CHARACTERIZATIONS OF ω-LIKE CLOSED SETS AND SEPARATION AXIOMS IN TOPOLOGICAL SPACES

H. Maki, N. Rajesh and S. Shanthi

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Abstract. One of the aim of the present paper is introduce the concept of ω^ρ-closed sets in topological space (X, τ) (cf. Definition 1.4) and study topological properties of their classes of sets, where $\rho : SO(X, \tau) \to P(X)$ is a function defined by $\rho(V) := V, \rho(V) := Int(V)$ or $\rho(V) := Int(Cl(V))$ for every semi-open set V of (X, τ). Furthermore, their relation ships with other generalied closed sets are investigated (cf. Remark 2.2). Using some analogous concept of the Jankovic-Reilly decomposition of sets ([2]), the concept of ω^ρ-closed sets is completely characterized (cf. Theorem 4.8(iii)). In Section 5 and Section 6, some new separation axioms are introduced and investigated (i.e. $(\omega^{\rho 1}, \omega^{\rho 2})$-$T^\rho_{1/2}$-separation axioms (cf. Definition 5.3(I)(i), Theorem 5.11, Theorem 5.13, Theorem 5.15), where $\rho 1, \rho 2, \rho \in \{id, \circ, \circ -\}$ (cf. Definition 1.3). Throughout the present paper, examples are almost stated from topics of the digital line $(\mathbb{Z}, \kappa)$ due to E. D. Khalimsky (cf. Definition 2.3).

1 Introduction and preliminaries Throughout the present paper, (X, τ) represents a nonempty topological space on which no separation axioms are assumed unless otherwise mentioned and $P(X)$ denotes the power set of X. For a subset A of (X, τ), $Cl(A)$, $Int(A)$ and $Ker(A)$ denote the closure, interior and kernel of A with respect to the topological space (X, τ) respectively; i.e., $Cl(A) := \cap\{F | A \subset F \text{ and } X \setminus F \in \tau\}$, $Int(A) := \cup\{U | U \subset A \text{ and } U \in \tau\}$ and $Ker(A) := \cap\{V | A \subset V \text{ and } V \in \tau\}$. A subset B of (X, τ) is said to be *semi-open* ([13, in 1963],[8]), if $B \subset Cl(Int(B))$ holds in (X, τ). And, a subset E of (X, τ) is said to be *preopen* ([19, in 1982]), if $E \subset Int(Cl(E))$ holds in (X, τ). The family of all semi-open sets (resp. preopen sets) of (X, τ) is denoted by $SO(X, \tau)$ (resp. $PO(X, \tau)$). For a subset A of (X, τ), $pCl(A)$ denotes the preclosure of A with respect to (X, τ), i.e., $pCl(A) := \cap\{F | A \subset F \text{ and } X \setminus F \in PO(X, \tau)\}$.

We recall the following concepts of two classes of generalized closed sets of a topological space (X, τ).

Definition 1.1 (i) ([27, in 1995], [28, in 2000;Definition 2.1],[26, in 2002]) A subset A of (X, τ) is said to be *ω-closed* in (X, τ), if $Cl(A) \subset U$ whenever $A \subset U$ and $U \in SO(X, \tau)$.

(ii) ([22, in 2005]) A subset A of (X, τ) is said to be *weakly ω-closed* in (X, τ), if $Cl(Int(A)) \subset U$ whenever $A \subset U$ and $U \in SO(X, \tau)$.

(iii) A subset B of (X, τ) is said to be *ω-open* ([27]) (resp. *weakly ω-open* ([22, Definition 3.22])) in (X, τ), if $X \setminus B$ is ω-closed (resp. weakly ω-closed) in (X, τ).

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We use the following notation and definition.

Notation 1.2 ($\bullet$1) $\omega C(X,\tau) := \{A|\ A$ is ω-closed in $(X,\tau)\}$;
($\bullet 1'$) $\omega O(X,\tau) := \{B|\ B$ is ω-open in $(X,\tau)\}$;
($\bullet$2) ${}^{w}\omega C(X,\tau) := \{A|\ A$ is weakly ω-closed in $(X,\tau)\}$;
($\bullet 2'$) ${}^{w}\omega O(X,\tau) := \{B|\ B$ is weakly ω-open in $(X,\tau)\}$.

Definition 1.3 Let $\mathcal{E}_X$ be a subfamily of $P(X)$. The following function $\rho : \mathcal{E}_X \to P(X)$ is used on the present paper: for every set $U \in \mathcal{E}_X$ and a topological space (X,τ),
(i) $\rho := \circ : \mathcal{E}_X \to P(X)$ defined by $\circ(U) := Int(U)$;
(ii) $\rho := \circ- : \mathcal{E}_X \to P(X)$ defined by $\circ-(U) := Int(Cl(U))$;
(iii) $\rho := \circ-\circ : \mathcal{E}_X \to P(X)$ defined by $\circ-\circ(U) := Int(Cl(Int(U)))$;
(iv) $\rho := -\circ : \mathcal{E}_X \to P(X)$ defined by $-\circ(U) := Cl(Int(U))$;
(v) $\rho := -\circ- : \mathcal{E}_X \to P(X)$ defined by $-\circ-(U) := Cl(Int(Cl(U)))$;
(vi) $\rho := id : \mathcal{E}_X \to P(X)$ defined by $id(U) := U$.

We define some related classes of ω-like closed sets (cf. Definition 1.4, Notation 1.5).

Definition 1.4 Let A and B be subsets of a topological space (X,τ). And, let $\rho : SO(X,\tau) \to P(X)$ be a function such that $\rho \in \{id, \circ, -\circ, -\circ-, \circ-, \circ-\circ\}$ (cf. Definition 1.3 above for $\mathcal{E}_X := SO(X,\tau)$). A subset A is said to be ω^{ρ}*-closed* in (X,τ), if $Cl(A) \subset \rho(U)$ holds whenever $A \subset U$ and $U \in SO(X,\tau)$. The complemet $X \setminus B$ of an ω^{ρ}-closed set B is called an ω^{ρ}*-open set* of (X,τ).

We have the following equivalent expression: a subset A is ω^{id}-closed (resp. ω^{id}-open) in (X,τ) if and only if A is ω-closed (resp. ω-open) in (X,τ).

Notation 1.5 (i) For each function $\rho : SO(X,\tau) \to P(X)$ with $\rho \in \{id, \circ, -\circ, -\circ-, \circ-, \circ-\circ\}$ (cf. Definition 1.3 above for $\mathcal{E}_X := SO(X,\tau)$), we use the following notation:
($\bullet 3^{\rho}$) $\omega^{\rho}C(X,\tau) := \{A|\ A$ is ω^{ρ}-closed in $(X,\tau)\})$;
($\bullet 3'^{\rho}$) $\omega^{\rho}O(X,\tau) := \{U|\ U$ is ω^{ρ}-open in $(X,\tau)\}$ (cf. Definition 1.4 above).
(ii) ($\bullet$4) $psC(X,\tau) := \{A|\ A$ is ps-closed in $(X,\tau)\}$;
($\bullet 4'$) $psO(X,\tau) := \{U|\ U$ is ps-open in $(X,\tau)\}$.
The concept of *ps-closed sets* of (ii) above (cf. [3, Definition 2.1]) is defined as follows: a subset A is called a *ps-closed set* of (X,τ) if $pCl(A) \subset U$ whenever $A \subset U$ and $U \in SO(X,\tau)$; and its complement $X \setminus A$ is called a ps-open set of (X,τ).
(iii) We note that $\omega^{id}C(X,\tau) = \omega C(X,\tau)$ and $\omega^{id}O(X,\tau) = \omega O(X,\tau)$ (cf. Notations 1.2, 1.5(i)).
(iv) ($\bullet$5) $C(X,\tau) := \{F|\ F$ is closed in (X,τ),i.e., $X \setminus F \in \tau\}$;
($\bullet$6) $PC(V,\tau) := \{F|\ F$ is preclosed in (X,τ),i.e., $X \setminus F \in PO(X,\tau)\}$.

The purposes of the present paper are to characterlize the ω-like closed sets of a topological space (cf. Theorem 2.1, Theorem 3.7, Proposition 4.4, Theorem 4.8) and to investigate the $(\omega^{\rho 1}, \omega^{\rho 2})$-$T^{\rho}_{1/2}$ separation axioms where $\rho 1, \rho 2, \rho \in \{id, \circ, \circ-\}$ (cf. Theorem 5.11, Theorem 5.13, Theorem 5.15). Moreover, in Section 6, it is shown that the digital line $(\mathbb{Z},\kappa)$ is $\omega^{\circ-}$-T_1 except $\mathbb{Z}_\kappa$ (cf. Definition 2.3, Theorem 6.1(iv)).

2 Properties on ω-like closed sets

For the families in Notation 1.5($\bullet 3^{\rho}$), ($\bullet$4), ($\bullet$6) and Notation 1.2 ($\bullet$1), ($\bullet$2), we have the following properties.

Theorem 2.1 (i) $\omega^{\circ}C(X,\tau) \subset \omega C(X,\tau) \subset \omega^{-\circ}C(X,\tau)$.
(ii) $\omega^{-\circ}C(X,\tau) = \omega^{-\circ-}C(X,\tau) = P(X)$.
(iii) $\omega^{\circ}C(X,\tau) \subset \omega^{\circ-}C(X,\tau) \subset \omega^{-\circ}C(X,\tau)$.
(iv) $\omega^{\circ-}C(X,\tau) = \omega^{\circ-\circ}C(X,\tau)$.
(v) ([3, Corollary 2.6 (iv), Table 1]) $psC(X,\tau) = PC(X,\tau)$.
(vi) ([26],[27],[3]) $C(X,\tau) \subset \omega C(X,\tau) \subset {}^{w}\omega C(X,\tau)$.
(vii) ${}^{w}\omega C(X,\tau) = PC(X,\tau)$.

Proof. **(i) - (iv)** They are proved by definitions.

(vii) *Proof of the equality* ${}^{w}\omega C(X,\tau) = psC(X,\tau)$: let $A \in {}^{w}\omega C(X,\tau)$. For any subset $U \in SO(X,\tau)$ such that $A \subset U$, we have that $Cl(Int(A)) \subset U$ and so $pCl(A) = A \cup Cl(Int(A)) \subset U$; and so we see $pCl(A) \subset U$. Thus, we have that ${}^{w}\omega C(X,\tau) \subset psC(X,\tau)$. Conversely, supppose that $A \in psC(X,\tau)$. Let $U \in SO(X,\tau)$ such that $A \subset U$. Then, $pCl(A) = A \cup Cl(Int(A)) \subset U$ and hence $Cl(Int(A)) \subset U$. Therefore, A is ${}^{w}\omega C(X,\tau)$. We proved that $psC(X,\tau) \subset {}^{w}\omega C(X,\tau)$.

Thus we show the required equality using (v). □

Remark 2.2 By Theorem 2.1 above, the following diagram of implications is obtained. All implications in the following diagram are not reversible (cf. Example 2.4 (i) - (v) below); and two concepts of $C(X,\tau)$ and $\omega^{\circ}C(X,\tau)$ are independent (cf. Example 2.4(vi) below).

$$\begin{array}{lll}
A \in C(X,\tau) & & A \in {}^{w}\omega C(X,\tau) = psC(X,\tau) = PC(X,\tau) \\
& \searrow \qquad \nearrow & \\
\not\uparrow\ \not\downarrow & A \in \omega C(X,\tau) & \\
& \nearrow \qquad \searrow & \\
A \in \omega^{\circ}C(X,\tau) & & A \in \omega^{-\circ}C(X,\tau) = \omega^{-\circ-}C(X,\tau) = P(X) \\
\downarrow & \nearrow & \\
A \in \omega^{\circ-}C(X,\tau) = \omega^{\circ-\circ}C(X,\tau) & &
\end{array}$$

The concept of *the digital line* $(\mathbb{Z},\kappa)$ is initiatived by E.D. Khalimsky and sometimes it is called the Khalimsky line ([10, in 1990]).

Definition 2.3 ([10, in 1990] and references there;[11, in 1991;p.905]; e.g.,[17, in 2014;Section 3]). *The digital line* or so called *Khalimsky line* $(\mathbb{Z},\kappa)$ is the set $\mathbb{Z}$ of all integers, equipped with the topology κ having $\{\{2m-1, 2m, 2m+1\} |\ m \in \mathbb{Z}\}$ as a subbase. *The digital plane* or *Khalimsky plane* is the Cartesian product of 2-copies of the digital line $(\mathbb{Z},\kappa)$; this topological space is denoted by $(\mathbb{Z}^2,\kappa^2)$ (cf. [12, in 1994;Definition 4])

Example 2.4 (i) An ω-closed set need not be ω°-closed (i.e., $\omega^{\circ}C(X,\tau) \not\leftarrow \omega C(X,\tau)$): we give two examples as follows.

(i-1) Let $(X,\tau) := (\mathbb{Z},\kappa)$ be the digital line (cf. Definition 2.3 above) and $A := \{2m\}$, where $m \in \mathbb{Z}$. Then, by definition of the topology $\kappa, A := \{2m\}$ is closed and so $A \in \omega C(\mathbb{Z},\kappa)$. We show $A \notin \omega^{\circ}C(\mathbb{Z},\kappa)$. Indeed, there exists a semi-open set $U := \{2m, 2m+1\}$ such that $A \subset U$; and so we have that $Int(U) = \{2m+1\}$ and $Cl(A) = \{2m\} \not\subset \{2m+1\} = Int(U)$. This shown that the set A is not ω°-closed in $(\mathbb{Z},\kappa)$.

(i-2) We can give an example on the Euclidean line $(X,\tau) := (\mathbb{R},\epsilon)$. Let $A := \{x, y\}$, where x and y are distinct point of $(\mathbb{R},\epsilon)$. There exists a semi-open set $U := \{t \in \mathbb{R} | x \leq t < z\} \cup \{t \in \mathbb{R} | z < t \leq y\}$, where z is a point with a relation $x < z < y$. Then, $A \subset U$

and $Cl(A) = \{x, y\} \not\subset Int(U)$, because $Int(U) = \{t \in \mathbb{R} | x < t < z\} \cup \{t \in \mathbb{R} | z < t < y\}$; and so $A \notin \omega^{\circ}C(\mathbb{R}, \epsilon)$. And A is closed and so $A \in \omega C(\mathbb{R}, \epsilon)$.

(ii) A ${}^{w}\omega$-closed set (=preclosed set; cf. Theorem 2.1 (v)(vii)) need not be ω-closed (i.e., $\omega C(X, \tau) \not\leftarrow {}^{w}\omega C(X, \tau)$): let $(X, \tau) := (\mathbb{Z}^2, \kappa^2)$ be the digital plane (cf. Definition 2.3 above) and $A := \{x, y\}$ a subset of $(\mathbb{Z}^2, \kappa^2)$, where $x = (2m, 2s)$ and $y = (2m + 1, 2s)$ for some integers m and s. Then, first we show that $Cl(Int(A)) = Cl(\emptyset) = \emptyset \subset A$; and so $A \in PC(\mathbb{Z}^2, \kappa^2)$ and hence $A \in {}^{w}\omega C(\mathbb{Z}^2, \kappa^2)$ (cf. Theorem 2.1(iii)). We note that the subset A of the present example (ii) is a preclosed set which is not closed in $(\mathbb{Z}^2, \kappa^2)$.

Secondly, we show that $A \notin \omega C(\mathbb{Z}^2, \kappa^2)$. Indeed, we take a subset $U := A \cup \{(2m + 1, 2s + 1)\}$; then U is semi-open in $(\mathbb{Z}^2, \kappa^2)$. Indeed since $\kappa^2 := \kappa \times \kappa$, we see that $Cl(Int(U)) = Cl(\{(2m + 1, 2s + 1)\}) = \{2m, 2m + 1, 2m + 2\} \times \{2s, 2s + 1, 2s + 2\} \supset U$ hold and so $U \subset Cl(Int(U))$ (i.e., $U \in SO(\mathbb{Z}^2, \kappa^2)$). Finally, we have that $A \subset U$ and $Cl(A) \not\subset U$. Indeed, $Cl(A) = Cl(\{x\}) \cup Cl(\{y\}) = A \cup \{(2m + 2, 2s)\} \not\subset U$ hold, because $\{x\} = \{(2m, 2s)\}$ is closed and $Cl(\{y\}) = Cl(\{(2m + 1, 2s)\} = \{(2m, 2s), (2m + 1, 2s), (2m + 2, 2s)\}$ holds in $(\mathbb{Z}^2, \kappa^2)$. Therefore, A is not ω-closed in $(\mathbb{Z}^2, \kappa^2)$. Moreover, we have a digital geometric example in Remark 4.5(ii).

(iii) An $\omega^{\circ-}$-closed set need not be ω°-closed (i.e., $\omega^{\circ}C(X, \tau) \not\leftarrow \omega^{\circ-}C(X, \tau)$): let (X, τ) be a topological space defined by $X := \{a, b, c\}$ and $\tau := \{\emptyset, \{a\}, \{a, b\}, X\}$. Then, we have $SO(X, \tau) = \{\emptyset, \{a\}, \{a, b\}, \{a, c\}, X\}$. Let $A := \{a, c\}$ and $U \in SO(X, \tau)$ with $A \subset U$; and so $U = \{a, c\}$ or X. Then, $Cl(A) = X \subset Int(Cl(U))$, because $Int(Cl(U)) = X$ for each subset U; hence we show $A \in \omega^{\circ-}C(X, \tau)$. Moreover, we show that the subset A is not ω°-closed in (X, τ). Indeed, the subset A is a semi-open set with $Cl(A) = X \not\subset Int(A) = \{a\}$. In addtion, in Remark 4.5(ii) below, we have a geometric example of the present topic.

(iv) An $\omega^{-\circ}$-closed set need not be $\omega^{\circ-}$-closed (i.e., $\omega^{\circ-}C(X, \tau) \not\leftarrow \omega^{-\circ}C(X, \tau)$): let $A := \{2m + 1, 2m + 2, 2m + 3, 2m + 4\}$ be a subset of $(\mathbb{Z}, \kappa)$. Since $A \in P(\mathbb{Z}) = \omega^{-\circ}C(\mathbb{Z}, \kappa)$ (cf. Theorem 2.1(ii)), we should show $A \notin \omega^{\circ-}C(\mathbb{Z}, \kappa)$. Indeed, let $U := A$; and $Cl(Int(U)) = Cl(\{2m+1, 2m+2, 2m+3\}) = \{2m, 2m+1, 2m+2, 2m+3, 2m+4\} \supset U$ and so $U \in SO(\mathbb{Z}, \kappa)$ such that $A \subset U$. For this semi-open set U, we have that :

- $Cl(A) = \{2m, 2m + 1, 2m + 2, 2m + 3, 2m + 4\}$ and;
- $Int(Cl(U)) = \{2m + 1, 2m + 2, 2m + 3\}$.

Thus, it is shown that $Cl(A) \not\subset Int(Cl(U))$, i.e., A is not $\omega^{\circ-}$-closed set in $(\mathbb{Z}, \kappa)$.

(v) An ω-closed set need not be a closed set (i.e., $C(X, \tau) \not\leftarrow \omega C(X, \tau)$): such example is shown by [26].

(vi) Two families $C(X, \tau)$ and $\omega^{\circ}C(X, \tau)$ are independent.

· *Proof of* $\omega^{\circ}C(X, \tau) \not\leftarrow C(X, \tau)$: the subset $A := \{2m\}$ of $(\mathbb{Z}, \kappa)$ in (i)(i-1) is a closed singleton, where $m \in \mathbb{Z}$, and it is not ω°-closed in $(\mathbb{Z}, \kappa)$ (cf. (i)(i-1)).

· *Proof of* $C(X, \tau) \not\leftarrow \omega^{\circ}C(X, \tau)$: let (X, τ) be a topological space defined by $X := \{a, b, c\}$ and $\tau := \{\emptyset, \{a\}, \{b, c\}, X\}$. Let $A := \{b\}$ be a not closed singleton. Let U be a semi-open set containing A; then $U = \{b, c\}$ or X and so $Int(U) = U$. Then, $Cl(A) = \{b, c\} \subset Int(U) = U$ hold and so we show that $A \in \omega^{\circ}C(X, \tau)$.

3 More characterizations of ${}^{w}\omega$-closed sets and related Janković Reilly decompositions

In the present section, we give more characterizations of ${}^{w}\omega$-closed sets (resp. ps-closed sets) by Theorem 3.7 (i)(1)(2)(3) (resp. (i) (4)(5)(6)(7)) below, even if we know that ${}^{w}\omega(X, \tau) = psC(X, \tau) = PC(X, \tau)$ hold for a topological space (X, τ) (cf. Theorem 2.1 (v)(vii)). They are done by an analogy of the Janković Reilly decomposition method; and so we recall them as follows (cf. Theorem 3.1, Notation 3.2, Lemma 3.4, Lemma 3.6 below).

Theorem 3.1 (i) ([9, Lemma 2]) *Every singleton $\{x\}$ of a topological space (X,τ) is either preopen* (i.e., $\{x\} \subset Int(Cl(\{x\}))$) *or nowhere dense* (i.e., $Int(Cl(\{x\})) = \emptyset$).

(ii) (Janković Reilly decompostion;[2, p. 40, line +10]; cf. Theorem 3.3 below) *Any topological space (X,τ) has the following decomposition:*

$X = X_1 \bigcup X_2$ *with* $X_1 \bigcap X_2 = \emptyset$, *where X_1 and X_2 are defined respectively by:*

(1a) $X_1 := \{x \in X |\ \{x\}$ *is nowhere dense in* $(X,\tau)\}$;

(1b) $X_2 := \{x \in X |\ \{x\}$ *is preopen in* $(X,\tau)\}$.

The decomposition $X = X_1 \cup X_2$(disjoint union) of Theorem 3.1 is usefull and it is called the *Janković Reilly decomposition* of X (e.g., [2, p. 40, line +10]). Moreover, we use the following convenient notation, because we want to investigate more decompositions.

Notation 3.2 For a subset E of (X,τ), we define the following subsets of E:

(•2a) $E_{\mathcal{ND}} := \{x |\ x \in E$ and $\{x\}$ is nowhere dense in $(X,\tau)\}$,

(i.e., $E_{\mathcal{ND}} = X_1 \cap E$ and $X_1 = X_{\mathcal{ND}}$, cf. (1a) of Theorem 3.1(ii) above);

(•2b) $E_{\mathcal{PO}} := \{x |\ x \in E$ and $\{x\}$ is preopen in $(X,\tau)\}$,

(i.e., $E_{\mathcal{PO}} = X_2 \cap E$ and $X_2 = X_{\mathcal{PO}}$, cf. (1b) of Theorem 3.1(ii) above);

(•2c) $E_{\mathcal{SC}} := \{x |\ x \in E$ and $\{x\}$ is semi-closed in $(X,\tau)\}$;

(•2d) $E_{\omega\mathcal{O}} := \{x |\ x \in E$ and $\{x\}$ is ω-open in $(X,\tau)\}$;

(•2e) $E_{\tau} := \{x |\ x \in E$ and $\{x\}$ in open in $(X,\tau)\}$;

(•2f) $E_{\mathcal{C}} := \{x |\ x \in E$ and $\{x\}$ in closed in $(X,\tau)\}$;

(•2g) $E_{\mathcal{RO}} := \{x |\ x \in E$ and $\{x\}$ in regular-open in (X,τ), i.e., $\{x\} = Int(Cl(\{x\}))\}$.

By using Notation 3.2 (•2a),(•2b) above, the *Janković Reilly decomposition* in Theorem 3.1(ii) is stated as follows.

Theorem 3.3 (Theorem 3.1(ii) above, [9, Lemma 2]) *For any subset E of (X,τ), $E = E_{\mathcal{PO}} \cup E_{\mathcal{ND}}$ and $E_{\mathcal{PO}} \cap E_{\mathcal{ND}} = \emptyset$ hold.*

Lemma 3.4 (i) *For any subset E of (X,τ), $E = E_{\mathcal{SC}} \cup E_{\omega\mathcal{O}}$ holds.*

(ii) *For a topological space (X,τ) and a subset E of (X,τ),*

(1) $X_{\mathcal{PO}} \cap X_{\mathcal{SC}} = (X_{\mathcal{PO}})_{\mathcal{SC}} = X_{\mathcal{RO}}$ *and* $X_{\mathcal{ND}} \cap X_{\omega\mathcal{O}} = (X_{\mathcal{ND}})_{\omega\mathcal{O}} \subset X_{\tau}$ *hold, and*

(2) $E_{\mathcal{PO}} \cap E_{\mathcal{SC}} = (E_{\mathcal{PO}})_{\mathcal{SC}} = E_{\mathcal{RO}}$ *and* $E_{\mathcal{ND}} \cap E_{\omega\mathcal{O}} = (E_{\mathcal{ND}})_{\omega\mathcal{O}} \subset E_{\tau}$ *hold.*

(iii) *Suppose one of the following properties:*

(a) $E_{\mathcal{ND}} = \emptyset$ *and* $E_{\mathcal{RO}} = \emptyset$; (b) $E_{\tau} = \emptyset$ *and* $(E_{\mathcal{PO}})_{\omega\mathcal{O}} = \emptyset$.

Then, $E_{\mathcal{SC}} \cap E_{\omega\mathcal{O}} = \emptyset$ holds; and so the union $E_{\mathcal{SC}} \cup E_{\omega\mathcal{O}}$ of (i) *is a disjoint union under the assumptions* (a) *or* (b) *above.*

Proof. **(i)** Let $x \in E$. We consider the following two cases.

Case 1. $\{x\}$ is not semi-closed in (X,τ): for this case, we show that $x \in E_{\omega\mathcal{O}}$. Indeed, X is a unique semi-open set containing $X \setminus \{x\}$. Thus, $X \setminus \{x\}$ is ω-closed in (X,τ) and so $\{x\}$ is an ω-open set (i.e., $x \in E_{\omega\mathcal{O}}$).

Case 2. $\{x\}$ is semi-closed: for this case, it is shown that $x \in E_{\mathcal{SC}}$, by definition.

Therefore, using two cases, we have $E \subset E_{\mathcal{SC}} \cup E_{\omega\mathcal{O}}$; the converse inequality is trivial, by the definition of (•2c) and (•2d) in Notation 3.2. Thus we show the equality: $E = E_{\mathcal{SC}} \cup E_{\omega\mathcal{O}}$.

(ii) They are shown by using definitions.

(iii) In general, by using Theorem 3.1 (i.e., Theorem 3.3), it is shown that: $E_{\mathcal{SC}} \cap E_{\omega\mathcal{O}} = \{(E_{\mathcal{PO}} \cup E_{\mathcal{ND}})_{\mathcal{SC}}\} \cap \{(E_{\mathcal{PO}} \cup E_{\mathcal{ND}})_{\omega\mathcal{O}}\} = \{(E_{\mathcal{PO}})_{\mathcal{SC}} \cup (E_{\mathcal{ND}})_{\mathcal{SC}}\} \cap \{(E_{\mathcal{PO}})_{\omega\mathcal{O}} \cup (E_{\mathcal{ND}})_{\omega\mathcal{O}}\}$. We prove that $E_{\mathcal{SC}} \cap E_{\omega\mathcal{O}} = \emptyset$ holds under one of our assumptions (a), (b).
Case 1. we assume (a): for this case, we show that $E_{\mathcal{SC}} \cap E_{\omega\mathcal{O}} \subset (E_{\mathcal{PO}})_{\mathcal{SC}} \cup (E_{\mathcal{ND}})_{\mathcal{SC}} \subset$

$(E_{\mathcal{PO}})_{\mathcal{SC}} \cup E_{\mathcal{ND}} = E_{\mathcal{RO}} \cup E_{\mathcal{ND}} = \emptyset$ (cf. (ii)(2) above and the assumption (a)).
Case 2. we assume (b): for this case, we show that $E_{\mathcal{SC}} \cap E_{\omega\mathcal{O}} \subset (E_{\mathcal{PO}})_{\omega\mathcal{O}} \cup (E_{\mathcal{ND}})_{\omega\mathcal{O}} \subset (E_{\mathcal{PO}})_{\omega\mathcal{O}} \cup E_{\tau} = \emptyset$(cf. (ii)(2) above and the assumption (b)). □

Remark 3.5 (i) The property ($X = X_{\mathcal{SC}} \cup X_{\omega\mathcal{O}}$) of Lemma 3.4 (i) above does not imply a disjoint union in general. For example, let (X,τ) be a topological space defined by $X := \{a,b,c\}$ and $\tau := \{\emptyset, \{a\}, \{b,c\}, X\}$. Then, a sigleton $\{a\}$ is semi-closed and ω-open; and so $a \in X_{\mathcal{SC}} \cap X_{\omega\mathcal{O}}$.

(ii) For the digital line $(\mathbb{Z},\kappa)$, we have the following datum on the subsets defined Lemma 3.4: $\mathbb{Z}_{\mathcal{PO}} = \{2m+1|m \in \mathbb{Z}\}$=$\mathbb{Z}_{\kappa}$ (e.g. [6, Theorem 2.1(i)(a)]), $\mathbb{Z}_{\mathcal{ND}} = \{2m|m \in \mathbb{Z}\}$; and so we have the decomposition $\mathbb{Z} = \mathbb{Z}_{\mathcal{PO}} \cup \mathbb{Z}_{\mathcal{ND}}$. On the other hands, we have that $\mathbb{Z}_{\mathcal{SC}} = \mathbb{Z}$, $\mathbb{Z}_{\omega\mathcal{O}} = \{2m+1|m \in \mathbb{Z}\}$; for a nonempty set E, $E_{\mathcal{ND}} = \{2m \in E|m \in \mathbb{Z}\}$ and $E_{\mathcal{RO}} = \{2m+1 \in E|m \in \mathbb{Z}\}$=$E_{\kappa}$ and $(E_{\mathcal{PO}})_{\omega\mathcal{O}} = E_{\mathcal{PO}}$.

We need the following lemma in order to prove Theorem 3.7 below; Lemma 3.6 (iii) and (iv) are applied; we recall the definitions of $sKer(\bullet)$ and $pKer(\bullet)$: for a subset A of (X,τ), $sKer(A) := \bigcap\{U|\ U \in SO(X,\tau)$ and $A \subset U\}$ and $pKer(A) := \bigcap\{V|\ V \in PO(X,\tau)$ and $A \subset V\}$.

Lemma 3.6 (cf. [4, Proposition 2.1]) *Let B be a subset of (X,τ). Then, we have following properties.*

(i) [4, Proposition 2.1] $(sCl(B))_{\mathcal{PO}} \subset sKer(B)$.
(ii) [26, Proposition 2.2.18] $(Cl(B))_{\mathcal{PO}} \subset sKer(B)$.
(iii) $(Cl(Int(B)))_{\mathcal{PO}} \subset sKer(B)$.
(iv) $(pCl(B))_{\mathcal{PO}} \subset sKer(B)$.
(v) $(sKer(B))_{\mathcal{SC}} \subset B \subset sKer(B)$.
(vi) $((Cl(B))_{\mathcal{PO}})_{\mathcal{C}} \subset pKer(B)$.
(vi)' $((sCl(B))_{\mathcal{PO}})_{\mathcal{C}} \subset pKer(B)$.
(vi)" $((pCl(B))_{\mathcal{PO}})_{\mathcal{C}} \subset pKer(B)$.

Proof. **(iii)** Since $Cl(Int(B)) \subset Cl(B)$ holds and $E_{\mathcal{PO}} \subset F_{\mathcal{PO}}$ holds if $E \subset F$ in general, we have that $(Cl(Int(B)))_{\mathcal{PO}} \subset (Cl(B))_{\mathcal{PO}}$; and so, by (ii), it is shown that $(Cl(Int(B)))_{\mathcal{PO}} \subset sKer(B)$ holds.

(iv) This is proved by using (ii), because $pCl(E) \subset Cl(E)$ holds for any subset E of (X,τ).

(v) We prove only the implication $(sKer(B))_{\mathcal{SC}} \subset B$. Let $x \in (sKer(B))_{\mathcal{SC}}$ and assume that $x \notin B$. Since $X \setminus \{x\} \in SO(X,\tau)$ and $B \subset X \setminus \{x\}$, it is shown that $sKer(B) \subset X \setminus \{x\}$. Then we have that $\{x\} \subset sKer(B) \subset X \setminus \{x\}$; and hence this is a contradiction.

(vi) Let $x \in ((Cl(B))_{\mathcal{PO}})_{\mathcal{C}}$. Suppose that $x \notin pKer(B)$. There exists a set $V \in PO(X,\tau)$ such that $B \subset V$ and $x \notin V$. Taking the set $X \setminus V$, then $X \setminus V$ is preclosed in (X,τ) and $x \in X \setminus V$. Then, we have that:
$\{x\} \cup Cl(Int(\{x\})) = pCl(\{x\}) \subset pCl(X \setminus V)$=$X \setminus V$; and so
$(\cdot\ 1)$ $Cl(Int(\{x\})) \subset X \setminus V$. Since $x \in Cl(B)$ and $B \subset V$, $(\cdot 2)$ $x \in Cl(\{x\}) \subset Cl(V)$. Since $x \in X_{\mathcal{PO}}$, we have that $(\cdot 3)$ $\{x\} \subset Int(Cl(\{x\}))$; and so we have that: $(\cdot 4)$ the set $Int(Cl(\{x\}))$ is an open set containing x such that $x \in Cl(V)$.

By $(\cdot 2)$ and $(\cdot 4)$, it is shown that: $(\cdot 5)$ $Int(Cl(\{x\})) \cap V \neq \emptyset$. By using $(\cdot 1)$ and an assumption that $x \in X_{\mathcal{C}}$, it is shown that $Int(Cl(\{x\})) \cap V \subset Cl(Int(Cl(\{x\}))) \cap V = Cl(Int(\{x\})) \cap V \subset (X \setminus V) \cap V = \emptyset$. Therefore, we have that $Int(Cl(\{x\})) \cap V$=$\emptyset$; this contradicts the property $(\cdot 5)$ above.

(vi)' (resp. **(vi)"**) Since $sCl(B) \subset Cl(B)$ (resp. $pCl(B) \subset Cl(B)$) holds for every set B of (X,τ), (vi)' (resp.(vi)") is obtaned by (vi). □

Finally, we have the following characterizations of weakly ω-closed sets (i.e., ${}^{w}\omega$-closed sets) and ps-closed sets as follows.

Theorem 3.7 (i) (cf. Theorem 2.1(v)(vi)) *For a subset B of (X,τ), the following properties are equivalent:*
(1) *B is ${}^{w}\omega$-closed in (X,τ);*
(2) $(Cl(Int(B)))_{\mathcal{ND}} \subset B$*;*
(3) $Cl(Int(B)) \subset sKer(B)$*;*
(4) *B is ps-closed in (X,τ)* (i.e., *B is $(SO(X,\tau), PO(X,\tau))^{id}$-closed);*
(5) $(pCl(B))_{\mathcal{ND}} \subset B$*;*
(6) $pCl(B) \subset sKer(B)$*;*
(7) *B is preclosed in (X,τ).*
(ii) *For a topological space (X,τ), ${}^{w}\omega O(X,\tau)$ forms a generalized topology of X in the sense of Lugojan* ([15]) *such that $\tau \subset \omega O(X,\tau) \subset {}^{w}\omega O(X,\tau) = PO(X,\tau)$.*

Proof. **(i) (1)⇒(2)** Let $x \in (Cl(Int(B)))_{\mathcal{ND}}$. Suppose that $x \notin B$. The singleton $\{x\}$ is semi-closed, because $\{x\}$ is nowhere dense (i.e., $Int(Cl(\{x\})) = \emptyset$) and so $X \setminus \{x\}$ is a semi-open set containing B. By (1), $Cl(Int(B)) \subset X \setminus \{x\}$. We have a contradiction that $x \in X \setminus \{x\}$.
(2)⇒(3) Using Theorem 3.3, Lemma 3.6(iii) and (2), we have $Cl(Int(B))$-$=(Cl(Int(B)))_{\mathcal{PO}} \cup (Cl(Int(B)))_{\mathcal{ND}} \subset sKer(B) \cup B = sKer(B)$.
(3)⇒(1) Let $U \in SO(X,\tau)$ such that $B \subset U$. By definition of the concept of $sKer(\cdot)$ and (3), it is shown that $sKer(B) \subset U$ and so $Cl(Int(B)) \subset U$. Therefore, the set B is ${}^{w}\omega$-closed in (X,τ).
(4)⇒(5) Let $x \in (pCl(B))_{\mathcal{ND}}$. Suppose that $x \notin B$. The singleton $\{x\}$ is semi-closed and so $X \setminus \{x\}$ is a semi-open set containing B. By (4), $pCl(B) \subset X \setminus \{x\}$. We have a contradiction that $x \in X \setminus \{x\}$.
(5)⇒(6) Using Theorem 3.3, Lemma 3.6(iv) and the assumption (5), we have that: $pCl(B)=(pCl(B))_{\mathcal{PO}} \cup (pCl(B))_{\mathcal{ND}} \subset sKer(B) \cup B = sKer(B)$.
(6)⇒(4) Let $U \in SO(X,\tau)$ such that $B \subset U$. By definition of the concept of $sKer(\cdot)$ and (6), it is shown that $sKer(B) \subset U$ and so $pCl(B) \subset U$. Therefore, the set B is ps-closed in (X,τ).
(6)⇒(7) It follow from definition and (6) that the set B is a ps-closed set. Indeed, let $U \in SO(X,\tau)$ such that $B \subset U$; and so $pCl(B) \subset sKer(B) \subset U$; thus $B \in psC(X,\tau)$. Using Theorem 2.1 (v), B is preclosed.
(7)⇒(1) and (1)⇒ (4) They are obtained by using Theorem 2.1 (v),(vii).
(ii) These properties are obviously obtained by properties on $PC(X,\tau)$, because ${}^{w}\omega C(X,\tau) = PC(X,\tau)$ holds (cf.(i)). However, we attempt to prove them from the Janković Reilly decompositions method point of view. Let $\{B_i |\ i \in \Gamma\}$ be a family of ${}^{w}\omega$-closed sets in (X,τ) and let $B := \bigcap\{B_i |\ i \in \Gamma\}$. We have $Cl(Int(B)) \subset Cl(Int(B_i))$ for each $i \in \Gamma$ and so $(Cl(Int(B)))_{\mathcal{ND}} \subset \bigcap\{(Cl(Int(B_i)))_{\mathcal{ND}} |\ i \in \Gamma\} \subset \bigcap\{B_i |\ i \in \Gamma\} = B$ (cf. (i) (1)⇒(2)). Namely, by the equivalente property (2)⇔(1) in (i), the set B is ${}^{w}\omega$-closed in (X,τ). It is obvious that $\emptyset \in {}^{w}\omega O(X,\tau)$ and X *in* ${}^{w}\omega O(X,\tau)$. Thus, it is shown that ${}^{w}\omega O(X,\tau)$ is a generalized topology of X in the sense of Lugojan ([15]). □

Remark 3.8 Using Janković Reilly decomposition method (cf. Theorem 3.3), we show an alternative proof of Theorem 2.1(v), i.e., $psC(X,\tau) = PC(X,\tau)$ hold (cf. [3, Corollary 2.6 (iv), Table 1]). First we show that $psC(X,\tau) \subset PC(X,\tau)$. Let $A \in psC(X,\tau)$ and $x \in pCl(A)$. We claim that $x \in A$. We recall that $pCl(A) = (pCl(A))_{\mathcal{PO}} \cup (pCl(A))_{\mathcal{ND}}$. When $x \in (pCl(A))_{\mathcal{PO}}$, $\{x\}$ is preopen and so $\{x\} \cap A \neq \emptyset$ (i.e., $x \in A$). When $x \in (pCl(A))_{\mathcal{ND}}$, it is obtained that $x \in A$, by Theorem 3.7 (i)(4)⇒(5). Therefore, for both cases, we have $x \in A$ whenever $x \in pCl(A)$, i.e., $A \in PC(X,\tau)$ and so $psC(X,\tau) \subset PC(X,\tau)$. The converse implication is obvious.

In the end of the present Section 3, we apply Lemma 3.4 (i) to an alternative characterization of the ω-closed sets; the equivalent property (3) $\Leftrightarrow$ (4) in Theorem 3.9 below is shown by using Lemma 3.4(i).

Theorem 3.9 (Sheik John [26] for (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3)) *For a subset B of (X,τ), the following properties are equivalent:*
(1) *B is ω-closed in (X,τ);*
(2) *$(Cl(B))_{\mathcal{ND}} \subset B$;*
(3) *$Cl(B) \subset sKer(B)$;*
(4) (a) *$(Cl(B))_{\mathcal{SC}} \subset B$ and* (b) *$(Cl(B))_{\omega\mathcal{O}} \subset sKer(B)$ hold.*

Proof. (3)$\Rightarrow$(4) First we claim that $(sKer(B))_{\mathcal{SC}} \subset B$. Indeed, let $x \in (sKer(B))_{\mathcal{SC}}$ and assume that $x \notin B$. Since the set $X \setminus \{x\} \in SO(X,\tau)$ and $B \subset X \setminus \{x\}$, $sKer(B) \subset X \setminus \{x\}$. Then, we have that $\{x\} \subset X \setminus \{x\}$ and so this is a contradiction. Thus, we show that $(sKer(B))_{\mathcal{SC}} \subset B$. By using (3), it is shown that $(Cl(B))_{\mathcal{SC}} \subset (sKer(B))_{\mathcal{SC}} \subset B$; and so (a) is proved. The property (b) is obtained by (3), because $(Cl(B))_{\omega\mathcal{O}} \subset Cl(B) \subset sKer(B)$ hold.

(4)$\Rightarrow$(3): Using Lemma 3.4 (i) and (4), we have that $Cl(B) = (Cl(B))_{\mathcal{SC}} \cup (Cl(B))_{\omega\mathcal{O}} \subset B \cup sKer(B) = sKer(B)$. That is, $Cl(B) \subset sKer(B)$ holds. $\square$

4 Some properties of ω^{ρ}-closed sets, where $\rho \in \{\circ, \circ-\}$ After some characteriations of ω^{ρ}-closedness (cf. Proposition 4.4), we add a complete characterization of the ω^{ρ}-closedness, where $\rho : SO(X,\tau) \to P(X)$ is a function such that $\rho \in \{\circ, \circ-\}$ (cf. Theorem 4.8(iii)).

Theorem 4.1 (i) *The union of two ω°-closed* (resp. *$\omega^{\circ-}$-closed*) *sets is ω°-closed* (resp. *$\omega^{\circ-}$-closed*).

(ii) *If A is ω°-closed* (resp. *$\omega^{\circ-}$-closed*) *and $A \subset B \subset Cl(A)$, then B is ω°-closed* (resp. *$\omega^{\circ-}$-closed*).

(iii) *If A is ω°-closed* (resp. *$\omega^{\circ-}$-closed*), *then $Cl(A) \setminus A$ does not contain any nonempty semi-closed* (resp. *semi-closed and semi-open set*).

Proof. **(i)** Let $A, B \in \omega^{\circ}C(X,\tau)$ (resp. $A, B \in \omega^{\circ-}C(X,\tau)$) and $U \in SO(X,\tau)$ such that $A \cup B \subset U$. Then, it follows from assumptions that $Cl(A \cup B) = Cl(A) \cup Cl(B) \subset Int(U)$ (resp. $Cl(A \cup B) \subset Int(Cl(U))$), because $Cl(A) \subset Int(U)$ and $Cl(B) \subset Int(U)$ hold (resp. $Cl(A) \subset Int(Cl(U))$ and $Cl(B) \subset Int(Cl(U))$ hold). Thus, we show that $A \cup B \in \omega^{\circ}C(X,\tau)$ (resp. $A \cup B \in \omega^{\circ-}C(X,\tau)$).

(ii) Let $U \in SO(X,\tau)$ such that $B \subset U$. Then, by assumptions, it is shown that $Cl(B) = Cl(A), A \subset U$ and so $Cl(B) \subset Int(U)$ (resp. $Cl(B) \subset Int(Cl(U))$), i.e., $B \in \omega^{\circ}C(X,\tau)$ (resp. $B \in \omega^{\circ-}C(X,\tau)$).

(iii) Case 1. $A \in \omega^{\circ}C(X,\tau)$: suppose that $Cl(A) \setminus A$ contains a semi-closed set F. Since $A \subset X \setminus F$ and $X \setminus F \in SO(X,\tau), Cl(A) \subset Int(X \setminus F)$ holds. Thus, we have that $Cl(F) = X \setminus (Int(X \setminus F)) \subset X \setminus Cl(A)$ and so $Cl(A) \subset X \setminus Cl(F)$. We have that $F \subset Cl(F) \cap Cl(A) \subset (X \setminus Cl(A)) \cap Cl(A)$, because $F \subset Cl(A)$ holds; and hence $F = \emptyset$.

Case 2. $A \in \omega^{\circ-}C(X,\tau)$: suppose that $Cl(A) \setminus A$ contains a semi-closed and semi-open set F. Since $A \subset X \setminus F$ and $X \setminus F \in SO(X,\tau), Cl(A) \subset Int(Cl(X \setminus F))$ holds. Thus, we have that $Cl(Int(F)) = X \setminus (Int(Cl(X \setminus F))) \subset X \setminus Cl(A)$ and so $Cl(A) \subset X \setminus Cl(Int(F))$. Then, we have that $F \subset Cl(Int(F)) \cap Cl(A) \subset (X \setminus Cl(A)) \cap Cl(A)$, because $F \subset Cl(A)$ and F is semi-open; and hence $F = \emptyset$. □

Moreover, as continuation of Notation 3.2, we prepare the following notation.

Notation 4.2 For a subset E of (X, τ), we define the following families: (cf. Defintion 1.4)
($\bullet$3a) $E_{\omega^\circ\mathcal{O}} := \{x |\ x \in E$ and $\{x\}$ is ω°-open set of $(X, \tau)\}$;
($\bullet$3b) $E_{\omega^{\circ-}\mathcal{O}} := \{x |\ x \in E$ and $\{x\}$ is $\omega^{\circ-}$-open set of $(X, \tau)\}$;
($\bullet$3c) $E_{\mathcal{PC}} := \{x |\ x \in E$ and $\{x\}$ is preclosed in $(X, \tau)\}$.

Lemma 4.3 *For a topological space (X, τ) and a subset E of (X, τ), we have the following properties* (cf. Notation 3.2, Notation 4.2)*.*
(i) *$X = X_{\mathcal{SC}} \cup X_{\omega^\circ\mathcal{O}}$ and $E = E_{\mathcal{SC}} \cup E_{\omega^\circ\mathcal{O}}$ hold.*
(ii) *$X = (X_{\mathcal{SC}} \cap X_\tau) \cup X_{\omega^{\circ-}\mathcal{O}}$ and $E = (E_{\mathcal{SC}} \cap E_\tau) \cup E_{\omega^{\circ-}\mathcal{O}}$ hold.*
(iii) *$X = (X_{\mathcal{SC}} \cap X_{\mathcal{PO}}) \cup X_{\omega^{\circ-}\mathcal{O}}$ and $E = (E_{\mathcal{SC}} \cap E_{\mathcal{PO}}) \cup E_{\omega^{\circ-}\mathcal{O}}$ hold.*

Proof. **(i)** First, let $x \in X$. Suppose that $x \notin X_{\mathcal{SC}}$. We claim that $x \in X_{\omega^\circ\mathcal{O}}$. Indeed, let U be any semi-open set containing $X \setminus \{x\}$. Then, $U = X$, because $X \setminus \{x\}$ is not semi-open and so X is a unique semi-open set containing $X \setminus \{x\}$. Thus, $Cl(X \setminus \{x\}) \subset U = X = Int(U)$, i.e., $X \setminus \{x\}$ is ω°-closed, i.e. $x \in X_{\omega^\circ\mathcal{O}}$. Therefore, we have that $X = X_{\mathcal{SC}} \cup X_{\omega^\circ\mathcal{O}}$ holds. And, for the final property that $E = E_{\mathcal{SC}} \cup E_{\omega^\circ\mathcal{O}}$, the proof is obvious, because of the facts that $E_{\mathcal{SC}}{=}E \cap X_{\mathcal{SC}}$ and $E_{\omega^\circ\mathcal{O}}{=}E \cap X_{\omega^\circ\mathcal{O}}$ for any subset E of (X, τ).

(ii) First, let $x \in X$ and suppose that $x \in X \setminus (X_{\mathcal{SC}} \cap X_\tau)$. We claim that $x \in X_{\omega^{\circ-}\mathcal{O}}$. Let $U \in SO(X, \tau)$ such that $X \setminus \{x\} \subset U$. Then, $U = X$ or $U = X \setminus \{x\}$.

Case 1. $x \notin X_{\mathcal{SC}}$: by similar argument of the proof of (i), it is shown that $X \setminus \{x\} \notin SO(X, \tau)$ and so $U = X$ and $Cl(X \setminus \{x\}) \subset X = Int(Cl(U))$.

Case 2. $x \notin X_\tau$: for this case, if $U = X$, then $Cl(X \setminus \{x\}) \subset X = Int(Cl(X)) = Int(Cl(U))$; if $U = X \setminus \{x\}$, then $X \setminus \{x\} \neq Cl(X \setminus \{x\}){=}X$-
$=Int(X){=}Int(Cl(X \setminus \{x\})) = Int(Cl(U))$.

By both cases, $X \setminus \{x\}$ is $\omega^{\circ-}$-closed in (X, τ), i.e., $x \in \omega^{\circ-}O(X, \tau)$ under the assumption that the point x satiesfies Case 1 or Case 2 above. Therefore, we show that, for a point $x \in X, x \in X_{\mathcal{SC}} \cap X_\tau$ or $x \in X_{\omega^{\circ-}\mathcal{O}}$, i.e., $X \subset (X_{\mathcal{SC}} \cap X_\tau) \cup X_{\omega^{\circ-}\mathcal{O}}$ holds; and hence we have the required first equality. Since $E_{\mathcal{E}} = E \cap X_{\mathcal{E}}$ holds where the symbol $\mathcal{E} \in \{\mathcal{SC}, \tau, \omega^{\circ-}\mathcal{O}\}$, we have the final equality using the firsr property above.

(iii) By using (ii) above and the following fact that $E_\tau \subset E_{\mathcal{PO}}$ holds, it is shown that $E = (E_{\mathcal{SC}} \cap E_\tau) \cup E_{\omega^{\circ-}\mathcal{O}} \subset (E_{\mathcal{SC}} \cap E_{\mathcal{PO}}) \cup E_{\omega^{\circ-}\mathcal{O}}$ hold. Hence, we have the required equalities. □

We have the following property: ($\bullet$) For a subset A of (X, τ), $(Cl(A))_\tau \subset A$ holds. Indeed, let $x \in (Cl(A))_\tau$. Suppose that $x \notin A$. Since $A \subset X \setminus \{x\}$ and $\{x\}$ is open, i.e., $X \setminus \{x\}$ is closed, we have that $Cl(A) \subset Cl(X \setminus \{x\}) = X \setminus \{x\}$; and so we have that $x \in Cl(A) \subset X \setminus \{x\}$; this contradicts $x \notin X \setminus \{x\}$. (□)

For an ω^ρ-closed set A, where $\rho \in \{\circ, \circ-\}$, we have an analogouse form of the property ($\bullet$) above and Theorem 3.7 (cf. Proposition 4.4 and Remark 4.5 below).

Proposition 4.4 (i) *If A is an $\omega^{\circ-}$-closed set of (X, τ), then*
$((Cl(A))_{\mathcal{PC}})_{\mathcal{SO}} \subset A$ (cf. Notations 3.2($\bullet$2e),4.2($\bullet$3d); Remark 4.5 (i), (ii))*.*
(ii) *If A is an ω°-closed set of (X, τ), then $(Cl(A))_{\mathcal{SC}} \subset A$* (cf. Remark 4.5 (iii),(iv))*.*
(iii) *If A is an $\omega^{\circ-}$-closed set of (X, τ), then $((Cl(A))_{\mathcal{SC}})_{\mathcal{SO}} \subset A$* (cf. Remark 4.5 (vii),(viii))*.*

(iv) *If A is an $\omega^{\circ -}$-closed set of (X,τ), then $((Cl(A))_{ND})_{\mathcal{SO}} \subset A$* (cf. Remark 4.5 (v),(vi)).

Proof. **(i)** First, we recall that $(E_{\mathcal{PC}})_{\mathcal{SO}} = E_{\mathcal{PC}} \cap E_{\mathcal{SO}}$ holds for any set $E \subset X$. Let $x \in ((Cl(A))_{\mathcal{PC}})_{\mathcal{SO}}$. Suppose that $x \notin A$. Since $A \subset X \setminus \{x\}$ and $X \setminus \{x\}$ is preopen (i.e., $X \setminus \{x\} \subset Int(Cl(X \setminus \{x\})))$, the set $Int(Cl(X \setminus \{x\}))$ is a semi-open set containing A. Since A is $\omega^{\circ -}$-closed, we have that $Cl(A) \subset Int(Cl(\{Int(Cl(X \setminus \{x\}))\})) = Int(Cl(X \setminus \{x\})) = X \setminus Cl(Int(\{x\}))$; and so $x \in X \setminus Cl(Int(\{x\}))$, i.e., $(*)$ $x \notin Cl(Int(\{x\}))$. On the other hand, it follows from the assumption $(x \in ((Cl(A))_{\mathcal{PC}})_{\mathcal{SO}} \subset X_{\mathcal{SO}})$ for the point x that $\{x\} \subset Cl(Int(\{x\}))$ holds; this contradicts the property $(*)$ above.

(ii) Let $x \in (Cl(A))_{\mathcal{SC}}$. And suppose that $x \notin A$. Then, $A \subset X \setminus \{x\}$ and $X \setminus \{x\} \in SO(X,\tau)$, we have $Cl(A) \subset Int(X \setminus \{x\})$; and so $x \in Int(X \setminus \{x\}) = X \setminus Cl(\{x\})$, i.e., $x \notin Cl(\{x\})$; this contradicts the property: $E \subset Cl(E)$ for any subset E.

(iii) Let $x \in ((Cl(A))_{\mathcal{SC}})_{\mathcal{SO}}$ such that $x \notin A$. Since $A \subset X \setminus \{x\}$ and $X \setminus \{x\} \in SO(X,\tau)$ and A is $\omega^{\circ -}$-closed, we have that $Cl(A) \subset Int(Cl(X \setminus \{x\})) = X \setminus Cl(Int(\{x\}))$. Since $X \setminus x \in X_{\mathcal{SC}}$, $Int(Cl(X \setminus \{x\})) \subset X \setminus \{x\}$ and so $x \in X \setminus \{x\}$; this is a contradiction.

(iv) It is known that $E_{ND} \subset E_{\mathcal{SC}}$ holds for any set E of a topological space (X,τ). Then, for the given $\omega^{\circ -}$-closed set A, by (iii) above, it is obtained that $((Cl(A))_{ND})_{\mathcal{SO}} \subset ((Cl(A))_{\mathcal{SC}})_{\mathcal{SO}} \subset A$. □

Remark 4.5 (i) The converse of Proposition 4.4 (i) is not true from the following example. Let $A := \{2m+1\}$ be a subset of the digital line $(\mathbb{Z},\kappa)$. First, we claim that A is not $\omega^{\circ -}$-closed in $(\mathbb{Z},\kappa)$. Indeed, the set A is semi-open; and, take $U := A \in SO(\mathbb{Z},\kappa)$; then, we have that $Cl(A) = \{2m, 2m+1, 2m+2\} \not\subset Int(Cl(U)) = \{2m+1\}$; and so A is not $\omega^{\circ -}$-closed in $(\mathbb{Z},\kappa)$. Finally, we show that $((Cl(A))_{\mathcal{PC}})_{\mathcal{SO}} = (\{2m, 2m+2\})_{\mathcal{SO}} = \emptyset \subset A$ hold.

(ii) Let $A := \{0\} \cup \{2s+1 \in \mathbb{Z} | \ s \in \mathbb{Z}\}$ be an open set of $(\mathbb{Z},\kappa)$. Then, A is an example of the $\omega^{\circ -}$-closed set which satisfies Proposition 4.4(i)). Indeed, let $U \in SO(\mathbb{Z},\kappa)$ such that $A \subset U$. Since $A \in \kappa \subset SO(\mathbb{Z},\kappa)$, we have that $Cl(A) = \mathbb{Z} = Int(\mathbb{Z}) = Int(Cl(A)) \subset Int(Cl(U))$; and so A is $\omega^{\circ -}$-closed in $(\mathbb{Z},\kappa)$. Moreover, $(((Cl(A))_{\mathcal{PC}})_{\mathcal{SO}} = (\mathbb{Z}_{\mathcal{PC}})_{\mathcal{SO}} = (\{2s | \ s \in \mathbb{Z}\})_{\mathcal{SO}} = \emptyset \subset A$ hold in $(\mathbb{Z},\kappa)$. On the other hand, the present set A is an example which is not ω°-closed in $(\mathbb{Z},\kappa)$. Indeed, take $U := A \in SO(\mathbb{Z},\kappa)$; and so $Cl(A) = \mathbb{Z} \not\subset Int(U) = A$; by Definition 1.4, A is not ω°-closed. Moreover, since $(Cl(A))_{\mathcal{SC}} = \mathbb{Z} \not\subset A$ holds, the set A is not ω°-closed in $(\mathbb{Z},\kappa)$ (cf. Proposition 4.4(ii)).

(iii) The converse of Proposition 4.4 (ii) is not true from the following example. Let $A := \{2m, 2m+1, 2m+2\}$ be a subset of $(\mathbb{Z},\kappa)$ and the semi-open set $U := A$. It is shown that $Cl(A) = A \not\subset Int(U) = \{2m+1\}$; and so A is not ω°-closed. On the other hands, $(Cl(A))_{\mathcal{SC}} = \mathbb{Z}_{\mathcal{SC}} \cap Cl(A) = \mathbb{Z} \cap A = A$ hold in $(\mathbb{Z},\kappa)$.

(iv) Using contraposition of Proposition 4.4(ii), we can find any examples of non-ω°-closed sets. For example, the subset $A := \{2m+1\}$ given by (i) above is not ω°-closed in $(\mathbb{Z},\kappa)$. Indeed, $(Cl(A))_{\mathcal{SC}} = \mathbb{Z}_{\mathcal{SC}} \cap Cl(A) = \mathbb{Z} \cap Cl(A) = \{2m, 2m+1, 2m+2\} \not\subset A$; and so A is not ω°-closed in $(\mathbb{Z},\kappa)$.

(v) We have an example of an $\omega^{\circ -}$-closed set A which satisfies Proposition 4.4 (iii). We consider the $\omega^{\circ -}$-closed set A of (ii) above, say $A := \{0\} \cup \{2s+1 \in \mathbb{Z} | \ s \in \mathbb{Z}\}$. Indeed, since $(Cl(A))_{\mathcal{SC}} = \mathbb{Z}_{\mathcal{SC}} = \mathbb{Z}$, we have that $(((Cl(A))_{\mathcal{SC}})_{\mathcal{SO}} = \mathbb{Z}_{\mathcal{SO}} = \{2s+1 \in \mathbb{Z} | \ s \in \mathbb{Z}\} \subset A$.

(vi) The converse of Proposition 4.4 (iii) is not true. Let $A := \{2s+1 \in \mathbb{Z} | \ s \in \mathbb{Z}\} \setminus \{1\}$ be an open set of $(\mathbb{Z},\kappa)$. Then, we have that $Cl(A) = \mathbb{Z} \setminus \{1\}$ and so $((Cl(A))_{\mathcal{SC}})_{\mathcal{SO}} =$ - $(Cl(A))_{\mathcal{SO}} = (\mathbb{Z} \setminus \{1\})_{\mathcal{SO}} = A$, because any singleton $\{x\}$ is semi-closed, any odd singleton $\{2s+1\}$ is semi-open and any even singleton $\{2s\}$ is not semi-open in $(\mathbb{Z},\kappa)$, where

$s \in \mathbb{Z}$. And, the set A is not $\omega^{\circ -}$-closed in $(\mathbb{Z}, \kappa)$. Indeed, there exists a semi-open set $U := A$ such that $A \subset U$; and so $Cl(A)$=$\mathbb{Z} \setminus \{1\} \not\subset \{z \in \mathbb{Z} | z \leq -1\} \cup \{z \in \mathbb{Z} | 3 \leq z\}$=$Int(\mathbb{Z} \setminus \{1\})$=$Int(Cl(A))$; and hence the set A is not $\omega^{\circ -}$-closed in $(\mathbb{Z}, \kappa)$.

(vii) The converse of Propositon 4.4(iv) is not true. The following subset $A := \{2m-2, 2m-1, 2m+1, 2m+2\}$ of $(\mathbb{Z}, \kappa)$ is an example of non-$\omega^{\circ -}$-closed sets. Indeed, we know that $A \in SO(\mathbb{Z}, \kappa)$ such that $A \subset A$; and so $Cl(A)$=$A \cup \{2m\} \not\subset \{2m-1, 2m, 2m+1\}$=$Int(Cl(A))$; thus A is not $\omega^{\circ -}$-closed. Moreover, $((Cl(A))_{\mathcal{ND}})_{\mathcal{SO}}$=$(\{2m-2, 2m, 2m+2\})_{\mathcal{SO}}$=$\emptyset \subset A$ hold.

(viii) (cf. Proposition 4.4(iv)) For the $\omega^{\circ -}$-closed set $A :=\{0\} \cup \{2s+1 | \ s \in \mathbb{Z}\}$ of (ii) above, we check the following property: $((Cl(A))_{ND})_{\mathcal{SO}} \subset A$. Indeed, $((Cl(A))_{\mathcal{ND}})_{\mathcal{SO}}$ -=$(\mathbb{Z}_{\mathcal{ND}})_{\mathcal{SO}}$=$(\{2s | \ s \in \mathbb{Z}\})_{\mathcal{SO}}$=$\emptyset \subset A$ hold.

We define some analogouse concepts of the sets $Ker(\bullet)$ and $sKer(\bullet)$ (cf. Definition 4.6) and we characterize the ω^{ρ}-closedness of a subset, where $\rho : SO(X, \tau) \to P(X)$ is a function such that $\rho \in \{\circ, \circ-\}$ (cf. Theorem 4.8(iii) below).

Definition 4.6 For a subset A of (X, τ) and a function $\rho : SO(X, \tau) \to P(X)$ with $\rho \in \{id, \circ-, \circ\}$, we define the following subsets:

$(\cdot)$ $s^{\rho}Ker(A) := \bigcap\{W | \ W \in SO(X, \tau)$ and $A \subset \rho(W)\}$;

$(\cdot)$' $s^{\rho}Ker^{'}(A) := \bigcap\{\rho(W) | \ W \in SO(X, \tau)$ and $A \subset \rho(W)\}$;

$(\cdot)$" $s^{\rho}Ker_1(A) := \bigcap\{\rho(W) | \ W \in SO(X, \tau)$ and $A \subset W\}$.

We note that $s^{id}Ker(A)$=$s^{id}Ker^{'}(A)$=$s^{id}Ker_1(A)$=$sKer(A)$ hold.

Proposition 4.7 (i) *For any subset A of a topological space (X, τ), we have the following properties:*

(i-1) *$s^{\circ}Ker_1(A) \subset s^{\circ}Ker'(A) \subset s^{\circ}Ker(A)$;*

(i-2) *$s^{\circ -}Ker(A) \subset s^{\circ}Ker(A)$;*

(i-3) *$A \subset s^{\circ}Ker(A)$.*

(ii) (ii-1) *There exists a subset A of $(\mathbb{Z}, \kappa)$ such that $s^{\circ -}Ker(A) \subsetneqq A$.*

(ii-2) *There exists a subset A of $(\mathbb{Z}, \kappa)$ such that $s^{\circ}Ker_1(A) \subsetneqq A$ and $s^{\circ -}Ker_1(A) \subsetneqq A$.*

Proof **(i) (i-1)** Let $\rho := \circ$ throughout the present proof of (i-1).

Proof of $s^{\circ}Ker_1(A) \subset s^{\circ}Ker'(A)$: let x be any point such that $x \notin s^{\circ}Ker'(A)$. Then, by Definition 4.6$(\cdot)$', there exists a subset $W \in SO(X, \tau)$ such that $x \notin \rho(W) = Int(W)$ and $A \subset \rho(W) = Int(W)$; and so $x \notin s^{\circ}Ker_1(A)$ (cf. Definition 4.6$(\cdot)$"), because $\rho(W) \subset W$ holds for $\rho = \circ$.

Proof of $s^{\circ}Ker'(A) \subset s^{\circ}Ker(A)$: let x be any point such that $x \notin s^{\circ}Ker(A)$. Then, by Definition 4.6$(\cdot)$, there exists a subset $W \in SO(X, \tau)$ such that $x \notin W$ and $A \subset \rho(W)$; and so $x \notin s^{\circ}Ker'(A)$, because $\rho(W) \subset W$ and so $x \notin \rho(W)$ holds for $\rho = \circ$.

(i-2) Let x be any pont such that $x \notin s^{\circ}Ker(A)$. Then, by Definition 4.6$(\cdot)$, there exists a subset $W \in SO(X, \tau)$ such that $x \notin W$ and $A \subset Int(W)$; and so $x \notin s^{\circ -}Ker(A)$ (cf. Definition 4.6$(\cdot)$), because $A \subset Int(W) \subset Int(Cl(W))$ holds.

(i-3) Let x be any point such that $x \notin s^{\circ}Ker(A)$. Then, by Definition 4.6$(\cdot)$, there exists a subset $W \in SO(X, \tau)$ such that $x \notin W$ and $A \subset Int(W)$; and so $x \notin A$, because $Int(W) \subset W$.

(ii) (ii-1) We prepare the following notation: $K^{\rho}_A(X, \tau) := \{S | \ S \in SO(X.\tau)$ and $A \subset \rho(S)\}$, where $\rho : SO(X, \tau) \to P(X)$ be a function and a subset A of a topological space (X, τ). Then, $(*)$ $s^{\rho}Ker(A)$=$\bigcap\{W | \ W \in K^{\rho}_A(X, \tau)\}$ holds.

Let (X, τ) be the digital line $(\mathbb{Z}, \kappa)$ and $\rho := \circ-$. Let $A := \{0\} \cup \{2s+1 | \ s \in \mathbb{Z}\}$ and $W_0 := A \setminus \{0\}$. Then, since $A \subset \rho(W_0) = Int(Cl(W_0)) = \mathbb{Z}$, $A \subset \rho(A) = \mathbb{Z}$ and

$W_0, A \in SO(\mathbb{Z}, \kappa)$, we have that $W_0 \in K^{\rho}_{A}(\mathbb{Z}, \kappa)$ and $A \in K^{\rho}_{A}(\mathbb{Z}, \kappa)$. Therefore, we have that $s^{\circ-}Ker(A) \subset W_0 \subsetneqq A$ holds for the set A.

(ii-2) Let $\rho \in \{\circ, \circ-\}$ and let $A := \{-5, 0, 1, 5\}$ be a subset of $(\mathbb{Z}, \kappa)$. Then, $A \in SO(\mathbb{Z}, \kappa)$ and $\rho(A) = \{-5, 1, 5\}$ for the function $\rho \in \{\circ, \circ-\}$. We are able to take the set $W := A$ as a semi-open set W in the set $s^{\rho}Ker_1(A) := \bigcap\{\rho(W) |\ W \in SO(\mathbb{Z}, \kappa)$ and $A \subset W\}$, then it is obtained that $s^{\rho}Ker_1(A) \subset \rho(A) = \{-5, 1, 5\} \subsetneqq \{-5, 0, 1, 5\} = A$; and hence $s^{\rho}Ker_1(A) \subsetneqq A$ for the present set A and $\rho \in \{\circ, \circ-\}$. $\square$

Theorem 4.8 *Let A be a subset of (X, τ).*

(i) *If A is ω°-closed in (X, τ), then $Cl(A) \subset s^{\circ}Ker(A)$* (cf. Remark 4.9 (i) below).

(ii) *If A is $\omega^{\circ-}$-closed in (X, τ), then $(Cl(A))_{\mathcal{PO}} \subset s^{\circ-}Ker(A)$* (cf. Remark 4.9 (ii) below).

(iii) *A is an ω^{ρ}-closed set of (X, τ) if and only if $Cl(A) \subset s^{\rho}Ker_1(A)$ holds, where $\rho : SO(X, \tau) \rightarrow P(X)$ is a function such that $\rho \in \{\circ, \circ-\}$.*

Proof **(i)** Throughout the present proof, let $\rho := \circ : SO(X, \tau) \rightarrow P(X)$ be the function defined by $\rho(U) := Int(U)$ for every set $U \in SO(X, \tau)$. Let $x \in Cl(A)$. Suppose that $x \notin s^{\rho}Ker(A)$. There exists a subset $V \in SO(X, \tau)$ such that $x \notin V$ and $A \subset \rho(V)$ (cf. Definition 4.6 (i)). Since A is ω°-closed and $\rho(V) = Int(V) \in \tau \subset SO(X, \tau)$, we have that $Cl(A) \subset Int(\rho(V)) = Int(Int(V)) \subset V$ and so $x \in V$; and hence this is a contradiction.

(ii) Throughout the present proof, let $\rho := \circ- : SO(X, \tau) \rightarrow P(X)$ be the function defined by $\rho(U) := Int(Cl(U))$ for every set $U \in SO(X, \tau)$. Let $x \in (Cl(A))_{\mathcal{PO}}$. Suppose that $x \notin s^{\circ-}Ker(A)$. There exists a subset $V \in SO(X, \tau)$ such that $x \notin V$ and $A \subset \rho(V)$ (cf. Definition 4.6 (i)). Since A is $\omega^{\circ-}$-closed and $\rho(V)) \in \tau \subset SO(X, \tau)$, we have that $Cl(A) \subset Int(Cl(\rho(V)))) = Int(Cl(Int(Cl(V)))) \subset Cl(V)$ and so $x \in Cl(V)$. Thus, it is proved that $(*1)$: $Int(Cl(\{x\})) \cap V \neq \emptyset$, because $x \in Cl(V)$, $x \in Int(Cl(\{x\}))$ and $Int(Cl(\{x\})) \in \tau$. On the other hands, since $x \in X \setminus V$ and $X \setminus V \in SC(X, \tau)$ hold, we have that $\{x\} \cup Int(Cl(\{x\})) = sCl(\{x\}) \subset sCl(X \setminus V) = X \setminus V$; and so $Int(Cl(\{x\})) \subset X \setminus V$; and hence we have that $Int(Cl(\{x\})) \cap V \subset (X \setminus V) \cap V = \emptyset$; this contradicts $(*1)$ above.

(iii) (*Necessity*) Let $x \in Cl(A)$. Suppose that $x \notin s^{\rho}Ker_1(A)$ (cf. Definition 4.6($\cdot$)"). There exists a subset $V \in SO(X, \tau)$ such that $x \notin \rho(V)$ and $A \subset V$. Since A is ω^{ρ}-closed, we have that $Cl(A) \subset \rho(V)$; and so $x \in \rho(V)$; and hence this is a contradiction.

(*Sufficiency*) Assume that $Cl(A) \subset s^{\rho}Ker_1(A)$. Let $V \in SO(X, \tau)$ such that $A \subset V$. Then, by definition, it is shown that $s^{\rho}Ker_1(A) \subset \rho(V)$ holds, where $s^{\rho}Ker_1(A) := \bigcap\{\rho(W) |\ W \in SO(X, \tau)$ and $A \subset W\}$. Therefore, $Cl(A) \subset \rho(V)$ hold, whenever $V \in SO(X, \tau)$ and $A \subset V$; thus A is ω^{ρ}-closed in (X, τ) (cf. Definition 4.6($\cdot$)"). $\square$

Remark 4.9 (i) The converse of Theorem 4.8(i) is not true from the same example given by Remark 4.5(iii). Namely, let $A := \{2m, 2m + 1, 2m + 2\}$ be a subset of the digital line $(\mathbb{Z}, \kappa)$, where $m \in \mathbb{Z}$; then A is not ω°-closed in $(\mathbb{Z}, \kappa)$ (cf. Remark 4.5 (iii)). And, it is obtained that $Cl(A) \subset s^{\circ}Ker(A)$ holds, because $Cl(A) = A$ for the present set A and $B \subset s^{\circ}Ker(B)$ holds, in general, for every set B of a topological space (X, τ).

(ii) The converse of Theorem 4.8(ii) is not true from the same example given by Remark 4.5(i). Indeed, let $A := \{2m + 1\}$ be a subset of the digital line $(\mathbb{Z}, \kappa)$, where $m \in \mathbb{Z}$; then A is not $\omega^{\circ-}$-closed in $(\mathbb{Z}, \kappa)$. And, we note that $(Cl(A))_{\mathcal{PO}} = (\{2m, 2m + 1, 2m + 2\})_{\mathcal{PO}} = A$. If $W \in SO(\mathbb{Z}, \kappa)$ and $A \subset Int(Cl(W))$, then $A \subset W$; and so we show that $A \subset s^{\circ-}Ker(A)$. Therefore, we have that $(Cl(A))_{\mathcal{PO}} \subset s^{\circ-}Ker(A)$ holds in $(\mathbb{Z}, \kappa)$.

Remark 4.10 Using the concepts of $(Cl(\bullet))_{\mathcal{PO}}$, it is possible to define the following ω^ρ*-like closed sets*, where $\rho : SO(X,\tau) \to P(X)$ is a function such that $\rho \in \{\circ, \circ-\}$:
$(\cdot 1)$ a subset A of (X,τ) is said to be $\omega^\rho_{(\mathcal{PO})}$*-closed, if* $(Cl(A))_{\mathcal{PO}} \subset \rho(V)$ *holds whenever* $A \subset V$ *and* $V \in SO(X,\tau)$.
$(\cdot 2)$ $\omega^\rho_{(\mathcal{PO})}C(X,\tau) := \{A|\ A$ is $\omega^\rho_{(\mathcal{PO})}$-closed in $(X,\tau)\}$, where $\rho \in \{\circ, \circ-\}$. Then, we prove the following properties:
$(\cdot 3)$ $\omega^\rho_{(\mathcal{PO})}C(X,\tau)=P(X)$ holds (i.e. every set is $\omega^\rho_{(\mathcal{PO})}$-closed in (X,τ)). Namely, let A be a set of (X,τ). Then $(Cl(A))_{\mathcal{PO}} \subset \rho(W)$ holds whenever $A \subset W$ and $W \in SO(X,\tau)$, where $\rho \in \{\circ, \circ-\}$.
$(\cdot 4)$ $(Cl(A))_{\mathcal{PO}} \subset s^\circ Ker_1(A) \subset s^{\circ-}Ker_1(A)$ hold (cf. Definition 4.6 (i)").

Proof of $(\cdot 3)$. Let A be a subset of (X,τ). By Lemma 3.6 (ii), it is well known that, $(*1)$ $(Cl(A))_{\mathcal{PO}} \subset sKer(A)$ holds. Let $W \in SO(X,\tau)$ such that $A \subset W$. Take a point $x \in (Cl(A))_{\mathcal{PO}}$ (i.e., $x \in Cl(A)$ and $\{x\} \subset Int(Cl(\{x\}))$).

Case 1. $\rho = \circ$: we suppose that $x \notin \rho(W) = Int(W)$. Since $x \in X \setminus Int(W) = Cl(X\setminus W)$, $Cl(X\setminus W)$ is semi-closed and $x \in Int(Cl(\{x\}))$, we have that $Int(Cl(\{x\})) = \{x\} \cup Int(Cl(\{x\})) = sCl(\{x\}) \subset sCl(Cl(X \setminus W)) = Cl(X \setminus W)$; and so $Int(Cl(\{x\})) \subset X \setminus Int(W)$. Thus, we show that $(*2)$ $Int(Cl(\{x\})) \cap Int(W) = \emptyset$. On the other hands, we use the property that $(*)$ $(Cl(A))_{\mathcal{PO}} \subset sKer(A)$; and so $x \in sKer(A)$. Then, for the given set $W \in SO(X,\tau)$ such that $A \subset W$, we show that $x \in sKer(A) \subset W$; and so $x \in W$. Since $x \in W \subset Cl(Int(W))$ and $x \in Int(Cl(\{x\})) \in \tau$, it is obtained that $(*3)$ $Int(Cl(\{x\}))\cap Int(W) \neq \emptyset$; and hence $(*3)$ contradicts $(*2)$ above. Therefore, we proved that the property that $x \in \rho(W) = Int(W)$ holds for any point $x \in (Cl(A))_{\mathcal{PO}}$. Namely, $(Cl(A))_{\mathcal{PO}} \subset \rho(W) = Int(W)$ holds for any set $W \in SO(X,\tau)$ such that $A \subset W$.

Case 2. $\rho = \circ-$: by the result for Case 1 above, it is obtained that $(Cl(A))_{\mathcal{PO}} \subset Int(W) \subset Int(Cl(W)) = \rho(W)$ holds for any set $W \in SO(X,\tau)$ such that $A \subset W$. $(\diamond)$.

Proof of $(\cdot 4)$. Let $A \in P(X)$. First, we recall that (cf. Definiton 4.6) $s^\circ Ker_1(A) = \bigcap\{Int(S)|\ S \in \mathcal{K}_{1,A}\}$, where $\mathcal{K}_{1,A} := \{S'|S' \in SO(X,\tau)$ and $A \subset S'\}$. Then, by $(\cdot\ 3)$ for $\rho = \circ$, it is obtained that $(Cl(A))_{\mathcal{PO}} \subset Int(W)$ holds for any set $W \in \mathcal{K}_{1,A}$; and hence $(Cl(A))_{\mathcal{PO}} \subset \bigcap\{Int(W)|\ W \in \mathcal{K}_{1,A}\} = s^\circ Ker_1(A)$ holds. And, we prove the last implication:$(*)$ $s^\circ Ker_1(A) \subset s^{\circ-}Ker_1(A)$ for any subset A of (X,τ). Indeed, let $x \notin s^{\circ-}Ker_1(A)$. There exists a set $W \in SO(X,\tau)$ such that $x \notin Int(Cl(W))$ and $A \subset W$. Since $x \notin Int(W), A \subset W$ and $W \in SO(X,\tau)$, we have that $x \notin \bigcap\{Int(W')|W' \in SO(X,\tau)$ and $A \subset W'\} = s^\circ Ker_1(A)$. $(\diamond)$

5 (ω,ω)-$T^\rho_{1/2}$ spaces and related separation axioms, where $\rho \in \{id, \circ, \circ-\}$

We recall that, by definition due to Levine [14], a topological space (X,τ) is said to be $T_{1/2}$ if every generalized closed set (shortly, g.closed set) is closed in (X,τ). And, by Dunham [5], it is shown that (X,τ) is $T_{1/2}$ if and only if every singleton $\{x\}$ is closed or open in (X,τ), where $x \in X$ (cf. [5], e.g., [7]). Moreover, it is well known that the separation axiom $T_{1/2}$ is placed between the axioms T_0 and T_1 ([14]).

In order to introduce the concept of (ω,ω)-$T^\rho_{1/2}$ spaces (cf. Definition 5.3) and related separation axioms, we prepare the concept of a general form of "g.closed sets" (cf. Definition 5.2). The purpose of the present section is to prove Theorem 5.11, Theorem 5.13 and Theorem 5.15.

Throughout the present paper, let $(\mathcal{E}_X, \mathcal{E}'_X)$ be an ordered pair of two families $\mathcal{E}_X$ and $\mathcal{E}'_X$ of subsets in a topological space (X,τ) such that
$(\bullet 1)$ $\{\emptyset, X\} \subset \mathcal{E}_X$ and $\{\emptyset, X\} \subset \mathcal{E}'_X$.

Notation 5.1 (i) (e.g., [18, in 1996; (2.1)], [16, in 1999;Definition 2.1], [20, in 2003;Definiton 3.2]) Let A be a subset of (X,τ) and $(\mathcal{E}_X, \mathcal{E}'_X)$ be an ordered pair satisfying $(\bullet 1)$ above.

($\bullet$2) $\mathcal{E}_X$-$Cl(A) := \bigcap\{F|\ A \subset F$ and $X \setminus F \in \mathcal{E}_X\}$;
($\bullet$2)' $\mathcal{E}'_X$-$Cl(A) := \bigcap\{F|\ A \subset F$ and $X \setminus F \in \mathcal{E}'_X\}$.

(ii) ([26, in 2002]) ($\bullet$3) $\omega Cl(A) := \omega O(X,\tau)$-$Cl(A)$ (cf. (i)($\bullet$2) above for the case where $\mathcal{E}_X = \omega O(X,\tau)$) ([27, in 1995], [28, in 2000;Defintion 3.1]));
($\bullet$4) $\omega^\mu Cl(A) :=\omega^\mu O(X,\tau)$-$Cl(A)$, where $\mu : SO(X,\tau) \to P(X)$ is a function such that $\mu \in \{id, \circ, \circ-\}$ and $A \subset X$ (cf. (i)($\bullet$2) above for the case where $\mathcal{E}_X = \omega^\mu O(X,\tau)$, Notation 1.5($\bullet 3^\mu)'$).

Definition 5.2 (I) Let $\rho 1 : SO(X,\tau) \to P(X)$ and $\rho 2 : SO(X,\tau) \to P(X)$ be two functions such that $\rho 1 \in \{id, \circ, \circ-\}$ and $\rho 2 \in \{id, \circ, \circ-\}$; and $\rho : \omega^{\rho 1}O(X,\tau) \to P(X)$ be a function such that $\rho \in \{id, \circ, \circ-\}$.

A subset A of a topological space (X,τ) is said to be:
$(\omega^{\rho 1}, \omega^{\rho 2})$-$g^\rho$*.closed* in (X,τ), if $\omega^{\rho 2}Cl(A) \subset \rho(V)$ holds whenever $V \in \omega^{\rho 1}O(X,\tau)$ with $A \subset V$ (cf. Notation 1.5($\bullet$3'$^\rho$)); this may be called as $(\omega^{\rho 1}, \omega^{\rho 2})$-*generalized closed set with degree* ρ. Sometimes, an "$(\omega^{id}, \omega^{id})$-$g^{id}$.closed" set is said simply to be "(ω, ω)-g.closed".

(II) (cf. [18, Definition 2.10] for $\rho = id$) Let $\rho : \mathcal{E}_X \to P(X)$ be a function with $\rho \in \{id, \circ, \circ-\}$. A subset A of (X,τ) is said to be:
$(\mathcal{E}_X, \mathcal{E}'_X)$-$g^\rho$*.closed* in (X,τ), if $\mathcal{E}'_X$-$Cl(A) \subset \rho(V)$ holds whenever $A \subset V$ and $V \in \mathcal{E}_X$; this may be called as $(\mathcal{E}_X, \mathcal{E}'_X)$-*generalized closed with degree* ρ .

We note that: a subset A is $(\omega^{\rho 1}, \omega^{\rho 2})$-$g^\rho$.closed in (X,τ) if and only if A is -$(\omega^{\rho 1}O(X,\tau),\omega^{\rho 2}O(X,\tau))$-$g^\rho$.closed in (X,τ) in the sense of Definition 5.2 (II) for $\mathcal{E}_X$:= -$\omega^{\rho 1}O(X,\tau)$, $\mathcal{E}'_X$:=$\omega^{\rho 2}O(X,\tau)$). The above pairs $(\omega^{\rho 1}, \omega^{\rho 2})$ and $(\omega^{\rho 1}O(X,\tau),\omega^{\rho 2}O(X,\tau))$ imply the ordered pairs.

First, using Definition 5.2 above, we define the concept on $(\omega^{\rho 1},\omega^{\rho 2})$-$T^\rho_{1/2}$ spaces and also it's general forms $(\mathcal{E}_X,\mathcal{E}'_X)$-$T^\rho_{1/2}$ spaces. Especially, the concept of $(\mathcal{E}_X,\mathcal{E}'_X)$-$T^{id}_{1/2}$ spaces is defined in [18, in 1996;Definition 2.19].

Definition 5.3 (I) Let $\rho 1 : SO(X,\tau) \to P(X)$ and $\rho 2 : SO(X,\tau) \to P(X)$ be two functions such that $\rho 1 \in \{id, \circ, \circ-\}$ and $\rho 2 \in \{id, \circ, \circ-\}$; and let $\rho : \omega^{\rho 1}O(X,\tau) \to P(X)$ be a function such that $\rho \in \{id, \circ, \circ-\}$.

For the fixed functions $\rho 1$, $\rho 2$ and ρ, a topological space (X,τ) is said to be:

(i) $(\omega^{\rho 1},\omega^{\rho 2})$-$T^\rho_{1/2}$, if A is $\omega^{\rho 2}$-closed (cf. Definition 1.4;i.e., $X \setminus A \in \omega^{\rho 2}O(X,\tau)$) for every $(\omega^{\rho 1},\omega^{\rho 2})$-$g^\rho$.closed set A, (cf. Definition 5.2(I));

(ii) *weak* $(\omega^{\rho 1},\omega^{\rho 2})$-$T^\rho_{1/2}$, where $\rho 2 \neq id$, if $\omega^{\rho 2}Cl(A) = A$ holds for every $(\omega^{\rho 1},\omega^{\rho 2})$-$g^\rho$.closed set A, where $\omega^{\rho 2}Cl(A)$:=$\omega^{\rho 2}O(X,\tau)$-$Cl(A)$ (cf. Definition 5.2(I), Notation 5.1).

(II) Let $(\mathcal{E}_X, \mathcal{E}'_X)$ be an ordered pair and let $\rho : \mathcal{E}_X \to P(X)$ be a fixed function such that $\rho \in \{id, \circ, \circ-\}$. A topological space (X,τ) is said to be:

(i) an $(\mathcal{E}_X,\mathcal{E}'_X)$-$T^\rho_{1/2}$ *space*, if $X \setminus A \in \mathcal{E}'_X$ holds for every $(\mathcal{E}_X,\mathcal{E}'_X)$-$g^\rho$.closed set A (cf. [18, Definition 2.19] for $\rho = id$).

(ii) a *weak* $(\mathcal{E}_X,\mathcal{E}'_X)$-$T^\rho_{1/2}$ *space*, if $\mathcal{E}'_X$-$Cl(A) = A$ holds for every $(\mathcal{E}_X,\mathcal{E}'_X)$-$g^\rho$.closed set A (cf. Definition 5.2(II), Notation 5.1).

We investigate some relations between "*weak* $(\mathcal{E}_X,\mathcal{E}'_X)$-$T^\rho_{1/2}$ *spaces*" and "$(\mathcal{E}_X,\mathcal{E}'_X)$-$T^\rho_{1/2}$ *spaces*" (cf. Lemma 5.5), applying the following Lemma 5.4 due to Noiri and Popa ([20, in 2003;Lemma 3.3], [21, in 2000]).

Lemma 5.4 ([20, in 2003;Lemma 3.3], [21, in 2000]) *For a minmal structure m_X on a nonempty set X (i.e., $\emptyset \in m_X, X \in m_X$ and $m_X \subset P(X)$), the following are equivalent:*
(1) *m_X has property, say $(\mathcal{B})_{m_X}$: if the union of any family of subsets belonging to m_X belongs to m_X;*
(2) *if m_X-$Int(V) = V$, then $V \in m_X$;*
(3) *if m_X-$Cl(F) = F$, then $X \setminus F \in m_X$.*

Lemma 5.5 *Let (X, τ) be a topological space and $(\mathcal{E}_X, \mathcal{E}'_X)$ an ordered pair of given familes $\mathcal{E}_X$ and $\mathcal{E}'_X$ such that $\{\emptyset, X\} \subset \mathcal{E}_X \cap \mathcal{E}'_X$.*

For each function $\rho : \mathcal{E}_X \to P(X)$ with $\rho \in \{id, \circ, \circ-\}$, we have the following properties.

(i) *Every $(\mathcal{E}_X, \mathcal{E}'_X)$-$T^{\rho}_{1/2}$ space (X, τ) is weak $(\mathcal{E}_X, \mathcal{E}'_X)$-$T^{\rho}_{1/2}$.*

(ii) *Suppose that $\mathcal{E}'_X$ has property $(\mathcal{B})_{\mathcal{E}'_X}$: the union of any family of subsets belonging to $\mathcal{E}'_X$ belongs to $\mathcal{E}'_X$* (cf. Lemma 5.4). *Then, every weak $(\mathcal{E}_X,\mathcal{E}'_X)$-$T^{\rho}_{1/2}$ space (X, τ) is $(\mathcal{E}_X,\mathcal{E}'_X)$-$T^{\rho}_{1/2}$.*

Proof. **(i)** Let A be an $(\mathcal{E}_X, \mathcal{E}'_X)$-$g^{\rho}$ closed set in (X, τ). Then, by assumption, it is obtained that $X \setminus A \in \mathcal{E}'_X$; and so $\mathcal{E}'_X$-$Cl(A) := \bigcap\{F|\ A \subset F$ and $X \setminus F \in \mathcal{E}'_X\} = A$ hold. Therefore, (X, τ) is weak $(\mathcal{E}_X,\mathcal{E}'_X)$-$T^{\rho}_{1/2}$.

(ii) Let A be an $(\mathcal{E}_X,\mathcal{E}'_X)$-$g^{\rho}$ closed set in (X, τ). Since (X, τ) is weak $(\mathcal{E}_X,\mathcal{E}'_X)$-$T^{\rho}_{1/2}$, we have $\mathcal{E}'_X$-$Cl(A) = A$. Since $(\mathcal{B})_{\mathcal{E}'_X}$ is supposed, by Lemma 5.4, it is obtained that $X \setminus A \in \mathcal{E}'_X$. Therefore, (X, τ) is $(\mathcal{E}_X,\mathcal{E}'_X)$-$T^{\rho}_{1/2}$. □

Remark 5.6 (i) The following properties on a topological space (X, τ) are equivalent for a fixed function $\rho : \mathcal{E}_X \to P(X)$ with $\rho \in \{id, \circ, \circ-\}$ and a fixed function $\rho 1 : SO(X, \tau) \to P(X)$ with $\rho 1 \in \{id, \circ, \circ-\}$:

(1) (X, τ) is $(\omega^{\rho 1},\omega^{id})$-$T^{\rho}_{1/2}$ (cf. Definition 5.3(I)(i);

(2) (X, τ) is *weak* $(\omega^{\rho 1}O(X, \tau),\omega O(X, \tau))$-$T^{\rho}_{1/2}$ (cf. Definition 5.3(II)(ii));

(3) (X, τ) is $(\omega^{\rho 1}O(X, \tau),\omega O(X, \tau))$-$T^{\rho}_{1/2}$ (cf. Definition 5.3(II)(ii)).

Indeed, they are obtained by definitions and the well known fact that, for a subset A of (X, τ), $X \setminus A \in \omega O(X, \tau)$ if and only if $\omega Cl(A) = A$ holds, where $\omega Cl(A)$:=$\omega O(X, \tau)$-$Cl(A)$. By [26], $\omega O(X, \tau)$ has property $(\mathcal{B})_{\omega O(X,\tau)}$; and so the equivalences are obtained by Lemma 5.5.

(ii) The concept of an $(\omega^{id},\omega^{id})$-$T^{id}_{1/2}$ space is called an (ω,ω)-$T_{1/2}$ space or an ω-$T_{1/2}$ space.

Lemma 5.7 (i) *The following properties on a topological space (X, τ) are equivalent: let $\mathcal{E}_X$ and $\mathcal{E}'_X$ be two families satisfying the condition that $\{\emptyset, X\} \subset \mathcal{E}_X \cap \mathcal{E}'_X$.*

(1) *(X, τ) is weak $(\mathcal{E}_X, \mathcal{E}'_X)$-$T^{\circ}_{1/2}$;*

(2) *$(*1)$: if $x \in X$, then $X \setminus \{x\} \in \mathcal{E}_X$ or $\mathcal{E}'_X$-$Cl(X \setminus \{x\}) = X \setminus \{x\}$ hold;*

(3) *(X, τ) is weak $(\mathcal{E}_X, \mathcal{E}'_X)$-$T^{id}_{1/2}$.*

(ii) *Every weak $(\mathcal{E}_X, \mathcal{E}'_X)$-$T^{\circ-}_{1/2}$ topological space (X, τ) is weak $(\mathcal{E}_X, \mathcal{E}'_X)$-$T^{\rho}_{1/2}$, where $\rho \in \{id, \circ\}$.*

(iii) *Suppose that $(*2)$: if $x \in X$, then $X \setminus \{x\} \in \mathcal{E}_X \cap SC(X, \tau)$ or $\mathcal{E}'_X$-$Cl(X \setminus \{x\}) = X \setminus \{x\}$ hold. Then, (X, τ) is weak $(\mathcal{E}_X, \mathcal{E}'_X)$-$T^{\circ-}_{1/2}$.*

Proof. **(i)** (1)⇒(2) We suppose that $X \setminus \{x\} \notin \mathcal{E}_X$. Let $U \in \mathcal{E}_X$ be any set such that $X \setminus \{x\} \subset U$. Then we have that $U = X$ only; and so $\mathcal{E}'_X$-$Cl(X \setminus \{x\}) \subset \mathcal{E}'_X$-$Cl(X) = X = Int(U)$. Thus, we have that $X \setminus \{x\}$ is $(\mathcal{E}_X, \mathcal{E}'_X)$-$g^{\circ}$.closed (cf. Definition 5.2(II)).

By assumption (cf. Defintion 5.3(II)(ii)), it is shown that $\mathcal{E}'_X$-$Cl(X \setminus \{x\}) = X \setminus \{x\}$ holds. Therefore, we have $(*1)$.

(2)$\Rightarrow$ (3) Let A be an $(\mathcal{E}_X, \mathcal{E}'_X)$-$g^{id}$.closed set. We claim that $\mathcal{E}'_X$-$Cl(A) = A$. Let $x \in \mathcal{E}'_X$-$Cl(A)$; and we suppose that $x \notin A$; and so $A \subset X \setminus \{x\}$.

Case 1. $\mathcal{E}'_X$-$Cl(X \setminus \{x\}) = X \setminus \{x\}$: for this case, we have that $x \in \mathcal{E}'_X$-$Cl(A) \subset \mathcal{E}'_X$-$Cl(X \setminus \{x\}) = X \setminus \{x\}$; and so $x \in X \setminus \{x\}$; this is a contradiction.

Case 2. $X \setminus \{x\} \in \mathcal{E}_X$: for this case, since $A \subset X \setminus \{x\}$, where $X \setminus \{x\} \in \mathcal{E}_X$, and A is $(\mathcal{E}_X, \mathcal{E}'_X)$-$g^{id}$.closed, we have that $x \in \mathcal{E}'_X$-$Cl(A) \subset X \setminus \{x\}$ (cf. Definition 5.2(II)); and so $x \in X \setminus \{x\}$; this is also a contradiction.

By all cases, we have contradictions; and so we prove that $\mathcal{E}'_X$-$Cl(A) \subset A$ holds. Since $A \subset \mathcal{E}'_X$-$Cl(A)$, we have the required equality $\mathcal{E}'_X$-$Cl(A) = A$; and hence (X, τ) is weak $(\mathcal{E}_X, \mathcal{E}'_X)$-$T^{id}_{1/2}$ (cf. Defintion 5.3(II)).

(3)$\Rightarrow$(1) Let A be an $(\mathcal{E}_X, \mathcal{E}'_X)$-$g^{\circ}$.closed set of (X, τ). Then, by Definition 5.2(II), it is shown that the set A is $(\mathcal{E}_X, \mathcal{E}'_X)$-$g^{id}$.closed. Using the assumption (3), we have that $\mathcal{E}'_X$-$Cl(A) = A$; and so (X, τ) is $(\mathcal{E}_X, \mathcal{E}'_X)$-$T^{\circ}_{1/2}$ (cf. Definition 5.3(II)).

(ii) We prove the property $(*1)$ of (i) above. Indeed, we suppose that $X \setminus \{x\} \notin \mathcal{E}_X$. Let $U \in \mathcal{E}_X$ be any set such that $X \setminus \{x\} \subset U$. Then we have that $U = X$ only; and so $\mathcal{E}'_X$-$Cl(X \setminus \{x\}) \subset \mathcal{E}'_X$-$Cl(U)$=$\mathcal{E}'_X$-$Cl(X) = X = Int(Cl(U))$. Thus, we have that $X \setminus \{x\}$ is $(\mathcal{E}_X, \mathcal{E}'_X)$-$g^{\circ-}$.closed (cf. Definition 5.2(II)). It is shown that $\mathcal{E}'_X$-$Cl(X \setminus \{x\}) = X \setminus \{x\}$ holds, because (X, τ) is weak $(\mathcal{E}_X, \mathcal{E}'_X)$-$T^{\circ-}_{1/2}$. Therefore, we have $(*1)$; and so (X, τ) is $(\mathcal{E}_X, \mathcal{E}'_X)$-$T^{\rho}_{1/2}$, where $\rho \in \{id, \circ\}$ (cf. (i) above).

(iii) Let A be an $(\mathcal{E}_X, \mathcal{E}'_X)$-$g^{\circ-}$.closed set. We claim that $\mathcal{E}'_X$-$Cl(A) = A$. Indeed, let $x \in \mathcal{E}'_X$-$Cl(A)$. And we suppose that $x \notin A$; and so $A \subset X \setminus \{x\}$.

Case 1. $\mathcal{E}'_X$-$Cl(X \setminus \{x\}) = X \setminus \{x\}$: for this case, we have that $x \in \mathcal{E}'_X$-$Cl(A) \subset \mathcal{E}'_X$-$Cl(X \setminus \{x\}) = X \setminus \{x\}$; and so $x \in X \setminus \{x\}$; this is a contradiction.

Case 2. $X \setminus \{x\} \in \mathcal{E}_X \cap SC(X, \tau)$: for this case, since $A \subset X \setminus \{x\}$, $X \setminus \{x\} \in \mathcal{E}_X$ and A is $(\mathcal{E}_X, \mathcal{E}'_X)$-$g^{\circ-}$.closed, we have that $x \in \mathcal{E}'_X$-$Cl(A) \subset Int(Cl(X \setminus \{x\})) = X \setminus Cl(Int(\{x\}))$. We have that $x \in X \setminus Cl(Int(\{x\}))$. Namely, we have that $\{x\} \not\subset Cl(Int(\{x\}))$,i.e., $\{x\}$ is not semi-open in (X, τ). This contradicts one of the assumptions: $X \setminus \{x\} \in SC(X, \tau)$ (i.e., $\{x\}$ is semi-open in (X, τ)).

Thus, for both cases, we have contradictions; and so we show that $\mathcal{E}'_X$-$Cl(A) \subset A$; and so $A = \mathcal{E}'_X$-$Cl(A)$; and hence (X, τ) is weak $(\mathcal{E}_X, \mathcal{E}'_X)$-$T^{\circ-}_{1/2}$. $\square$

Remark 5.8 The following diagram shows the implications in Lemma 5.7 above: under the assumption that $\{\emptyset, X\} \subset \mathcal{E}_X \cap \mathcal{E}'_X$.

$$\begin{array}{lll}
 & \textit{weak } (\mathcal{E}_X, \mathcal{E}'_X)\text{-}T^{id}_{1/2} \rightleftharpoons & (*1) \text{ of Lemma 5.7(i)(2)} \\
 & \nearrow & \\
\textit{weak } (\mathcal{E}_X, \mathcal{E}'_X)\text{-}T^{\circ-}_{1/2} & & \downarrow \uparrow \\
\uparrow & \searrow & \\
(*2) \text{ of Lemma 5.7(iii)} & \textit{weak } (\mathcal{E}_X, \mathcal{E}'_X)\text{-}T^{\circ}_{1/2} &
\end{array}$$

We investigate the following properties on "$(\mathcal{E}_X, \mathcal{E}'_X)$-$T^{\rho}_{1/2}$", corresponding to Lemma 5.7 above.

Lemma 5.9 (i) *Let $\rho : \mathcal{E}_X \to P(X)$ be a fixed function such that $\rho \in \{id, \circ, \circ-\}$. Suppose that (X, τ) is an $(\mathcal{E}_X, \mathcal{E}'_X)$-$T^{\rho}_{1/2}$ topological space. Then,*
$(*1')$: *if $x \in X$ then $X \setminus \{x\} \in \mathcal{E}_X$ or $\{x\} \in \mathcal{E}'_X$.*

(ii) *Suppose that $\mathcal{E}'_X$ has property $(\mathcal{B})_{\mathcal{E}'_X}$* (cf. Lemma 5.5(ii)). *If* $(*1')$ *of* (i) *above holds, then (X, τ) is $(\mathcal{E}_X, \mathcal{E}'_X)$-$T^{\rho}_{1/2}$, where $\rho \in \{id, \circ\}$. And, every $(\mathcal{E}_X, \mathcal{E}'_X)$-$T^{\circ-}_{1/2}$ topological space is $(\mathcal{E}_X, \mathcal{E}'_X)$-$T^{id}_{1/2}$ and $(\mathcal{E}_X, \mathcal{E}'_X)$-$T^{\circ}_{1/2}$.*

(iii) *Suppose that $\mathcal{E}'_X$ has property $(\mathcal{B})_{\mathcal{E}'_X}$ and that $(*2')$: if $x \in X$ then $X \setminus \{x\} \in \mathcal{E}_X \cap SC(X,\tau)$ or $\{x\} \in \mathcal{E}'_X$. Then, (X,τ) is $(\mathcal{E}_X, \mathcal{E}'_X)$-$T^{\circ -}_{1/2}$.*

Proof. **(i)** Let $\{x\}$ be a singleton in (X,τ). We suppose that $X \setminus \{x\} \notin \mathcal{E}_X$. Let $U \in \mathcal{E}_X$ be any set such that $X \setminus \{x\} \subset U$. Then, $U = X$ holds only; and so $\mathcal{E}'_X$-$Cl(X \setminus \{x\}) \subset \mathcal{E}_X$-$Cl(U) = \mathcal{E}_X$-$Cl(X) = X = \rho(U)$, where $\rho \in \{id, \circ, \circ-\}$. Thus, we have that $X \setminus \{x\}$ is $(\mathcal{E}_X, \mathcal{E}'_X)$-$g^\rho$.closed (cf. Definition 5.3(II)). By assumption, it is shown that $X \setminus (X \setminus \{x\}) \in \mathcal{E}'_X$ and so $\{x\} \in \mathcal{E}'_X$.

(ii) First, suppose that $(*1')$ holds. For a singleton $\{x\}$ such that $\{x\} \in \mathcal{E}'_X$, it is shown that $\mathcal{E}'_X$-$Cl(X \setminus \{x\}) = X \setminus \{x\}$ holds (cf. Notation 5.1(I)(i)). Then, the given assumption $(*1')$ implies the assumption $(*1)$ of Lemma 5.7(i)(2), i.e., $X \setminus \{x\} \in \mathcal{E}_X$ or $\mathcal{E}'_X$-$Cl(X \setminus \{x\}) = X \setminus \{x\}$ hold. Thus, by Lemma 5.7(i), (X,τ) is weak $(\mathcal{E}_X, \mathcal{E}'_X)$-$T^\rho_{1/2}$, where $\rho \in \{id, \circ\}$; and, by Lemma 5.5(ii), (X,τ) is $(\mathcal{E}_X, \mathcal{E}'_X)$-$T^\rho_{1/2}$, where $\rho \in \{id, \circ\}$. Finally, suppose that (X,τ) is $(\mathcal{E}_X, \mathcal{E}'_X)$-$T^{\circ-}_{1/2}$. Then, by (i) above, it is shown that the property $(*1')$ holds; and so, by the first property of the present (ii), the space (X,τ) is $(\mathcal{E}_X, \mathcal{E}'_X)$-$T^\rho_{1/2}$, where $\rho \in \{id, \circ\}$.

(iii) Let $\{x\} \in \mathcal{E}'_X$. It is shown that $\mathcal{E}'_X$-$Cl(X \setminus \{x\}) = X \setminus \{x\}$ holds; and so the assumption $(*2')$ implies the assumption $(*2)$ of Lemma 5.7(iii), i.e., $X \setminus \{x\} \in \mathcal{E}_X \cap SC(X,\tau))$ or $\mathcal{E}'_X$-$Cl(X \setminus \{x\}) = X \setminus \{x\})$ hold. Thus, by Lemma 5.7(iii) and Lemma 5.5(ii), it is shown that (X,τ) is $(\mathcal{E}_X, \mathcal{E}'_X)$-$T^{\circ-}_{1/2}$. □

Remark 5.10 The following diagram is shown by the above implications in Lemma 5.9: under the assumption $(\mathcal{B})_{\mathcal{E}'_X}$;

$$\begin{array}{ccc} & & (\mathcal{E}_X, \mathcal{E}'_X)\text{-}T^{id}_{1/2} \\ & \nearrow & \\ (\mathcal{E}_X, \mathcal{E}'_X)\text{-}T^{\circ-}_{1/2} & & \downarrow \uparrow \\ & \searrow & \\ & & (\mathcal{E}_X, \mathcal{E}'_X)\text{-}T^{\circ}_{1/2} \end{array}$$

Using Lemma 5.7 for $\mathcal{E}_X := \omega^{\rho 1}O(X,\tau)$ and $\mathcal{E}'_X := \omega^{\rho 2}O(X,\tau)$, the concept of "weak $(\omega^{\rho 1}, \omega^{\rho 2})$-$T^\rho_{1/2}$ spaces" is characterized by the following Theorem 5.11, where $\rho 1 : SO(X,\tau) \to P(X)$ and $\rho 2 : SO(X,\tau) \to P(X)$ are functions such that $\rho 1 \in \{id, \circ, \circ-\}$ and $\rho 2 \in \{id, \circ, \circ-\}$ and $\rho : \omega^{\rho 1}O(X,\tau) \to P(X)$ is a function such that $\rho \in \{id, \circ, \circ-\}$; (cf. Definition 5.3 (I)(ii)).

Theorem 5.11 *Let $\rho 1 : SO(X,\tau) \to P(X)$ and $\rho 2 : SO(X,\tau) \to P(X)$ be two functions such that $\rho 1 \in \{id, \circ, \circ-\}$ and $\rho 2 \in \{id, \circ, \circ-\}$.*

(i) *The following properties are equivalent*:

(1) *a topological space (X,τ) is weak $(\omega^{\rho 1}, \omega^{\rho 2})$-$T^{\circ}_{1/2}$;*

(2) $(*1)$: *if $x \in X$ then $\{x\}$ is $\omega^{\rho 1}$-closed* (cf. Definition d75) (i.e., $X \setminus \{x\} \in \omega^{\rho 1}O(X,\tau)$) *or $\omega^{\rho 2}Cl(X \setminus \{x\}) = X \setminus \{x\}$;*

(3) *(X,τ) is weak $(\omega^{\rho 1}, \omega^{\rho 2})$-$T^{id}_{1/2}$.*

(ii) *Every weak $(\omega^{\rho 1}, \omega^{\rho 2})$-$T^{\circ-}_{1/2}$ topological space is weak $(\omega^{\rho 1}, \omega^{\rho 2})$-$T^\rho_{1/2}$, where $\rho \in \{id, \circ\}$.*

(iii) *Suppose that $(*2)$: if $x \in X$ then $X \setminus \{x\} \in \omega^{\rho 1}O(X,\tau) \cap SC(X,\tau)$ or $\omega^{\rho 2}Cl(X \setminus \{x\}) = X \setminus \{x\}$. Then, (X,τ) is weak $(\omega^{\rho 1}, \omega^{\rho 2})$-$T^{\circ-}_{1/2}$.* □

Remark 5.12 The following diagrams are obtained by Theorem 5.11(i) and (ii) above: for fixed functions $\rho 1 \in \{id, \circ, \circ-\}$ and $\rho 2 \in \{id, \circ, \circ-\}$ (cf. Remark 5.8),

$$\begin{array}{lcl}
\textit{weak } (\omega^{\rho 1},\omega^{\rho 2})\text{-}T^{id}_{1/2} & \rightleftharpoons & \{x\}\in\omega^{\rho 1}C(X,\tau) \text{ or} \\
\quad\nwarrow & & \omega^{\rho 2}Cl(X\setminus\{x\})=X\setminus\{x\}(\forall x\in X) \\
\downarrow\uparrow \quad \textit{weak } (\omega^{\rho 1},\omega^{\rho 2})\text{-}T^{\circ -}_{1/2} & & \\
\quad\swarrow & & \\
\textit{weak } (\omega^{\rho 1},\omega^{\rho 2})\text{-}T^{\circ}_{1/2} & &
\end{array}$$

In Definition 5.3 (II)(i), especially we consider the case where $\mathcal{E}_X := \omega^{\rho 1}O(X,\tau)$ ($\rho 1 : SO(X,\tau)\to P(X)$ is a function such that $\rho 1\in\{id,\circ,\circ -\}$) and $\mathcal{E}'_X{:=}\omega O(X,\tau)$; and so we have the following propeties on "$(\omega^{\rho 1},\omega)$-$T^{\rho}_{1/2}$" spaces using Lemma 5.9 above and Definition 5.3 (II)(i), where $\rho : \omega^{\rho 1}O(X,\tau)\to P(X)$ is a function such that $\rho\in\{id,\circ,\circ -\}$. We note that the family $\mathcal{E}'_X := \omega O(X,\tau)$ has property $(\mathcal{B})_{\mathcal{E}'_X}$ (cf. Remark 5.6 above;[26]).

Theorem 5.13 *For a fixed function $\rho 1 : SO(X,\tau)\to P(X)$ with $\rho 1\in\{id,\circ,\circ -\}$, we have the following properties.*

(i) *The following properties are equivalent:*
(1) *a topological space (X,τ) is $(\omega^{\rho 1},\omega)$-$T^{id}_{1/2}$;*
(2) *if $x\in X$ then $\{x\}$ is $\omega^{\rho 1}$-closed or $\{x\}$ is ω-open;*
(3) *(X,τ) is $(\omega^{\rho 1},\omega)$-$T^{\circ}_{1/2}$.*

(ii) *Every $(\omega^{\rho 1},\omega)$-$T^{\circ -}_{1/2}$ topological space is $(\omega^{\rho 1},\omega)$-$T^{id}_{1/2}$ and $(\omega^{\rho 1},\omega)$-$T^{\circ}_{1/2}$.*

(iii) *Suppose that if $x\in X$ then $\{x\}$ is $\omega^{\rho 1}$-closed and semi-open* (i.e. $X\setminus\{x\}\in\omega^{\rho 1}O(X,\tau)$ *and* $\{x\}\in SO(X,\tau)$)*, or $\{x\}$ is ω-open, then (X,τ) is $(\omega^{\rho 1},\omega)$-$T^{\circ -}_{1/2}$.* □

Remark 5.14 The following diagram is obtained by Theorem 5.13(i) and (ii) above:

$$\begin{array}{lcl}
 & (\omega^{\rho 1},\omega)\text{-}T^{id}_{1/2} \rightleftharpoons & \{x\}\in\omega^{\rho 1}C(X,\tau)\cup\omega O(X,\tau)\ (\forall x\in X) \\
 & \nearrow & \\
(\omega^{\rho 1},\omega)\text{-}T^{\circ -}_{1/2} & & \downarrow\uparrow \\
 & \searrow & \\
 & (\omega^{\rho 1},\omega)\text{-}T^{\circ}_{1/2} &
\end{array}$$

In Defintion 5.3(II)(i), especially we consider the case where $\mathcal{E}_X := \omega^{\rho 1}O(X,\tau)$ ($\rho 1 : SO(X,\tau)\to P(X)$ is function such that $\rho 1\in\{id,\circ,\circ -\}$), $\mathcal{E}'_X := \omega^{\circ}O(X,\tau)$ (resp. $\mathcal{E}'_X := \omega^{\circ -}O(X,\tau)$) and a function $\rho : SO(X,\tau)\to P(X)$ with $\rho\in\{id,\circ,\circ -\}$; and so we have the following properties on "$(\omega^{\rho 1},\omega^{\circ})$-$T^{\rho}_{1/2}$" (resp. "$(\omega^{\rho 1},\omega^{\circ -})$-$T^{\rho}_{1/2}$") spaces, using Lemma 5.9 and Definition 5.3 (I) above.

Theorem 5.15 *For fixed functions $\rho 1 : SO(X,\tau)\to P(X)$ with $\rho 1\in\{id,\circ,\circ -\}$ and $\mu : SO(X,\tau)\to P(X)$ with $\mu\in\{\circ,\circ -\}$, we have the following properties.*

(i) *For a fixed function $\rho : \omega^{\rho 1}O(X,\tau)\to P(X)$ with $\rho\in\{id,\circ,\circ -\}$, if (X,τ) is $(\omega^{\rho 1},\omega^{\mu})$-$T^{\rho}_{1/2}$, then $\{x\}\in\omega^{\rho 1}C(X,\tau)\cup\omega^{\mu}O(X,\tau)$ for each singleton $\{x\}$ of (X,τ).*

(ii) *Suppose that $\omega^{\mu}O(X,\tau)$ has property $(\mathcal{B})_{\omega^{\mu}O(X,\tau)}$ for $\mu\in\{\circ,\circ -\}$. Then, the following properties are equivalent:*
(1) *(X,τ) is $(\omega^{\rho 1},\omega^{\mu})$-$T^{id}_{1/2}$;*
(2) *if $x\in X$ then $\{x\}\in\omega^{\rho 1}C(X,\tau)\cup\omega^{\mu}O(X,\tau)$;*
(3) *(X,τ) is $(\omega^{\rho 1},\omega^{\mu})$-$T^{\circ}_{1/2}$.*

(iii) *Suppose that $\omega^{\mu}O(X,\tau)$ has property $(\mathcal{B})_{\omega^{\mu}O(X,\tau)}$ for $\mu\in\{\circ,\circ -\}$. Then, every $(\omega^{\rho 1},\omega^{\mu})$-$T^{\circ -}_{1/2}$ topological space is $(\omega^{\rho 1},\omega^{\mu})$-$T^{id}_{1/2}$ and $(\omega^{\rho 1},\omega^{\mu})$-$T^{\circ}_{1/2}$.*

(iv) *Suppose that $\omega^{\mu}O(X,\tau)$ has property $(\mathcal{B})_{\omega^{\mu}O(X,\tau)}$ for $\mu \in \{\circ, \circ-\}$. Then, if $\{x\} \in (\omega^{\rho 1}C(X,\tau) \cap SO(X,\tau)) \cup \omega^{\mu}O(X,\tau)$ for each $x \in X$, then (X,τ) is $(\omega^{\rho 1}, \omega^{\mu})$-$T^{\circ-}_{1/2}$.* □

($\bullet$) In the end of the present section, we define the concepts of ω°-T_i spaces, $\omega^{\circ-}$-T_i spaces and ω-T_i spaces for each integer $i \in \{1, 0\}$ (cf. Definition 5.16 (II) below). The following Definition 5.16 (I) (i.e., $\mathcal{E}_X$-T_i separation axioms, where $i \in \{0, 1\}$) are well known by many authors; for examples, they are defined on a generalized topology, say λ, due to [1, in 2002] and they are investigated on (X, λ) by [24, in 2011; for i=1],[25, in 2016;Definition 1.7 (for i=1), Definition 1.8 (for i=1/2), Defintion 3.1 (for i=3/4)]. We give Definition 5.16 (I) in order to explain the concepts of ω^{ρ}-$T_i (i \in \{0, 1\})$ accurately (cf. Definiton 5.16 (II)).

Let $X \times X$ be the direct product of X and $\triangle(X) := \{(x, x)|\ x \in X\}$ the diagonal set of X; and $(X \times X) \setminus \triangle(X) := \{(x, y) \in X \times X | x \neq y\}$.

Definition 5.16 (I)([1], [24],[25]) A topological space (X,τ) is said to be:

(i) $\mathcal{E}_X$-T_1, if for each $(x, y) \in (X \times X) \setminus \triangle(X)$ there exist subsets U and V belonging to $\mathcal{E}_X$ such that $x \in U$ but $y \notin U$ and $y \in V$ but $x \notin V$;

(ii) $\mathcal{E}_X$-T_0, if for each $(x, y) \in (X \times X) \setminus \triangle(X)$ there exists a subset U belonging to $\mathcal{E}_X$ such that $x \in U$ and $y \notin U$ or $y \in U$ and $x \notin U$ (i.e., $U \in \mathcal{E}_X$ contains exactly one of two points).

(II) For each integer $i \in \{0, 1\}$ and a function $\rho : SO(X,\tau) \to P(X)$ with $\rho \in \{id, \circ, \circ-\}$, a topological space (X,τ) is said to be ω^{ρ}-T_i , if (X,τ) is $\omega^{\rho}O(X,\tau)$-T_i (in the sense of (I) for $\mathcal{E}_X = \omega^{\rho}O(X,\tau)$) (cf. Notation 1.5 (i)). Sometimes, the separation axiom ω^{id}-T_i is denoted by ω-T_i, where $i \in \{0, 1\}$.

The followng properties are well known; (ii) is obtained by using (i) below and Lemma 5.4.

Theorem 5.17 (i) *The following properties* (1) *and* (2) *are equivalent:*

(1) *a topological space (X,τ) is $\mathcal{E}_X$-T_1;*

(2) *for each singleton $\{x\}, \mathcal{E}_X$-$Cl(\{x\}) = \{x\}$ holds.*

(ii) *Suppose that $\mathcal{E}_X$ has property $(\mathcal{B})_{\mathcal{E}_X}$. Then,* (1), (2) *above and the following property* (3) *are equivalent.*

(3) *For each singleton $\{x\}$, $X \setminus \{x\} \in \mathcal{E}_X$ holds.* □

We investigate some relations among $\omega^{\rho 1}$-T_i spaces for a function $\rho 1 : SO(X,\tau) \to P(X)$ with $\rho 1 \in \{id, \circ, \circ-\}$ and a fixed number i with $i \in \{0, 1/2, 1\}$.

Theorem 5.18 (i) *Every T_i space is ω-T_i for each $i \in \{0, 1/2, 1\}$, where a symbol ω-$T_{1/2}$ means the separation axiom:* (ω, ω)-$T^{id}_{1/2}$ (cf. Definition 5.3 (I)($*1$)).

(ii) *Every ω°-T_i space is ω-T_i and $\omega^{\circ-}$-T_i for each $i \in \{0, 1\}$* (cf. Theorem 5.13(ii) for the case where $i = 1/2$).

Proof **(i)** Since $\tau \subset \omega O(X,\tau)$, the case where of $i \in \{0, 1\}$ is proved by Definition 5.16 for $\mathcal{E}_X := \omega O(X,\tau)$. By [5, Theorem 2.5], it is shown that if (X,τ) is $T_{1/2}$ then every singleton $\{x\}$ of (X,τ) is open or closed; and so it is ω-open or ω-closed. Then, the proof of the case where of $i = 1/2$ is obtained by Theorem 5.13(i) for $\rho 1 = id$.

(ii) Since $\omega^{\circ}O(X,\tau) \subset \omega O(X,\tau)$ and $\omega^{\circ}O(X,\tau) \subset \omega^{\circ-}O(X,\tau)$ holds (cf. Theorem 2.1), the proof of (ii) is obtained by Definition 5.16. □

We investigate some relations among $\omega^{\rho1}$-T_0 spaces, $\omega^{\rho1}$-T_1 spaces and $(\omega^{\rho1},\omega^{\rho1})$-$T^{\rho}_{1/2}$ spaces, where $\rho1 : SO(X,\tau) \to P(X)$ is a function such that $\rho1 \in \{id, \circ, \circ-\}$ and $\rho = id : \omega^{\rho1}O(X,\tau) \to P(X)$ (cf. Definition 5.3(I) and Definition 5.16 (II)).

Theorem 5.19 *We have the following diagram of implications.*

(i) ω-$T_1 \Rightarrow (\omega,\omega)$-$T^{id}_{1/2}(=\omega$-$T_{1/2}) \Rightarrow \omega$-$T_0$.

(ii) *Let $\mu : SO(X,\tau) \to P(X)$ be a function such that $\mu \in \{\circ, \circ-\}$. Supose that $\omega^{\mu}O(X,\tau)$ has property property $(\mathcal{B})_{\omega^{\mu}O(X,\tau)}$. Then,*

ω^{μ}-$T_1 \Rightarrow (\omega^{\mu},\omega^{\mu})$-$T^{id}_{1/2}$.

(iii) *Let $b : SO(X,\tau) \to P(X)$ be a fixed function such that $b \in \{\circ, \circ-\}$. Then,*

(ω^{b},ω^{b})-$T^{id}_{1/2} \Rightarrow \omega^{b}$-$T_0$.

Proof **(i)** $\cdot(\omega$-$T_1 \Rightarrow (\omega,\omega)$-$T^{id}_{1/2})$: Suppose that (X,τ) is ω-T_1, i.e., $\omega O(X,\tau)$-T_1. By Theorem 5.17(i) for $\mathcal{E}_X := \omega O(X,\tau)$, it is shown that $\omega O(X,\tau)$-$\mathrm{Cl}(\{x\}) = \{x\}$ for each singleton $\{x\}$ of (X,τ); and so, by Theorem 5.17(ii) for $\mathcal{E}_X := \omega O(X,\tau)$, it is shown that every singleton $\{x\}$ is ω-closed (i.e., $X \setminus \{x\} \in \omega O(X,\tau)$), because $\omega O(X,\tau)$ has property $(\mathcal{B})_{\omega O(X,\tau)}$ (cf. Remark 5.6(i)). Using Theorem 5.13(i) for $\rho1 = id$, we have that the space (X,τ) is (ω,ω)-$T^{id}_{1/2}$ (cf. Remark 5.6(ii)).

$\cdot((\omega,\omega)$-$T^{id}_{1/2} \Rightarrow \omega$-$T_0)$: Suppose that (X,τ) is (ω,ω)-$T^{id}_{1/2}$. By Theorem 5.13(i) for $\rho1 = id$, every singleton $\{x\}$ is ω-closed or ω-open. For a pair of distinct points x and y, we consider the following cases:

Case 1. $\{x\} \in \omega O(X,\tau)$ and $\{y\} \in \omega O(X,\tau)$: for this case, $\{x\}$ is the required set belonging to $\mathcal{E}_X := \omega O(X,\tau)$ such that $x \in \{x\}$ and $y \notin \{x\}$.

Case 2. $\{x\} \in \omega O(X,\tau)$ and $\{y\} \in \omega C(X,\tau)$: for this case, $\{x\} \in \mathcal{E}_X := \omega O(X,\tau)$ such that $x \in \{x\}$ and $y \notin \{x\}$.

Case 2'. $\{x\} \in \omega C(X,\tau)$ and $\{y\} \in \omega O(X,\tau)$: for this case, $\{y\} \in \mathcal{E}_X := \omega O(X,\tau)$ such that $y \in \{y\}$ and $x \notin \{y\}$.

Case 3. $\{x\} \in \omega C(X,\tau)$ and $\{y\} \in \omega C(X,\tau)$: for this case, $X \setminus \{y\} \in \mathcal{E}_X := \omega O(X,\tau)$ such that $x \in X \setminus \{y\}$ and $y \notin X \setminus \{y\}$.

Therefore (X,τ) is ω-T_0 (cf. Definition 5.16(II) for $\rho = id$).

(ii) Let $x \in X$. By Theorem 5.17(ii) for $\mathcal{E}_X := \omega^{\mu}O(X,\tau)$, it is shown that the singleton $\{x\}$ is ω^{μ}-closed (i.e., $X \setminus \{x\} \in \omega^{\mu}O(X,\tau)$); and so, by Theorem 5.15(ii) for the case where $\rho1 = \mu$, (X,τ) is $(\omega^{\mu},\omega^{\mu})$-$T^{id}_{1/2}$.

(iii) Let (X,τ) be an (ω^{b},ω^{b})-$T^{id}_{1/2}$ space, where $b \in \{\circ, \circ-\}$. Let $x \neq y$ be two points of X. Then, by Theorem 5.15(i) for $\mathcal{E}_X := \omega^{b}O(X,\tau)$ and $\rho1 = \mu = b$, it is shown that, the singleton $\{x\}$ is ω^{b}-closed or $\{x\}$ is ω^{b}-open. Then, (X,τ) is ω^{b}-T_0. □

6 An example satisfying a separation axiom: "$\omega^{\circ-}$-T_1 except a subset A" of $(\mathbb{Z},\kappa)$

In the last section, we prove the following properties: Theorem 6.1 on some separation axioms of the digital line $(\mathbb{Z},\kappa)$.

Theorem 6.1 *Let $(\mathbb{Z},\kappa)$ be the digital line and $\mathbb{Z}_{\kappa} := \{2s+1 | s \in \mathbb{Z}\}$. We have the following properties of $(\mathbb{Z},\kappa)$.*

(i) *$(\mathbb{Z},\kappa)$ is (ω,ω)-$T^{id}_{1/2}$.*

(ii) *$(\mathbb{Z},\kappa)$ is not ω°-T_0.*

(iii) *$(\mathbb{Z},\kappa)$ is not $\omega^{\circ-}$-T_0.*

(iv) *$(\mathbb{Z},\kappa)$ is $\omega^{\circ-}$-T_1 except $\mathbb{Z}_{\kappa}$.*

In the end of the present section, we prove the Theorem 6.1 above, after recalling of definitions (i.e., Definitions 6.2, 6.3) and preparing some propositions (i.e., Propositions 6.4,6.5).

Definition 6.2 Suppose that $|X| > 1$. Let A be a proper subset of X. A topological space (X, τ) is said to be:

$\mathcal{E}_X$-T_1 *except* A, if the following properties (1) and (2) are satisfied:

(1) for every ordered pair $(x, y) \in (X \setminus A) \times (X \setminus A)$ such that $x \neq y$, there exists a set $V \in \mathcal{E}_X$ such that $x \in V$ and $y \notin V$ and there exists a set $V_1 \in \mathcal{E}_X$ such that $x \notin V_1$ and $y \in V_1$;

(2) for every ordered pair $(a, b) \in A \times A$ such that $a \neq b$, there does not exist any subsets $V \in \mathcal{E}_X$ and $V_1 \in \mathcal{E}_X$ such that $a \in V$ and $b \notin V$, and $b \in V_1$ and $a \notin V_1$.

Put $\mathcal{E}_X := \omega^{\circ-}O(X, \tau)$ in Defintion 6.2; then we have the following definition.

Definition 6.3 Suppose that $|X| > 1$ and A is a proper subset of X. A topologcal space (X, τ) is said to be $\omega^{\circ-}$-T_1 *except* A, if the space (X, τ) is $\omega^{\circ-}O(X, \tau)$-$T_1$ except A in the sense of Definition 6.2.

Proposition 6.4 *Let $(\mathbb{Z}, \kappa)$ be the digital line and $\{2m\}$ and $\{2s+1\}$ be two singletons of $(\mathbb{Z}, \kappa)$, where $m, s \in \mathbb{Z}$.*

(i) $\{2m\} \in \omega C(\mathbb{Z}, \kappa)$, $\{2m\} \notin \omega O(\mathbb{Z}, \kappa)$; $\{2s+1\} \notin \omega C(\mathbb{Z}, \kappa)$, $\{2s+1\} \in \omega O(\mathbb{Z}, \kappa)$.

(ii) $\{2m\} \notin \omega^{\circ-}C(\mathbb{Z}, \kappa)$, $\{2m\} \in \omega^{\circ-}O(\mathbb{Z}, \kappa)$; $\{2s+1\} \notin \omega^{\circ-}C(\mathbb{Z}, \kappa)$, $\{2s+1\} \notin \omega^{\circ-}O(\mathbb{Z}, \kappa)$.

(iii) *For every singleton $\{x\}$ of $(\mathbb{Z}, \kappa)$, $\{x\} \notin \omega^{\circ}C(\mathbb{Z}, \kappa)$ and $\{x\} \notin \omega^{\circ}O(\mathbb{Z}, \kappa)$.*

Proof. **(i)** It is well known that $\{2m\}$ is not open and it is closed in $(\mathbb{Z}, \kappa)$ and $\{2s+1\}$ is open and it is not closed in $(\mathbb{Z}, \kappa)$. Since $\omega O(\mathbb{Z}, \kappa) = \kappa$ holds by [17, Theorem 4.6], and hence we have that $\{2m\} \in \omega C(\mathbb{Z}, \kappa) \setminus \omega O(\mathbb{Z}, \kappa)$ and $\{2s+1\} \in \omega O(\mathbb{Z}, \kappa) \setminus \omega C(\mathbb{Z}, \kappa)$ hold.

(ii) $\cdot$ *Proof of* $\{2m\} \notin \omega^{\circ-}C(\mathbb{Z}, \kappa)$: there exists a semi-open set $V := \{2m, 2m+1\}$ such that $\{2m\} \subset V$ and $Cl(\{2m\}) = \{2m\} \not\subset Int(Cl(V))$, because of $Int(Cl(V)) = Int(\{2m, 2m+1, 2m+2\}) = \{2m+1\}$; and so $\{2m\}$ is not $\omega^{\circ-}$-closed in $(\mathbb{Z}, \kappa)$ (i.e., $\{2m\} \notin \omega^{\circ-}C(\mathbb{Z}, \kappa)$).

$\cdot$ *Proof of* $\{2m\} \in \omega^{\circ-}O(\mathbb{Z}, \kappa)$: let $E := \mathbb{Z} \setminus \{2m\}$. Let V be a semi-open set containing E; then $V = E$ or $V = \mathbb{Z}$. Since $Cl(E) = \mathbb{Z}$ and $Int(Cl(E)) = \mathbb{Z}$ hold, we have that $Cl(E) \subset Int(Cl(V))$; and so $E := \mathbb{Z} \setminus \{2m\}$ is $\omega^{\circ-}$-closed in $(\mathbb{Z}, \kappa)$. Hence $\{2m\}$ is $\omega^{\circ-}$-open (i.e., $\{2m\} \in \omega^{\circ-}O(\mathbb{Z}, \kappa)$).

$\cdot$ *Proof of* $\{2s+1\} \notin \omega^{\circ-}C(\mathbb{Z}, \kappa)$: there exists a semi-open set $V := \{2s+1\}$ such that $\{2s+1\} \subset V$ and $\mathrm{Cl}(\{2s+1\}) = \{2s, 2s+1, 2s+2\} \not\subset Int(Cl(V))$, because of $Int(Cl(V)) = Int(\{2s, 2s+1, 2s+2\}) = \{2s+1\}$; and so $\{2s+1\}$ is not $\omega^{\circ-}$-closed in $(\mathbb{Z}, \kappa)$ (i.e., $\{2s+1\} \notin \omega^{\circ-}C(\mathbb{Z}, \kappa)$).

$\cdot$ *Proof of* $\{2s+1\} \notin \omega^{\circ-}O(\mathbb{Z}, \kappa)$: let $E := \mathbb{Z} \setminus \{2s+1\}$. Let $V := E$; and so V is a semi-open set containing E. Since $Cl(E) = E$ and $Int(Cl(V)) = Int(E) = \mathbb{Z} \setminus \{2s, 2s+1, 2s+2\}$ hold, we have that $Cl(E) = E \not\subset Int(Cl(V))$; and so $E := \mathbb{Z} \setminus \{2s+1\}$ is not $\omega^{\circ-}$-closed in $(\mathbb{Z}, \kappa)$. Hence $\{2s+1\} \notin \omega^{\circ-}O(\mathbb{Z}, \kappa)$.

(iii) Let $x = 2m$ or $x = 2s+1$, where $m \in \mathbb{Z}$ and $s \in \mathbb{Z}$.

$\cdot$ *Proof of* $\{2m\} \notin \omega^{\circ}C(\mathbb{Z}, \kappa)$: by using the properties for $(\mathbb{Z}, \kappa)$ of Theorem 2.1(iii) (i.e., $\omega^{\circ}C(\mathbb{Z}, \kappa) \subset \omega^{\circ-}C(\mathbb{Z}, \kappa)$) and the corresponding property of the present (ii) (i.e., $\{2m\} \notin \omega^{\circ-}C(\mathbb{Z}, \kappa)$), it is shown that $\{2m\} \notin \omega^{\circ}C(\mathbb{Z}, \kappa)$.

$\cdot$ *Proof of* $\{2m\} \notin \omega^{\circ}O(\mathbb{Z}, \kappa)$: by using the property for (X, τ) of Theorem 2.1(i) (i.e., $\omega^{\circ}C(X, \tau) \subset \omega C(X, \tau)$) and definitions, it is shown that $\omega^{\circ}O(X, \tau) \subset \omega O(X, \tau)$ holds in general. By using the corresponding property of the proof of (i), it is obtained that $\{2m\} \notin \omega^\circ O(\mathbb{Z},\kappa)$.

$\cdot$ *Proof of* $\{2s+1\} \notin \omega^\circ C(\mathbb{Z},\kappa)$: by using the property for $(\mathbb{Z},\kappa)$ of Theorem 2.1(iii) (i.e., $\omega^\circ C(\mathbb{Z},\kappa) \subset \omega^{\circ -} C(\mathbb{Z},\kappa)$) and the corresponding property of the present (ii) (i.e., $\{2s+1\} \notin \omega^{\circ -} C(\mathbb{Z},\kappa)$), it is shown that $\{2s+1\} \notin \omega^\circ C(\mathbb{Z},\kappa)$.

$\cdot$ *Proof of* $\{2s+1\} \notin \omega^\circ O(\mathbb{Z},\kappa)$: by using the same property for $(\mathbb{Z},\kappa)$ of Theorem 2.1(iii) (cf. Proof of $\{2m\} \notin \omega^\circ O(\mathbb{Z},\kappa)$) and the corresponding property of the present (ii) (i.e., $\{2s+1\} \notin \omega^{\circ -} O(\mathbb{Z},\kappa)$), it is shown that $\{2s+1\} \notin \omega^\circ O(\mathbb{Z},\kappa)$. □

Proposition 6.5 (i) (i-1) *If* $U \in \omega^\circ O(\mathbb{Z},\kappa)$ *and* $2m \in U$ *for some integer* m*, then* $\{2m-1, 2m, 2m+1\} \subset U$.

(i-2) *If* $U \in \omega^\circ O(\mathbb{Z},\kappa)$ *and* $2s+1 \in U$ *for some integer* s*, then* $\{2s-1, 2s, 2s+1, 2s+2, 2s+3\} \subset U$.

(i-3) $\omega^\circ O(\mathbb{Z},\kappa) = \{\emptyset, \mathbb{Z}\}$ *holds.*

(ii) (ii-1) *If* $V \in \omega^{\circ -} O(\mathbb{Z},\kappa)$ *and* $2s+1 \in V$ *for some integer* s*, then* $\{2s-1, 2s, 2s+1, 2s+2, 2s+3\} \subset V$.

(ii-2) *The following properties on a nonempty subset* V *are equivalent:*
(1) $V \in \omega^{\circ -} O(\mathbb{Z},\kappa)$ *and* $2s+1 \in V$ *for some integer* s*;*
(2) $V = \mathbb{Z}$ *holds.*

(ii-3) $\omega^{\circ -} O(\mathbb{Z},\kappa) = \{E \mid E \subset \mathbb{Z}_F\} \cup \{\emptyset, \mathbb{Z}\}$ *holds, where* $\mathbb{Z}_F :=\{2m \mid m \in \mathbb{Z}\}$*. Especially,* $\mathbb{Z}_F \in \omega^{\circ -} O(\mathbb{Z},\kappa)$ *holds.*

(ii-4) *Every nonempty subset of* $\mathbb{Z}_\kappa$ *is not* $\omega^{\circ -}$*-open in* $(\mathbb{Z},\kappa)$(i.e., $\{2m+1 \mid m \in E\} \notin \omega^{\circ -} O(\mathbb{Z},\kappa)$*, where* $E \subset \mathbb{Z}$ *with* $E \neq \emptyset$*). Especially,* $\mathbb{Z}_\kappa \notin \omega^{\circ -} O(\mathbb{Z},\kappa)$ *holds.*

Proof. **(i) (i-1)** Since $\{2m\} \in SC(\mathbb{Z},\kappa)$ and so $\mathbb{Z} \setminus \{2m\}$ is a semi-open set. And, it follows from assumptions that $\mathbb{Z} \setminus \{2m\}$ contains the set $\mathbb{Z} \setminus U$ which is ω°-closed. Then, $Cl(\mathbb{Z} \setminus U) \subset Int(\mathbb{Z} \setminus \{2m\})$=$\mathbb{Z} \setminus \{2m\}$; and so we have that $\mathbb{Z} \setminus Int(U) \subset \mathbb{Z} \setminus \{2m\}$, i.e., $2m \in Int(U)$. There exists the smallest open set $\{2m-1, 2m, 2m+1\}$ containing $2m$ such that $\{2m-1, 2m, 2m+1\} \subset Int(U) \subset U$ (e.g., [17, Definition 3.3 and its near part]).

(i-2) Since $\mathbb{Z} = \mathbb{Z}_{\mathcal{SC}} \cup \mathbb{Z}_{\omega^\circ \mathcal{O}}$ (cf. Lemma 4.3(i)), we consider the following cases: $\{2s+1\} \in SC(\mathbb{Z},\kappa)$ or $\{2s+1\} \in \omega^\circ O(\mathbb{Z},\kappa)$. By Proposition 6.4(iii), $\{2s+1\} \notin \omega^\circ O(\mathbb{Z},\kappa)$; and so we consider the case where $\{2s+1\} \in SC(\mathbb{Z},\kappa)$. Since $\mathbb{Z} \setminus \{2s+1\}$ is a semi-open set containing $\mathbb{Z} \setminus U$ and the set $\mathbb{Z} \setminus U$ is an ω°-closed set, we have that $Cl(\mathbb{Z} \setminus U) \subset Int(\mathbb{Z} \setminus \{2s+1\}) = \mathbb{Z} \setminus Cl(\{2s+1\}) = \mathbb{Z} \setminus \{2s, 2s+1, 2s+2\}$. Thus, we have that $\{2s, 2s+1, 2s+2\} \in Int(U)$. Since $2s \in Int(U)$ (resp. $2s+2 \in Int(U)$), the minimal open set containing $2s$ (resp. $2s+2$) is included in $Int(U)$,i.e., $\{2s-1, 2s, 2s+1\} \subset Int(U)$ (resp. $\{2s+1, 2s+2, 2s+3\} \subset Int(U)$.

(i-3) Let $U \in \omega^\circ O(\mathbb{Z},\kappa)$ such that $U \neq \emptyset$. Then, by (i-1) and (i-2) above, it is shown that there exists an odd point, say $2u+1 \in U$, where $u \in \mathbb{Z}$. We claim that $\mathbb{Z} \subset U$. Indeed, let $z \in \mathbb{Z}$ be a point.

Case 1. $z = 2s$, where $s \in \mathbb{Z}$: for the present case, if $2s < 2u+1$, then we can take the following sequence of points, say $\{z_i\}_{i=1}^{k}$, where $k := 2(u-s+1)$ and $z_i := 2u+2-i$ $(1 \leq i \leq k)$, where ; then, $z_1 = 2u+1 \in U$ and $z_k = 2s = z$; and by using (i-1) and (i-2) above, we show inductively, that $z_i \in U$ $(2 \leq i \leq k)$ and hence $z \in U$. If $2s > 2u+1$, then we can take the following sequence of points, say $\{z'_i\}_{i=1}^{k'}$, $k' := 2(s-u)$ and $z'_i := 2u+i$ $(1 \leq i \leq k')$; then, $z'_1 = 2u+1 \in U$ and $z'_{k'} = z$; and by a similar arguments of the above case, it is shown that $z'_i \in U$ $(2 \leq i \leq k')$; and so $z \in U$. Thus, we proved that $z = 2s \in U$ holds for any cases.

Case 2. $z = 2t+1$, where $t \in \mathbb{Z}$: for the present case, let $z \neq 2u+1$. If $z < 2u+1$, then we can constract the following sequence of points, say $\{x_i\}_{i=1}^{k}$, where $k := u-t+1$, and $x_i := 2u + 1 - 2(i - 1)(1 \leq i \leq k)$; then $x_1 = 2u + 1 \in U$ and $x_k = z$; and by using (i-2) above, we show inductively, that $x_i \in U(2 \leq i \leq k)$; and so $z \in U$. If $2u + 1 < z$, then we can constract the following sequence of points, say $\{x'_i\}_{i=1}^{k'}$, where $k' := t - u + 1$, and $x'_i := 2u + 1 + 2(i - 1)(1 \leq i \leq k')$; then $x'_1 = 2u + 1 \in U$ and $x'_{k'} = z$; and by using (i-2) above, we show inductively, that $x'_i \in U(2 \leq i \leq k')$; and so $z \in U$.

Therefore, we prove that $\mathbb{Z} \subset U$ and so $U = \mathbb{Z}$.

(ii) (ii-1) $\cdot$ *Proof of* $\{2s, 2s+2\} \subset V$. Since $\mathbb{Z} \setminus \{2s+1\}$ is a semi-open set containing $\mathbb{Z} \setminus V$ and $\mathbb{Z} \setminus V$ is $\omega^{\circ -}$-closed, we have that $Cl(\mathbb{Z} \setminus V) \subset Int(Cl(\mathbb{Z} \setminus \{2s + 1\}))$=$\mathbb{Z} \setminus \{2s, 2s + 1, 2s + 2\}$. Thus, we have that $\{2s, 2s + 1, 2s + 2\} \subset Int(V)$.

Since $2s \in Int(V)$ (resp. $2s + 2 \in Int(V)$) and the set $\{2s - 1, 2s, 2s + 1\}$ (resp. $\{2s + 1, 2s + 2, 2s + 3\}$) is the ninimal open set containing the point $2s$ (resp. $2s + 2$), we have that $\{2s - 1, 2s, 2s + 1\} \subset V$ (resp. $\{2s + 1, 2s + 2, 2s + 3\} \subset V$). Therefore, we show the required property that $\{2s - 2 + j | 1 \leq j \leq 5\} \subset V$.

(ii-2) (1)$\Rightarrow$(2) In order to prove that $\mathbb{Z} \subset V$, let $z \in \mathbb{Z}$ be a point. First, it is claimed that:

$(*1)$ if $z = 2m + 1$ for some integer m, then $z \in V$.

Proof of $(*1)$: **(Case 1)** $z := 2m + 1$ and $z < 2s + 1$; for the present case, we apply (ii-1) for the point $2s + 1 \in V$ and $V \in \omega^{\circ -}O(X, \tau)$. And, it is shown inductively that there exists a finite sequence of points $\{y_i\}_{i=1}^k$ such that:

$(*2)_i$ $y_i \in V$ $(1 \leq i \leq k)$, where $y_i := 2s + 1 - 2i$ and $k := s - m$.

Indeed, by (ii-1) above for the odd point $2s + 1 \in V$, it is shown that $\{2s - 1, 2s, 2s + 1, 2s + 2, 2s + 3\} \subset V$. Thus, $2s - 1 \in V$; and so $y_1 = 2s + 1 - 2 \in V$. Then, we show that $(*2)_i$ holds for $i = 1$. In order to prove $(*2)_i$ by finite induction on i $(1 \leq i \leq k)$, suppose that $y_r \in V$, where $1 < r < k$ and $y_r := 2s + 1 - 2r$. Since y_r is odd and $y_r \in V$, by (ii-1) above, it is shown that $\{y_r - 2, y_r - 1, \ y_r, \ y_r + 1, y_r + 2\} \subset V$. Thus, we have that $y_{r+1} = 2s + 1 - 2(r + 1) = 2s + 1 - 2r - 2 = y_r - 2 \in V$, i.e., we have that $(*2)_i$ holds for $i = r + 1$. Then, by finte induction on i $(1 \leq i \leq k)$, it is shown that $y_k \in V$; and hence $z = 2m + 1 = 2s + 1 - 2(s - m) = y_{s-m} = y_k \in V$. Thus, we show that $z \in V$ for the present Case 1.

(Case 1'). $z = 2m+1 \in \mathbb{Z}$ and $2s+1 < z$: for the present case, we apply (ii-1) above for the point $2s + 1 \in V$ and $V \in \omega^{\circ -}O(X, \tau)$. By an argument similar to that in the proof of Case 1 above, it is shown inductively that there exists a sequence of points $\{y'_i\}_{i=1}^{k'}$ such that :

$(*2)'_i$ $y'_i \in V$ holds for each integer i with $1 \leq i \leq k'$, where $y'_i := (2s + 1) + 2i$ and $k' := m - s$. Thus, we show that $z \in V$ for the present Case 1'.

Finally, it is claimed that:

$(*3)$ if $z = 2m$ for some integer m, then $z \in V$.

Proof of $(*3)$: by $(*1)$ above, it is shown that $2u+1 \in V$ for any odd point $2u+1 \in \mathbb{Z}$. Then, take the odd point $z + 1 = 2m + 1$; and so $2m + 1 \in V$. Here, by using (ii-1) above for the point $2m+1 \in V$ and $V \in \omega^{\circ -}O(\mathbb{Z}, \kappa)$, it is shown that $\{2m-2, 2m, 2m+1, 2m + 2, 2m + 3\} \subset V$; and so $z := 2m \in V$.

Therefore, we conclude that $z \in V$ for any point $z \in \mathbb{Z}$ (i.e., $\mathbb{Z} = V$).

(2)$\Rightarrow$(1) Suppose $V = \mathbb{Z}$. By definitions, it is obvious that $\mathbb{Z} \in \omega^{\circ -}O(\mathbb{Z}, \kappa)$ and there exists an odd point $2s + 1 \in V = \mathbb{Z}$, where $s \in \mathbb{Z}$.

(ii-3) First, we prove that:

$(*4)$ $\omega^{\circ -}O(\mathbb{Z}, \kappa) \subset \{E | \ E \subset \mathbb{Z}_F\} \cup \{\emptyset, \mathbb{Z}\}$. Indeed, let $V \in \omega^{\circ -}O(\mathbb{Z}, \kappa)$ such that $V \notin \{\emptyset, \mathbb{Z}\}$. Then, by (ii-2) above, it is shown that $2s + 1 \notin V$ holds for every integer $s \in \mathbb{Z}$, i.e., $V \subset \mathbb{Z}_F$:=$\{2m | \ m \in \mathbb{Z}\}$. Thus, we proved $(*4)$. Secondly, we prove that:

$(*5)$ $\{E | \ E \subset \mathbb{Z}_F\} \cup \{\emptyset, \mathbb{Z}\} \subset \omega^{\circ -}O(\mathbb{Z}, \kappa)$ holds. Let $V \subset \mathbb{Z}_F$ with $V \notin \{\emptyset, \mathbb{Z}\}$. Then,

$V = \{2m|\ m \in A\}$, where $A \subset \mathbb{Z}$. In order to prove hat $\mathbb{Z} \setminus V \in \omega^{\circ -}C(\mathbb{Z}, \kappa)$, let U be a semi-open set such that $\mathbb{Z} \setminus V \subset U$. Since $\mathbb{Z}_\kappa = \{2s+1|s \in \mathbb{Z}\} \subset \mathbb{Z} \setminus V$, it is shown that $\mathbb{Z} = Cl(\mathbb{Z}_\kappa) \subset Cl(\mathbb{Z} \setminus V) \subset Cl(U)$; and so $\mathbb{Z} = Cl(U)$ and $Cl(\mathbb{Z} \setminus V) = \mathbb{Z} = Int(Cl(U))$ hold. Thus, we prove that $\mathbb{Z} \setminus V$ is $\omega^{\circ -}$-closed, i.e., $V \in \omega^{\circ -}O(\mathbb{Z}, \kappa)$.
Finally, by $(*5)$ above, it is especially shown that $\mathbb{Z}_F \in \omega^{\circ -}O(\mathbb{Z}, \kappa)$.

(ii-4) Let denote $V := \{2m+1|\ m \in A\}$, where $A \subset \mathbb{Z}$ with $A \neq \emptyset$. Then, $V \notin \{E|\ E \subset \mathbb{Z}_F\} \cup \{\emptyset, \mathbb{Z}\}$;and so, by (ii-3) above, $V \notin \omega^{\circ -}O(\mathbb{Z}, \kappa)$. Especially, $\mathbb{Z}_\kappa \notin \omega^{\circ -}O(\mathbb{Z}, \kappa)$. □

Remark 6.6 The converse of Proposition 6.5(ii)(ii-1) is not true. Indeed, Let $V := \{2s-1, 2s, 2s+1, 2s+2, 2s+3\}$ be a subset of $(\mathbb{Z}, \kappa)$, where $s \in \mathbb{Z}$. Then, there exists a semi-open set $W := \mathbb{Z} \setminus V$ such that $\mathbb{Z} \setminus V \subset W$ and $Cl(\mathbb{Z} \setminus V) = Cl(W) = W \not\subset Int(Cl(W))$. Then, $\mathbb{Z} \setminus V \notin \omega^{\circ -}C(\mathbb{Z}, \kappa)$,i.e., $V \notin \omega^{\circ -}O(\mathbb{Z}, \kappa)$ holds, even if $2s+1 \in V$ and $\{2s-1, 2s, 2s+1, 2s+2, 2s+3\} \subset V$.

Proof of Theorem 6.1:

Proof of (i) It is well known that $(\mathbb{Z}, \kappa)$ is $T_{1/2}$ and so it is (ω, ω)-$T^{id}_{1/2}$ (cf. [5, Theorem 2.5], Theorem 5.18 (i)).

Proof of (ii) Let $x := 2m \in \mathbb{Z}$ and U be any ω°-open set such that $x \in U$. By Proposition 6.5(i)(i-3), it is shown that $U = \mathbb{Z}$ and so $2m+1 \in U$. Thus, there exists a pair of distinct points $2m$ and $2m+1$ of $(\mathbb{Z}, \kappa)$ which does not satisfy the condition of the ω°-T_0 (cf. Definition 5.16 for $\mathcal{E}_\mathbb{Z} := \omega^\circ O(\mathbb{Z}, \kappa)$).

Proof of (iii) Let $x := 2s+1$ and $y := 2s+3$ be two points of $(\mathbb{Z}, \kappa)$, where $s \in \mathbb{Z}$. And, let V (resp. V_1) be any $\omega^{\circ -}$-open set containing the point x (resp. y). By Proposition 6.5(ii)(ii-2)(1)⇒(2), it is shown that $V = \mathbb{Z}$ (resp. $V_1 = \mathbb{Z}$), and so $y \in V$ (resp. $x \in V$). Thus, $(\mathbb{Z}, \kappa)$ is not $\omega^{\circ -}$-T_0 (cf. Definition 5.16).

Proof of (iv) First, we recall that $\mathbb{Z}_\kappa := \{2u+1|\ u \in \mathbb{Z}\}$. Let $(x, y) \in (\mathbb{Z} \setminus \mathbb{Z}_\kappa) \times (\mathbb{Z} \setminus \mathbb{Z}_\kappa)$ be an ordered pair of points such that $x \neq y$. Since $x = 2m$ for some integer m, there exists a set $V \in \omega^{\circ -}O(\mathbb{Z}, \kappa)$ (cf. Proposition 6.4(ii)), where $V := \{2m\}$, such that $x \in V$ and $y \notin V$. And, since $y = 2s$ for some integer s with $s \neq m$, there exists a set $V_1 \in \omega^{\circ -}O(\mathbb{Z}, \kappa)$, where $V_1 := \{2s\}$, such that $x \notin V_1$ and $y \in V_1$. Thus, one of the properties of $\omega^{\circ -}$-T_1-ness except $\mathbb{Z}_\kappa$ is satiesfied (cf. (1) of Definition 6.2 and Definition 6.3).

Finally, let $(a, b) \in \mathbb{Z}_\kappa \times \mathbb{Z}_\kappa$ be any ordered pair of points a and b such that $a \neq b$. Let V_a (resp. W_b) be any $\omega^{\circ -}$-open set such that $a \in V_a$ (resp. $b \in W_b$). Then, by Proposition 6.5(ii)(ii-2) (1)⇒(2), it is shown that $V_a = \mathbb{Z}$; and so $b \in V$ (resp. $W_b = \mathbb{Z}$ and so $a \in W_b$). Thus, the property (2) for $A := \mathbb{Z}_\kappa$ in Definition 6.2 of $\omega^{\circ -}$-T_1-ness except $\mathbb{Z}_\kappa$ is satisfied.

Therefore, the digital line $(\mathbb{Z}, \kappa)$ is $\omega^{\circ -}$-T_1 except $\mathbb{Z}_\kappa$. □

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Communicated by *Yasunao Hattori*

H. Maki; 2-10-13 Wakagidai, Fukutsu-shi, Fukuoka-ken, 811-3221, JAPAN
e-mail:makih@pop12.odn.ne.jp

N. Rajesh; Department of Mathematics, Rajah Serfoji Goverment College
Thanjavur 613 005, Tamil Nadu, INDIA

S. Shanthi; Department of Mathematics, Arignar Anna Govt. Arts College
Namakkal - 637 001, Tamil Nadu, INDIA