of Theorem 2.1 (see (2.1.1)), implicitly.

Next, we study the properties of $K_{\mathcal{M}}^{\Phi}$ used in the rest of this paper.

Lemma 3.6. *Let $\mathcal{M}$ be a closed subspace in H^2, and Φ be a subset of L^∞ which contains H^∞.*

(1) *$\mathcal{M} \in \operatorname{Lat} T_\Phi$ if and only if $\varphi\mathcal{M} \subset \mathcal{M} + \overline{K_{\mathcal{M}}^{\varphi}}$ for all $\varphi \in \Phi$.*

(2) *If $\mathcal{M} \in \operatorname{Lat} T_\Phi$, then $(I_{L^2} - P_{\mathcal{M}})\varphi\mathcal{M}^{\varphi} = \overline{K_{\mathcal{M}}^{\varphi}}$ for all $\varphi \in \Phi$.*

Proof. (1) First we show the 'if' part. For any $\varphi \in \Phi$ and $f \in \mathcal{M}$, there exist $g \in \mathcal{M}$ and $k \in K_{\mathcal{M}}^{\varphi}$ such that $\varphi f = g + \overline{k}$. From this equality, we have $T_\varphi f = g \in \mathcal{M}$. Hence we see that $\mathcal{M} \in \operatorname{Lat} T_\Phi$. Next, we show the 'only if' part. Suppose that $\mathcal{M}$ is in $\operatorname{Lat} T_\Phi$. For any $\varphi \in \Phi$ and $f \in \mathcal{M}$, there exist $g \in \mathcal{M}$, $h \in H^2 \ominus \mathcal{M}$ and $k \in K$ such that

$$\varphi f = g + h + \overline{k}.$$

From this equality, we have $P(\varphi f) = g + h$. Since $P(\varphi f)$ and g are in $\mathcal{M}$, h must be 0. Therefore we see that $\varphi f = g + \overline{k}$ and that $k \in K_{\mathcal{M}}^{\varphi}$ by the definition of $K_{\mathcal{M}}^{\varphi}$.
(2) Since $\mathcal{M}$ contains $\mathcal{M}^{\varphi}$, for any $f \in \mathcal{M}^{\varphi}$ there exist $g \in \mathcal{M}$ and $k \in K_{\mathcal{M}}^{\varphi}$ such that $\varphi f = g + \bar{k}$ by (1). Then we see

$$(I_{L^2} - P_{\mathcal{M}})\varphi f = (I_{L^2} - P_{\mathcal{M}})(g + \bar{k}) = \bar{k}.$$

Therefore we have $(I_{L^2} - P_{\mathcal{M}})\varphi\mathcal{M}^\varphi \subset \overline{K_{\mathcal{M}}^\varphi}$. On the other hand, for any $k \in K_{\mathcal{M}}^\varphi$ there exist f and $g \in \mathcal{M}$ such that $\varphi f = g + \bar{k}$ by the definition of $K_{\mathcal{M}}^\varphi$. In particular, we can write $f = f_1 + f_2$, where $f_1 \in \mathcal{M}_\varphi$ and $f_2 \in \mathcal{M}^\varphi$. Since $\varphi f_1 \in \mathcal{M}$, we have

$$\begin{aligned}\bar{k} &= (I_{L^2} - P_{\mathcal{M}})\bar{k} \\ &= (I_{L^2} - P_{\mathcal{M}})(\varphi f - g) \\ &= (I_{L^2} - P_{\mathcal{M}})(\varphi f_1 + \varphi f_2 - g) \\ &= (I_{L^2} - P_{\mathcal{M}})\varphi f_2,\end{aligned}$$

and which implies $\overline{K_{\mathcal{M}}^\varphi} \subset (I_{L^2} - P_{\mathcal{M}})\varphi\mathcal{M}^\varphi$. Hence we have

$$(I_{L^2} - P_{\mathcal{M}})\varphi\mathcal{M}^\varphi = \overline{K_{\mathcal{M}}^\varphi}.$$

Thus we obtain (2). □

4 Properties of $\mathrm{Lat}\, T_\Phi$ In this section, we study properties of $\mathrm{Lat}\, T_\Phi$ for some Φ as the union of H^∞ and some set. First we set Φ the union of H^∞ and the complex conjugate of functions in H^∞.

Proposition 4.1. *If* $\Phi = H^\infty \cup \overline{H^\infty}$, *then* $\mathrm{Lat}\, T_\Phi = \mathrm{Lat}\, T_{L^\infty}$.

Proof. It is obvious that $\mathrm{Lat}\, T_{L^\infty} \subset \mathrm{Lat}\, T_\Phi$. To prove the converse inclusion, suppose that $\mathcal{M} \in \mathrm{Lat}\, T_\Phi$. Then, since $T_{h_1\overline{h_2}} = T_{\overline{h_2}}T_{h_1}$ for any $h_1, h_2 \in H^\infty$, we see that $T_{h_1\overline{h_2}}\mathcal{M} \subset \mathcal{M}$. We note that L^∞ is the algebra generated by H^∞ and $\overline{H^\infty}$ in the w^*-topology. So for any $\varphi \in L^\infty$ we can choose a net $\{\varphi_\alpha\} \subset L^\infty$ converging in w^*-topology to φ, where each φ_α is a linear combination of products of functions in H^∞ and $\overline{H^\infty}$ and satisfies $T_{\varphi_\alpha}\mathcal{M} \subset \mathcal{M}$. For any f and $g \in H^2$ we have

$$\lim_{\alpha \in A} \langle T_{\varphi_\alpha} f, g\rangle = \lim_{\alpha \in A} \int_{\Gamma^2} \varphi_\alpha f \overline{g} d\mu = \int_{\Gamma^2} \varphi f \overline{g} d\mu = \langle T_\varphi f, g\rangle.$$

In particular, for any $f \in \mathcal{M}$ and $g \in H^2 \ominus \mathcal{M}$ we see that

$$\langle T_\varphi f, g\rangle = \lim_{\alpha \in A} \langle T_{\varphi_\alpha} f, g\rangle = 0.$$

Hence $T_\varphi f$ is in $\mathcal{M}$. Therefore we have $T_\varphi\mathcal{M} \subset \mathcal{M}$ and so we conclude that $\mathrm{Lat}\, T_\Phi \subset \mathrm{Lat}\, T_{L^\infty}$. □

Proposition 4.2. *Suppose that* F *is a non-constant function in* $H^\infty \cap q\overline{H^\infty}$ *for some inner function* q. *Let* $\Phi = H^\infty \cup \{\overline{F}\}$. *If* $\mathcal{M}$ *is in* $\mathrm{Lat}\, T_\Phi$, *then* $\mathcal{M}_\Phi = \mathcal{M}_{\overline{F}} \supseteq q\mathcal{M}$.

Proof. If $F \in H^\infty \cap q\overline{H^\infty}$ then there exists $f \in H^\infty$ such that $F = q\overline{f}$. Hence $\overline{F}q\mathcal{M} = f\mathcal{M} \subset \mathcal{M}$, and trivially, $q\mathcal{M} \subset \mathcal{M}$. Therefore we have that $q\mathcal{M} \subset \mathcal{M}_{\overline{F}}$. □

Next, we consider examples when Φ consists of all functions in H^∞ and the complex conjugate of an inner function.

Theorem 4.3. *Let* $\Phi = H^\infty \cup \{\overline{q}\}$ *for some non-constant inner function* q. *Suppose that* $\mathcal{M} \in \mathrm{Lat}\, T_\Phi$. *Then the following statements hold.*

(1) $\mathcal{M}_\Phi = q\mathcal{M}$ *and* $\mathcal{M}^\Phi = \mathcal{M} \ominus q\mathcal{M}$.

(2) $\mathcal{M}_\Phi \subset (H^2)_\Phi$ *and* $\mathcal{M}^\Phi \subset (H^2)^\Phi$.

(3) $\overline{K_{\mathcal{M}}^{\Phi}} = \overline{q}(\mathcal{M} \ominus q\mathcal{M})$.

Proof. (1) It is sufficient to prove $\mathcal{M}_{\overline{q}} = q\mathcal{M}$ since $\mathcal{M}_{\Phi} = \mathcal{M}_{\overline{q}}$. If $f \in \mathcal{M}_{\overline{q}}$, then $\overline{q}f \in \mathcal{M}$ from the definition of $\mathcal{M}_{\overline{q}}$. The assumption that q is an inner function implies that $f \in q\mathcal{M}$, and hence we see that $\mathcal{M}_{\overline{q}} \subset q\mathcal{M}$. Conversely, if $f \in q\mathcal{M}$, then $f \in \mathcal{M}$ since $q\mathcal{M} \subset \mathcal{M}$. Moreover, that q is inner implies that $\overline{q}f \in \mathcal{M}$. Therefore we see that $q\mathcal{M} \subset \mathcal{M}_{\overline{q}}$, which implies that the first statement. The second statement follows from the first statement.
(2) The first statement follows from the definition of $\mathcal{M}_{\Phi}$ and $(H^2)_{\Phi}$. To show the second statement, suppose that $f \in \mathcal{M}^{\Phi}$. By (1) we have $f \in \mathcal{M}$ and $f \perp q\mathcal{M}$. Moreover, since $\mathcal{M}$ is invariant under $T_{\overline{q}}$, we see that $T_q(H^2 \ominus \mathcal{M}) \subset H^2 \ominus \mathcal{M}$, that is, $q(H^2 \ominus \mathcal{M}) \subset H^2 \ominus \mathcal{M}$. This implies that $\mathcal{M} \perp q(H^2 \ominus \mathcal{M})$. For any $g \in H^2$, there exist $g_1 \in \mathcal{M}$ and $g_2 \in H^2 \ominus \mathcal{M}$ such that $g = g_1 + g_2$. Then we have

$$\begin{aligned} \langle f, qg \rangle &= \langle f, qg_1 + qg_2 \rangle \\ &= \langle f, qg_1 \rangle + \langle f, qg_2 \rangle \\ &= 0 \end{aligned}$$

since $f \perp q\mathcal{M}$ and $\mathcal{M} \perp q(H^2 \ominus \mathcal{M})$. Therefore we see that $f \perp qH^2$, that is, $f \in (H^2)^{\Phi}$. Hence the second statement holds.
(3) By (2) of Lemma 3.6, it is obvious that

$$\overline{q}(\mathcal{M} \ominus q\mathcal{M}) \supset (I_{L^2} - P_{\mathcal{M}})\overline{q}(\mathcal{M} \ominus q\mathcal{M}) = \overline{K_{\mathcal{M}}^{\overline{q}}}.$$

Next, we will show the converse inclusion. For any $f \in \mathcal{M} \ominus q\mathcal{M}$, there exist $g \in \mathcal{M}$ and $k \in K_{\mathcal{M}}^{\overline{q}}$ such that $\overline{q}f = g + \overline{k}$ by (1) of Lemma 3.6. Then we have

$$\begin{aligned} \|g\|^2 &= \langle g, g \rangle \\ &= \langle \overline{q}f - \overline{k}, g \rangle \\ &= \langle \overline{q}f, g \rangle - \langle \overline{k}, g \rangle \\ &= \langle f, qg \rangle - \langle \overline{k}, g \rangle \\ &= 0, \end{aligned}$$

since $f \perp q\mathcal{M}$ and $g \perp \overline{K_{\mathcal{M}}^{\overline{q}}}$. So we see that $g = 0$, which implies that $\overline{q}f = \overline{k} \in \overline{K_{\mathcal{M}}^{\overline{q}}}$. Therefore we have $\overline{q}(\mathcal{M} \ominus q\mathcal{M}) \subset \overline{K_{\mathcal{M}}^{\overline{q}}}$. Hence we obtain

$$\overline{q}(\mathcal{M} \ominus q\mathcal{M}) = (I_{L^2} - P_{\mathcal{M}})\overline{q}(\mathcal{M} \ominus q\mathcal{M}) = \overline{K_{\mathcal{M}}^{\overline{q}}}$$

Since $\overline{K_{\mathcal{M}}^{\Phi}} = \overline{K_{\mathcal{M}}^{\overline{q}}}$, the statement holds. □

More generally, we are able to consider the case when Φ is the union of H^∞ and a set of the complex conjugate of inner functions. In Corollary 4.4, we denote by Λ a subset of $\mathbb{R}$.

Corollary 4.4. *Let $\Phi = H^\infty \cup \{\overline{q_\alpha} : q_\alpha$ is inner$, \alpha \in \Lambda\}$. Suppose that $\mathcal{M} \in \operatorname{Lat} T_{\Phi}$. Then the following statements hold.*

(1) $\mathcal{M}_{\Phi} = \bigcap_{\alpha \in \Lambda} q_\alpha \mathcal{M}$ *and* $\mathcal{M}^{\Phi} = \mathcal{M} \ominus \bigcap_{\alpha \in \Lambda} q_\alpha \mathcal{M}$.

(2) $\mathcal{M}_{\Phi} \subset (H^2)_{\Phi}$ *and* $\mathcal{M}^{\Phi} \subset (H^2)^{\Phi}$.

(3) $\overline{K_{\mathcal{M}}^{\Phi}} = \bigcup_{\alpha \in \Lambda} \overline{q_\alpha}(\mathcal{M} \ominus q_\alpha \mathcal{M})$.

Proof. (1) These statements follow from (1) of Theorem 4.3 and the definitions of $\mathcal{M}_\Phi$ and $\mathcal{M}^\Phi$.
(2) It is clear that $q_\alpha \mathcal{M} \subset q_\alpha H^2$ for all $\alpha \in \Lambda$. Hence we have

$$\mathcal{M}_\Phi = \bigcap_{\alpha\in\Lambda} q_\alpha \mathcal{M} \subset \bigcap_{\alpha\in\Lambda} q_\alpha H^2 = (H^2)_\Phi.$$

Moreover by (2) of Theorem 4.3, we see that if f is in $\mathcal{M} \ominus q_\alpha \mathcal{M}$, then $f \perp q_\alpha H^2$ for all $\alpha \in \Lambda$. Therefore the second statement holds.
(3) The statement follows from (3) of Theorem 4.3 and the definition of $K^\Phi_{\mathcal{M}}$. □

We will use Proposition 4.5 to determine $\mathrm{Lat}\, T_\Phi$ in some concrete case.

Proposition 4.5. *Let q be a non-constant inner function and $\psi = \frac{q-a}{1-\overline{a}q}$ for some $a \in \mathbb{C}$ with $|a| < 1$. If $\Phi = H^\infty \cup \{\overline{q}\}$ and $\Psi = H^\infty \cup \{\overline{\psi}\}$, then $\mathrm{Lat}\, T_\Phi = \mathrm{Lat}\, T_\Psi$.*

Proof. Suppose that $\mathcal{M} \in \mathrm{Lat}\, T_\Phi$. Since $\mathcal{M}$ is invariant under $T_{\overline{q}}$, we see that $T_q \mathcal{N} \subset \mathcal{N}$ where $\mathcal{N} = H^2 \ominus \mathcal{M}$. In particular, we have

$$q\mathcal{N} \subset \mathcal{N}.$$

Note that $\mathcal{N}$ is a closed subspace in H^2. We obtain

$$(q-a)\mathcal{N} \subset \mathcal{N} \quad \text{and} \quad (1-\overline{a}q)^{-1}\mathcal{N} \subset \mathcal{N}$$

for $|a| < 1$. Thus $T_\psi \mathcal{N} \subset \mathcal{N}$ and so $T_{\overline{\psi}} \mathcal{M} \subset \mathcal{M}$. This shows that $\mathrm{Lat}\, T_\Phi \subset \mathrm{Lat}\, T_\Psi$. Since $q = \frac{\psi + a}{1+\overline{a}\psi}$, we can prove the converse inclusion similarly. □

5 Examples In this section, we will describe $\mathrm{Lat}\, T_\Phi$ for some concrete Φ. To begin with, in Corollary 5.3, we will show the case that $\mathrm{Lat}\, T_\Phi$ is trivial. To show this, we consider when Φ is the union of H^∞ and $\{\overline{q}\}$ for a one variable inner function $q = q(z)$.

Theorem 5.1. *Let $\Phi = H^\infty \cup \{\overline{q(z)}\}$ for a one variable non-constant inner function $q = q(z)$. If $\mathcal{M} \in \mathrm{Lat}\, T_\Phi$, then there exists some one variable inner function $Q = Q(w)$ such that $\mathcal{M} = Q(w)H^2$.*

Proof. Since $q = q(z)$ is a one variable non-constant inner function, there exist some $a, b \in \mathbb{C}$ such that $q(b) = a$ and $|a| < 1, |b| < 1$. Put $\psi = \frac{q-a}{1-\overline{a}q}$. Since $\psi(b) = 0$, we write $\psi = q_0 q_1$ where $q_0 = \frac{z-b}{1-\overline{b}z}$ and $q_1(z)$ is inner. If we put $\Psi = H^\infty \cup \{\overline{\psi}\}$, then $\mathrm{Lat}\, T_\Phi = \mathrm{Lat}\, T_\Psi$ by Proposition 4.5. This implies that $\mathcal{M}$ is invariant under $T_{\overline{\psi}} = T_{\overline{q_0 q_1}}$. So we have that

$$T_{\overline{q_0}}\mathcal{M} = T_{\overline{q_0 q_1}} q_1 \mathcal{M} \subset T_{\overline{q_0 q_1}} \mathcal{M} \subset \mathcal{M}.$$

Therefore we obtain $T_{\overline{q_0}}\mathcal{M} \subset \mathcal{M}$. So if we put $\Omega = H^\infty \cup \{\overline{q_0}\}$, then $\mathrm{Lat}\, T_\Psi \subset \mathrm{Lat}\, T_\Omega$. Moreover, by Proposition 4.5, we obtain $\mathrm{Lat}\, T_\Omega = \mathrm{Lat}\, T_{\Omega'}$, where $\Omega' = H^\infty \cup \{\overline{z}\}$. Hence we have $T_{\overline{z}}\mathcal{M} \subset \mathcal{M}$. By (2) of Theorem 4.3, we see that

$$\mathcal{M} \ominus z\mathcal{M} \subset H^2 \ominus zH^2 = H^2(\Gamma_w)$$

and so $w(\mathcal{M} \ominus z\mathcal{M}) \subset \mathcal{M} \ominus z\mathcal{M} \subset H^2(\Gamma_w)$. The Beurling theorem implies that $\mathcal{M} \ominus z\mathcal{M} = QH^2(\Gamma_w)$, where $Q = Q(w)$. Thus we have $\mathcal{M} = Q(w)H^2$. □

Remark 5.2. Let $\Phi = H^\infty \cup \{\overline{q(w)}\}$ for a one variable non-constant inner function $q = q(w)$. Making the same argument for Theorem 5.1, we can show that if $\mathcal{M} \in \mathrm{Lat}\, T_\Phi$, then there exists some one variable inner function $Q = Q(z)$ such that $\mathcal{M} = Q(z)H^2$.

Corollary 5.3. *If* $\Phi = H^\infty \cup \{\overline{q_1(z)q_2(w)}\}$ *for one variable non-constant inner functions* $q_1 = q_1(z)$ *and* $q_2 = q_2(w)$*, then* $\operatorname{Lat} T_\Phi = \{\langle 0 \rangle, H^2\}$.

Proof. If $\mathcal{M} \in \operatorname{Lat} T_\Phi$, then we have that

$$T_{\overline{q_1}}\mathcal{M} = T_{\overline{q_1 q_2}}(q_2\mathcal{M}) \subset T_{\overline{q_1 q_2}}\mathcal{M} \subset \mathcal{M}.$$

Hence by Theorem 5.1, there exists some one variable inner function $Q_2 = Q_2(w)$ such that $\mathcal{M} = Q_2(w)H^2$. Similarly we have $T_{\overline{q}_2}\mathcal{M} \subset \mathcal{M}$ and so $\mathcal{M} = Q_1(z)H^2$ for some one variable inner function $Q_1 = Q_1(z)$. This happens only when Q_1 and Q_2 are constant. Therefore we obtain the corollary. □

Next, we will show the case that $\operatorname{Lat} T_\Phi$ is nontrivial. Now we study the case of $\Phi = H^\infty \cup \{\overline{q_1}q_2, q_1\overline{q_2}\}$ for some non-constant inner functions $q_1 = q_1(z)$ and $q_2 = q_2(w)$. We note that if $\mathcal{M} = \sum_{k=0}^n q_1^{n-k}q_2^k H^2$, then it is clear that $\mathcal{M}$ is in $\operatorname{Lat} T_\Phi$. Theorem 5.4 shows properties of $\operatorname{Lat} T_\Phi$.

Theorem 5.4. *Let* $\Phi = H^\infty \cup \{\overline{q_1}q_2, q_1\overline{q_2}\}$ *for some non-constant one variable inner functions* $q_1 = q_1(z)$ *and* $q_2 = q_2(w)$*. Suppose that* $\mathcal{M} \in \operatorname{Lat} T_\Phi$*. Then the following statements hold.*

(1) $q_1\mathcal{M} \subset q_2\mathcal{M} + H^2 \ominus q_2H^2$ *and* $q_2\mathcal{M} \subset q_1\mathcal{M} + H^2 \ominus q_1H^2$.

(2) *If there exists some natural number* n *such that* $q_1^n \in \mathcal{M}$ *and* $q_1^{n-1} \notin \mathcal{M}$*, then we have* $q_1^l q_2^m \notin \mathcal{M}$ *for* $l \geq 0, m \geq 0$ *and* $l + m < n$.

(3) *If there exists some natural number* n *such that* $q_1^n \in \mathcal{M}$*, then we have* $\mathcal{M} \supset \sum_{k=0}^n q_1^{n-k}q_2^k H^2$.

Proof. (1) By (1) of Lemma 3.6,

$$q_1\overline{q_2}\mathcal{M} \subset \mathcal{M} + \overline{K_\mathcal{M}^\Phi}.$$

Then we have

$$q_1\mathcal{M} \subset q_2\mathcal{M} + q_2\overline{K_\mathcal{M}^\Phi} \subset q_2\mathcal{M} + q_2\overline{K}$$

since $\overline{K_\mathcal{M}^\Phi}$ is a subset of K. Hence $q_1\mathcal{M} \subset q_2\mathcal{M} + q_2\overline{K} \cap H^2$. Moreover from the definition of $\overline{K}$, it is clear that $q_2\overline{K} \cap H^2 \subset H^2 \ominus q_2H^2$. Therefore we obtain

$$q_1\mathcal{M} \subset q_2\mathcal{M} + H^2 \ominus q_2H^2.$$

The same argument shows that $q_2\mathcal{M} \subset q_1\mathcal{M} + H^2 \ominus q_1H^2$.
(2) If $q_1^l q_2^m$ were in $\mathcal{M}$, then we would have

$$T_{\overline{q_1}}^{n-1-m-l} T_{q_1\overline{q_2}}^m (q_1^l q_2^m) = T_{\overline{q_1}}^{n-1-m-l}(q_1^{m+l}) = q_1^{n-1} \in \mathcal{M}.$$

This contradicts that $q_1^{n-1} \notin \mathcal{M}$. Hence we conclude that $q_1^l q_2^m \notin \mathcal{M}$ for $l \geq 0, m \geq 0$ and $l + m < n$.
(3) Since q_1^n is in $\mathcal{M}$, we have $T_{\overline{q_1}q_2}^j(q_1^n) = q_1^{n-j}q_2^j \in \mathcal{M}$ for $0 \leq j \leq n$. Let $\mathcal{P}_+$ be the set of analytic trigonometric polynomials. Then we see that $\sum_{j=0}^n q_1^{n-j}q_2^j\mathcal{P}_+ \subset \mathcal{M}$. Since H^2 is the closure in the L^2-norm of $\mathcal{P}_+$ and the multiplication by an inner function is continuous, we have

$$\sum_{j=0}^n q_1^{n-j}q_2^j H^2 \subset \mathcal{M}.$$

□

In [3], the first author studied $\operatorname{Lat} T_\Psi$ for $\Psi = \{z^n\overline{w}, \overline{z}^n w\}$ for a fixed natural number n. In this context, we consider the case when $\Phi = H^\infty \cup \{\overline{z}w, z\overline{w}\}$. In Theorem 5.5, we describe $\operatorname{Lat} T_\Phi$ completely and show that $\operatorname{Lat} T_\Phi$ is nontrivial. Moreover we provide a concrete example of invariant subspaces of T_z and T_w. We recall that $H^2(\Gamma_z)$ or $H^2(\Gamma_w)$ denotes a one variable Hardy space on the unit circle $\Gamma = \Gamma_z$ or Γ_w respectively.

Theorem 5.5. *Let $\Phi = H^\infty \cup \{\overline{z}w, z\overline{w}\}$. Then the following statements hold.*

(1) *If $\mathcal{M} \in \operatorname{Lat} T_\Phi$, then*

$$z\mathcal{M} \subset w\mathcal{M} + H^2(\Gamma_z) \quad \text{and} \quad w\mathcal{M} \subset z\mathcal{M} + H^2(\Gamma_w).$$

(2) *A closed subspace $\mathcal{M}$ is in $\operatorname{Lat} T_\Phi$ if and only if there exists the smallest natural number N such that z^N and w^N belong to $\mathcal{M}$ and $\mathcal{M} = \sum_{j=0}^N z^{N-j} w^j H^2$.*

Proof. (1) We note that equalities

$$H^2 \ominus zH^2 = H^2(\Gamma_w) \quad \text{and} \quad H^2 \ominus wH^2 = H^2(\Gamma_z)$$

hold. Applying (1) of Theorem 5.4, we obtain the conclusion.
(2) The 'if' part is not hard to prove. Now we show the 'only if' part. Assume that $\mathcal{M} \in \operatorname{Lat} T_\Phi$. It is clear that there exists the smallest natural number N satisfying the following condition; there exists $f \in \mathcal{M}$ such that $\frac{\partial^N}{\partial z^N} f(0,0) \neq 0$ but $\frac{\partial^k}{\partial z^k} g(0,0) = 0$ for all $g \in \mathcal{M}$ if $k < N$. In order to show that $z^N \in \mathcal{M}$, we consider the extremal problem

$$\sup\{\operatorname{Re} \frac{\partial^N}{\partial z^N} f(0,0); f \in \mathcal{M}, \|f\| \leq 1\}.$$

Note that the mapping $f \mapsto \frac{\partial^N}{\partial z^N} f(0,0)$ is a bounded linear functional on H^2. By the Riesz representation theorem, this extremal problem has a unique solution $G \in \mathcal{M}$ with $\|G\| = 1$ and $\frac{\partial^N}{\partial z^N} G(0,0) > 0$. We will see that $G = z^N$. Put

$$g_f = \frac{G + T_{z\overline{w}}^{N+1} f}{\|G + T_{z\overline{w}}^{N+1} f\|}$$

for each $f \in \mathcal{M}$. Since $\operatorname{Re} \frac{\partial^N}{\partial z^N} g_f(0,0) \leq \frac{\partial^N}{\partial z^N} G(0,0)$, it is easy to see that $\|G + T_{z\overline{w}}^{N+1} f\| \geq 1$ for any $f \in \mathcal{M}$. From this inequality, we obtain $G \perp T_{z\overline{w}}^{N+1} f$. Hence we have $T_{\overline{z}w}^{N+1} G = 0$. Similarly we have $T_{z\overline{w}} G = 0$. From these equalities, we obtain $G = z^N$. It is obvious that $w^N = T_{\overline{z}w}^N z^N$ is in $\mathcal{M}$.

By (3) of Theorem 5.4, we obtain $\mathcal{M} \supset \sum_{j=0}^N z^{N-j} w^j H^2$. Moreover, by (2) of Theorem 5.4, we see that $z^{k_1} w^{k_2} \notin \mathcal{M}$ for $0 \leq k_1 + k_2 < N$, which shows the converse inclusion. □

Corollary 5.6 shows that each $\mathcal{M}$ in $\operatorname{Lat} T_\Phi$ contains an invariant subspace $z^N H^2 + w^N H^2$ for some natural number N.

Corollary 5.6. *Let $\Phi = H^\infty \cup \{\overline{z}w, z\overline{w}\}$. If $\mathcal{M} \in \operatorname{Lat} T_\Phi$, then there exists some natural number N such that*

$$\mathcal{M} \supset z^N H^2 + w^N H^2.$$

Proof. By (2) of Theorem 5.5, there exists some natural number N such that

$$\mathcal{M} = \sum_{j=0}^N z^j w^{N-j} H^2.$$

Then we obtain

$$z^N H^2 + w^N H^2 \subset \sum_{j=0}^{N} z^j w^{N-j} H^2 = \mathcal{M}.$$

Hence the statement is clear. □

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