LOGICAL ANALYSIS OF RATIO INFERENCE BY CHILDREN

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Abstract. In this study, we applied propositional and predicate logic for mathematical explication of the processes of inference by children. This facilitated extraction and comparison of children-specific inference processes, which are difficult to derive from a child's protocol itself, and elucidation of the structure of children's ratio-related conceptual and procedural knowledge.

1 Introduction In arithmetic education, ascertaining the concepts of given domains in terms of conceptual and procedural knowledge is essential as a mechanism of knowledge change during knowledge acquisition. Conceptual knowledge consists of an implicit or explicit system of interlinked pieces of knowledge for a given domain, and procedural knowledge comprises systems of multiple execution series for problem solution [1], [2].

The concept of ratio is applied in ascertaining the relation between two quantities and in comparing the relative quantities of two sets. It differs in meaning from simple multiplication and is active in the sense of comparing the relative sizes (multiples) of given quantities and base quantities rather than directly comparing quantities [3]. In the present study, we therefore focus on comparison of the relations between quantities in two different sets. It has been noted that the concept of ratio can be investigated in a fairly pure form as a logical mathematical recognition [4], and in this light we treat this comparison as a probabilistic comparison task. Ratio and probability are different concepts, but for children unschooled in probability, the ratio concept can be utilized as an approach for probability settings. Studies that have utilized probability comparison tasks include A. Nakagaki [4], [5], N. Fujimura [6], G. Noelting [7], [8], J. Piaget and B. Inhelder [9], and R. S. Siegler [10], [11], which in relation to quantification of probability all share the view that recognition of equivalence based on recognition of multiple relationships provides the foundation for the intensive quantity concept, and formation of that concept begins at the age of 11 or 12 years [6]. A. Nakagaki [5] identifies the psychological stage of development of the ratio concept as a process of balancing in which the ability to compensate affirmation with negation becomes complete. Moreover, children inherently possess and apply the concept of " half " 1/2) as an intuitive approach for quantification of probability [12], and the " half " benchmark strategy [13] is of key significance during the stage in which children recognize and develop ratio inference leading to " part-whole " comparison in probability comparison problems.

Previous studies have not included integrated analyses of children's recognition in the three situational contexts of ratio, comparative quantity, and base quantity, and are generally protocol-based analyses of children's recognition of ratio rather than mathematical representations of children's thought processes, using test problems that include numbers, and thereby make it difficult to determine the relationship between children's conceptual and procedural knowledge in their ratio recognition.

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The present study was undertaken to develop test problems that distinguish between conceptual and procedural knowledge relating to ratios, express the thought processes of children mathematically, and elucidate the structures of ratio-related conceptual and procedural knowledge.

2 Development of test problems

(1) Symbolization of inference process by propositional and predicate logic In the development of each test problem, it is necessary to prove that a given inference process can derive the correct conclusion from the perspective of probability with the conditions given in the problem statement as assumptions. In analysis of the test results, moreover, it is essential to explain the children-specific logic used in the inference process mathematically. In the present study, we perform these proofs and analyses by using propositional logic and predicate logic with reference to the views of S. Tamura, K. Aragane, and T. Hirai [14] and K. Todayama [15]. The symbols and the rules and laws of inference as used in the present study are essentially as follows. Note that we express $A \Rightarrow B$, i.e., if $A \equiv \top$ then $B \equiv \top$, as inference schemata with a horizontal line of the form as below.

$A \Rightarrow B$	$\dfrac{A}{B}$

1) Inference rules and laws We let x, y, z, a, b, c, and d be nonnegative variables, and let $f(x)$ be $x = y, x > y$, or $x < y$. We refer to $f(x)$ containing variable x as the expression. The focus is on the thought processes of children, and we accordingly allow the use of operations on the variables and take the operation rules to be applicable to inference rules. Tables 1 through 3 show the unit element, zero element, and reflective, symmetric, and transitive laws, the inference rules, and the inference laws, respectively, for operations on the variables. The proofs of the inference laws are not shown.

$Unit\ Element(UE)$	If $x \times y = y \times x = x$, take y as a unit element and write $y = 1$.
$Zero\ Element(ZE)$	If $x + y = y + x = x$, take y as a zero element and write $y = 0$.
$Reflective\ Law(RL)$	$x = x$
$Symmetric\ Law(SL)$	$\dfrac{x = y}{y = x}$
$Transitive\ Law(TL)$	$\dfrac{x = y \quad y = z}{x = z}$ $\dfrac{x > y \quad y > z}{x > z}$ $\dfrac{x < y \quad y < z}{x < z}$ $\dfrac{x > y \quad y = z}{x > z}$ $\dfrac{x = y \quad y > z}{x > z}$ $\dfrac{x < y \quad y = z}{x < z}$ $\dfrac{x = y \quad y < z}{x < z}$

Table 1: Unit element, zero element, and reflective, symmetric, and transitive laws for operations on the variables

Rule name	Inference rule
$Operation-Inference(OI)$	Where $a \circ b = c$, allow $\dfrac{f(a \circ b)}{f(c)}$ and $\dfrac{f(c)}{f(a \circ b)}$
==	$\dfrac{a = b \quad c = d}{a \circ c = b \circ d}$ $\circ : +, -, \times$, or $\div$

>= 1	$\dfrac{a > b \quad c = d}{a \circ c > b \circ d}$	$\dfrac{a < b \quad c = d}{a \circ c < b \circ d}$	$\circ : +, -, \times,$ or $\div$
	$\dfrac{a = b \quad c > d}{a \circ c > b \circ d}$	$\dfrac{a = b \quad c < d}{a \circ c < b \circ d}$	$\circ : +$ or $\times$
>= 2	$\dfrac{a = b \quad c > d}{a \circ c < b \circ d}$	$\dfrac{a = b \quad c < d}{a \circ c > b \circ d}$	$\circ : -$ or $\div$
>> 1	$\dfrac{a > b \quad c > d}{a \circ c > b \circ d}$	$\dfrac{a < b \quad c < d}{a \circ c < b \circ d}$	$\circ : +$ or $\times$
>> 2	$\dfrac{a > b \quad c > d}{a \circ d > b \circ c}$	$\dfrac{a < b \quad c < d}{a \circ d < b \circ c}$	$\circ : -$ or $\div$
<>	$\dfrac{a > b}{b < a}$	$\dfrac{a < b}{b > a}$	

Table 2: Rules of inference for operations on variables

In all of the above operations, $\div$ is applicable so long as $c \neq 0$ and $d \neq 0$.

The following rules are allowed as operation-inference rules for $a \circ b = c$.
(1) $x \times 1 = 1 \times x = x$
(2) $x \times 1/x = 1/x \times x = x \div x = x/x = 1$
(3) $x + 0 = 0 + x = x$
(4) $x - x = 0$
(5) $x \circ y = y \circ x \quad (\circ : +$ or $\times)$ [Commutative Law]
(6) $(x \circ y) \circ z = x \circ (y \circ z) \quad (\circ : +$ or $\times)$ [Associative Law]
(7) $x \times (y \circ z) = x \times y \circ x \times z \quad (\circ : +$ or $-)$ [Distributive Law]
(8) $(y \circ z) \div x = y \div x \circ z \div x \quad (\circ : +$ or $-)$ [Distributive Law]

The following calculations are allowed as operation-inference rules for $a \circ b = c$.
(1) $x \times 1/y = x \div y = x/y$
(2) $a \div b = (a \times c) \div (b \times c)$
(3) $(a/b \times bd) \div (c/d \times bd) = (a \times d) \div (b \times c)$

Law name	Inference law
$= Substitution (= Sub)$	$\dfrac{f(a_1, a_2, \cdots, a_n) \quad a_1 = b_1, a_2 = b_2, \cdots, a_n = b_n}{f(b_1, b_2, \cdots, b_n)}$

Table 3: Laws of inference for operations on variables

The next four tables show the inference rules (Table 4) and inference laws (Table 5) for propositional logic, and the inference rules (Table 6) and inference law (Table 7) for predicate logic. $F(X)$ is a logical expression containing propositional variable X. The proofs of the inference laws are not shown.

Rule name	Inference rule	Rule name	Inference rule
$\rightarrow Introduction(\rightarrow Int)$	$\dfrac{\overset{(k)}{[A]}\quad\vdots\quad B}{A \rightarrow B}\ (k)$	$\vee Removal(\vee Rem)$	$\dfrac{A \vee B \quad \overset{(k)}{[A]}\ C \quad \overset{(k)}{[B]}\ C}{C}\ (k)$
$\rightarrow Removal(\rightarrow Rem)$	$\dfrac{A \quad A \rightarrow B}{B}$	$\vee Introduction(\vee Int)$	$\dfrac{A}{A \vee B}$
$Transition(Trn)$	$\dfrac{A \rightarrow B \quad B \rightarrow C}{A \rightarrow B}$	$\neg Removal(\neg Rem)$	$\dfrac{A \quad \neg\neg A}{\bot}$
$\wedge Introduction(\wedge Int)$	$\dfrac{A \quad B}{A \wedge B}$	$\neg Introduction(\neg Int)$	$\dfrac{[A]\quad \bot}{\neg A}$
$\wedge Removal(\wedge Rem)$	$\dfrac{A \wedge B}{A} \quad \dfrac{A \wedge B}{B}$	$\neg\neg Removal(\neg\neg Rem)$	$\dfrac{\neg\neg A}{A}$

Table 4: Rules of inference for propositional logic

Law name	Inference law	Law name	Inference law
$\equiv Removal(\equiv Rem)$	$\dfrac{A \equiv B}{A \rightarrow B} \quad \dfrac{A \equiv B}{B \rightarrow A}$	$\equiv Substiution(\equiv Sub)$	$\dfrac{F(A) \quad A \equiv B}{F(B)}$
$\wedge\wedge Introduction(\wedge\wedge Int)$	$\dfrac{A_1 \quad A_2 \quad A_3 \cdots A_n}{A_1 \wedge A_2 \wedge A_3 \wedge \cdots \wedge A_n}$	$\veebar \rightarrow \vee$	$\dfrac{A \veebar B}{A \vee B}$
$\neg\neg Introduction(\neg\neg Int)$	$\dfrac{A}{\neg\neg A}$	$Importation(Imp)$	$\dfrac{A \rightarrow (B \rightarrow C)}{A \wedge B \rightarrow C}$
$Contraposition(Cont)$	$\dfrac{A \rightarrow B}{\neg B \rightarrow \neg A}$		

Table 5: Laws of inference for propositional logic

Rule name	Inference rule
$\forall Removal(\forall Rem)$	$\dfrac{\forall x[P(x)]}{P(a_i)}$
$\exists Introduction(\exists Int)$	$\dfrac{P(a_i)}{\exists x[P(x)]}$
$\forall Introduction(\forall Int)$	$\dfrac{P(a_1) \quad P(a_2) \cdots P(a_n)}{\forall x[P(x)]}$
$\exists Removal(\exists Rem)$	$\dfrac{\exists x[P(x)] \quad \dfrac{P(a_1)}{C} \quad \dfrac{P(a_2)}{C} \cdots \dfrac{P(a_n)}{C}}{C}$

Table 6: Rules of inference for predicate logic

Law name	Inference law
$\exists\exists Introduction(\exists\exists Int)$	$\dfrac{P_1(a_{1_{i_1}}) \quad P_2(a_{2_{i_2}}) \cdots P_n(a_{n_{i_n}})}{\exists x_1 \exists x_2 \cdots \exists x_n[P_1(x_1) \wedge P_2(x_2) \wedge \cdots \wedge P_n(x_n)]}$

Table 7: Law of inference for predicate logic

2) Symbolization for single-lot drawing trials Table 8 shows the symbolization for the number of events, elementary events, and probabilities in a trial drawing of one lot from a set containing winning and losing lots and in a trial drawing of one lot each from sets A and B (thus an " A lot " and a " B lot ", respectively) with both sets containing winning and losing lots. Variable x may be $n(X), n(Y), n(S), P(X)$, or $P(Y)$, either alone or in combination.

X	Event: Drawing of winning lot	$n(S)$	Total number of lots
X_A	Event: Drawing of winning A lot	$n(S_A)$	Total number of A lots
X_B	Event: Drawing of winning B lot	$n(S_B)$	Total number of B lots
Y	Event: Drawing of losing lot	$P(X)$	Probability of drawing winning lot
Y_A	Event: Drawing of losing A lot	$P(X_A)$	Probability of drawing winning A lot
Y_B	Event: Drawing of losing B lot	$P(X_B)$	Probability of drawing winning B lot
S	All events	$P(Y)$	Probability of drawing losing lot
S_A	All A-lot events	$P(Y_A)$	Probability of drawing losing A lot
S_B	All B-lot events	$P(Y_B)$	Probability of drawing losing B lot
$n(X)$	Number of winning lots	$P(S)$	Probability of all events
$n(X_A)$	Number of winning A lots	$P(S_A)$	Probability of all events for A lots
$n(X_B)$	Number of winning B lots	$P(S_B)$	Probability of all events for B lots
$n(Y)$	Number of losing lots		
$n(Y_A)$	Number of losing A lots		
$n(Y_B)$	Number of losing B lots		

Table 8: Number and probability of events and elementary events in single-lot drawing trials

Table 9 shows the symbolization of comparative conditions in the settings, with the total number of lots, number of winning lots, number of losing lots, probability of winning, and probability of losing as the objects of comparison. The expression $(A \wedge \neg B) \vee (\neg A \wedge B)$ is abbreviated $A \veebar B$, and exclusive disjunction is symbolized as $\veebar$.

Condition	Symbolization
Equal total numbers of A and B lots	A_1: $n(S_A) = n(S_B)$
Larger total number of A lots	A_2: $n(S_A) > n(S_B)$
Larger total number of B lots	A_3: $n(S_A) < n(S_B)$
Different total numbers of A and B lots	$\neg A_1$: $\neg(n(S_A) = n(S_B))$
Equal numbers of winning A and B lots	B_1: $n(X_A) = n(X_B)$
Larger number of winning A lots	B_2: $n(X_A) > n(X_B)$
Larger number of winning B lots	B_3: $n(X_A) < n(X_B)$
Different numbers of winning A and B lots	$\neg B_1$: $\neg(n(X_A) = n(X_B))$
Equal numbers of losing A and B lots	C_1: $n(Y_A) = n(Y_B)$
Larger number of losing A lots	C_2: $n(Y_A) > n(Y_B)$

Larger number of losing B lots	C_3: $n(Y_A) < n(Y_B)$
Different numbers of losing A and B lots	$\neg C_1$: $\neg(n(Y_A) = n(Y_B))$
Equal chances of winning with A and B lots	D_1: $P(X_A) = P(X_B)$
Greater chance of winning with A lots	D_2: $P(X_A) > P(X_B)$
Greater chance of winning with B lots	D_3: $P(X_A) < P(X_B)$
Different chances of winning with A and B lots	$\neg D_1$: $\neg(P(X_A) = P(X_B))$
Equal chances of losing with A and B lots	E_1: $P(Y_A) = P(Y_B)$
Greater chance of losing with A lots	E_2: $P(Y_A) > P(Y_B)$
Greater chance of losing with B lots	E_3: $P(Y_A) < P(Y_B)$
Different chances of losing with A and B lots	$\neg E_1$: $\neg(P(Y_A) = P(Y_B))$

Table 9: Comparative setting conditions relating to probabilities

From $A_1 \veebar A_2 \veebar A_3$, $\neg A_1 \equiv A_2 \veebar A_3$; from $B_1 \veebar B_2 \veebar B_3$, $\neg B_1 \equiv B_2 \veebar B_3$; from $C_1 \veebar C_2 \veebar C_3$, $\neg C_1 \equiv C_2 \veebar C_3$; from $D_1 \veebar D_2 \veebar D_3$, $\neg D_1 \equiv D_2 \veebar D_3$; and from $E_1 \veebar E_2 \veebar E_3$, $\neg E_1 \equiv E_2 \veebar E_3$

3) Axioms, definitions, and theorems for single-lot drawing trials Table 10 shows the axioms, definitions, and theorems for the trials in which a single lot is drawn. The theorem proofs are not shown.

$Axiom1(Ax1)$	$P(S) = 1, P(\phi) = 0$
$Axiom2(Ax2)$	$P(S) = P(X) + P(Y)$
$Axiom3(Ax3)$	$0 \leqq P(X) \leqq 1,\ 0 \leqq P(Y) \leqq 1 \quad (X \subseteqq S, Y \subseteqq S)$
$Definition(Def)$	$P(Z) = n(Z) \div n(S) \quad (Z : X, Y)$
$Theorem1(Thm1)$	$P(Y) = 1 - P(X)$
$Theorem2(Thm2)$	$P(X) = 1 - P(Y)$
$Theorem3(Thm3)$	$n(Z) = n(S) \times P(Z) \quad (Z : X, Y)$
$Theorem4(Thm4)$	$n(S) = n(Z) \div P(Z) \quad (Z : X, Y)$
$Theorem5(Thm5)$	$n(S) = n(X) + n(Y)$
$Theorem6(Thm6)$	$n(Y) = n(S) - n(X)$
$Theorem7(Thm7)$	$n(X) = n(S) - n(Y)$

Table 10: Axioms, definitions, and theorems for single-lot drawing trials

(2) Test problems The test problems in the probability comparison tasks are in the two categories of ratio-related conceptual and ratio-related procedural knowledge. Each of the two categories includes the three contextual categories of ratio, comparative-quantity, and base-quantity. The conceptual-knowledge problems are those that contain no numbers and thus require approaches based primarily on concepts. The procedural-knowledge problems are those that contain numbers and thus allow approaches based primarily on procedures. In the following, we provide examples of ratio-context test problems that pertain to ratio-related conceptual and procedural knowledge. Tables 11 and 12 show the supposition and conclusion of each of these test problems. Please refer to Supplements 1 through 4 for test problems in the comparative-quantity and base-quantity contexts pertaining to ratio-related conceptual and procedural knowledge.

Example test problem for ratio-related conceptual knowledge in the ratio context

Sample question

In this lot drawing, some of the lots are winning lots and some of them are losing lots. There are two groups of lots. Lots from one group are called " A lots " and lots from the other group are called " B lots ". Both groups include winning lots and losing lots. The " total number of lots " in one group means all the winning and losing lots in that group. If a winning lot is easy to draw, we call the group an " easy winner ".

The total number of A lots is the same as the total number of B lots.
There are more winning A lots than winning B lots.
There are more losing B lots than losing A lots. (Supposition)

If just one lot is drawn, will it be easier to win with an A lot or a B lot, or will it be the same for an A lot and a B lot? Draw a circle in the box above any of the following answers that you think may be correct. Note that in some questions, a circle can be drawn in all of the boxes.

☐	☐	☐
It is easier to win with an A lot.	No difference between an A lot and a B lot.	It is easier to win with a B lot.

(Conclusion)

	Supposition	Correct conclusion
Question 1	A_1, B_2, C_3	D_2
Question 2	A_1, B_1, C_1	D_1
Question 3	$\neg A_1, B_2, C_3$	D_2
Question 4	$\neg A_1, B_2, C_2$	D_1, D_2, D_3
Question 5	$\neg A_1, B_2, C_1$	D_2
Question 6	$\neg A_1, B_1, C_2$	D_3

Table 11: Test problem suppositions and correct conclusions for ratio-related conceptual knowledge in the ratio context

Example test problem for ratio-related procedural knowledge in the ratio context

Sample question

In this lot drawing, some of the lots are winning lots and some of them are losing lots. There are two groups of lots. Lots from one group are called " A lots " and lots from the other group are called " B lots ". Both groups include winning lots and losing lots. The " total number of lots " in one group means all the winning and losing lots in that group. We call how easy it is to draw a winning lot " chance of winning ". If chance of winning is high, we call the group an " easy winner ".

The total number of A lots is 5, and 3 of them are winning lots.
The total number of B lots is 5, and 1 of them is a winning lot. (Supposition)

If just one lot is drawn, will it be easier to win with an A lot or a B lot, or will it be the same for an A lot and a B lot? Draw a circle in the box above any of the following answers that you think may be correct.

□	□	□	
It is easier to win with an A lot.	No difference between an A lot and a B lot.	It is easier to win with a B lot.	(Conclusion)

	Supposition	Correct conclusion
Question 1	$n(X_A) = 3, n(X_B) = 1, n(S_A) = 5, n(S_B) = 5$	D_2
Question 2	$n(X_A) = 1, n(X_B) = 3, n(S_A) = 2, n(S_B) = 6$	D_1
Question 3	$n(X_A) = 3, n(X_B) = 3, n(S_A) = 4, n(S_B) = 5$	D_2
Question 4	$n(X_A) = 1, n(X_B) = 3, n(S_A) = 4, n(S_B) = 4$	D_3
Question 5	$n(X_A) = 3, n(X_B) = 6, n(S_A) = 4, n(S_B) = 8$	D_1
Question 6	$n(X_A) = 2, n(X_B) = 2, n(S_A) = 4, n(S_B) = 5$	D_2
Question 7	$n(X_A) = 1, n(X_B) = 4, n(S_A) = 2, n(S_B) = 5$	D_3
Question 8	$n(X_A) = 1, n(X_B) = 3, n(S_A) = 4, n(S_B) = 6$	D_3
Question 9	$n(X_A) = 2, n(X_B) = 3, n(S_A) = 4, n(S_B) = 5$	D_3
Question 10	$n(X_A) = 2, n(X_B) = 3, n(S_A) = 8, n(S_B) = 10$	D_3
Question 11	$n(X_A) = 3, n(X_B) = 4, n(S_A) = 4, n(S_B) = 5$	D_3
Question 12	$n(X_A) = 4, n(X_B) = 3, n(S_A) = 10, n(S_B) = 6$	D_3

Table 12: Test problem suppositions and correct conclusions for ratio-related procedural knowledge in the ratio context

(3) Test-problem proofs We proved the validity of the correct conclusions given the problem descriptions and suppositions, by propositional logic in cases resulting in one correct answer and by predicate logic in cases not resulting in one correct answer. The following two examples are typical of the proof process. One is for a problem involving ratio-related conceptual knowledge in the ratio context and the other is for a problem involving ratio-related procedural knowledge in the ratio context.

1) Ratio-related conceptual knowledge in the ratio context
Case resulting in one correct answer: Question 1

Supposition	A_1, B_2, C_3
Correct conclusion	D_2

$$\frac{B_2 : n(X_A) > n(X_B) \quad A_1 : n(S_A) = n(S_B)}{n(X_A) \div n(S_A) > n(X_B) \div n(S_B)} \ (>= 1)$$

$$\frac{Def : P(Z) = n(Z) \div n(S) \qquad n(Z) = n(X_A) \ \ n(S) = n(S_A) \ \ P(Z) = P(X_A)}{\dfrac{P(X_A) = n(X_A) \div n(S_A)}{n(X_A) \div n(S_A) = P(X_A)}(SL)} \ (= Sub)$$

$$\frac{Def : P(Z) = n(Z) \div n(S) \qquad n(Z) = n(X_B) \ \ n(S) = n(S_B) \ \ P(Z) = P(X_B)}{\dfrac{P(X_B) = n(X_B) \div n(S_B)}{n(X_B) \div n(S_B) = P(X_B)}(SL)} \ (= Sub)$$

$$\frac{n(X_A)\div n(S_A)>n(X_B)\div n(S_B) \quad n(X_A)\div n(S_A)=P(X_A) \quad n(X_B)\div n(S_B)=P(X_B)}{D_2 : P(X_A) > P(X_B)}\ (=Sub)$$

This proves that D_2 is the correct conclusion, given supposition A_1 and B_2.

Case not resulting in one correct answer: Question 4

Supposition	$\neg A_1, B_2, C_2$
Correct conclusion	D_1, D_2, D_3

$$\frac{n(X_A)=2 \quad n(X_B)=1 \quad n(S_A)=4 \quad n(S_B)=2 \quad 2>1 \quad 4>2 \quad 4-2>2-1 \quad 2/4=1/2}{\begin{array}{l}\exists x\exists y\exists z\exists w[n(X_A)=x\wedge n(X_B)=y\wedge x>y\\ \wedge n(S_A)=z\wedge n(S_B)=w\wedge z>w\\ \wedge n(Y_A)=z-x\wedge n(Y_B)=w-y\wedge z-x>w-y\\ \wedge P(X_A)=x/z\wedge P(X_B)=y/w\wedge x/z=y/w]\cdots(a)\end{array}}\ (\exists\exists Int)$$

$$\frac{n(X_A)=3 \quad n(X_B)=1 \quad n(S_A)=5 \quad n(S_B)=2 \quad 3>1 \quad 5>2 \quad 5-3>2-1 \quad 3/5>1/2}{\begin{array}{l}\exists x'\exists y'\exists z'\exists w'[n(X_A)=x'\wedge n(X_B)=y'\wedge x'>y'\\ \wedge n(S_A)=z'\wedge n(S_B)=w'\wedge z'>w'\\ \wedge n(Y_A)=z'-x'\wedge n(Y_B)=w'-y'\wedge z'-x'>w'-y'\\ \wedge P(X_A)=x'/z'\wedge P(X_B)=y'/w'\wedge x'/z'>y'/w']\cdots(b)\end{array}}\ (\exists\exists Int)$$

$$\frac{n(X_A)=2 \quad n(X_B)=1 \quad n(S_A)=5 \quad n(S_B)=2 \quad 2>1 \quad 5>2 \quad 5-2>2-1 \quad 2/5<1/2}{\begin{array}{l}\exists x''\exists y''\exists z''\exists w''[n(X_A)=x''\wedge n(X_B)=y''\wedge x''>y''\\ \wedge n(S_A)=z''\wedge n(S_B)=w''\wedge z''>w''\\ \wedge n(Y_A)=z''-x''\wedge n(Y_B)=w''-y''\wedge z''-x''>w''-y''\\ \wedge P(X_A)=x''/z''\wedge P(X_B)=y''/w''\wedge x''/z''<y''/w'']\cdots(c)\end{array}}\ (\exists\exists Int)$$

$$\frac{(a) \quad (b) \quad (c)}{\begin{array}{l}(\exists x\exists y\exists z\exists w[n(X_A)=x\wedge n(X_B)=y\wedge x>y\\ \wedge n(S_A)=z\wedge n(S_B)=w\wedge z>w\\ \wedge n(Y_A)=z-x\wedge n(Y_B)=w-y\wedge z-x>w-y\\ \wedge P(X_A)=x/z\wedge P(X_B)=y/w\wedge x/z=y/w])\\ \wedge(\exists x'\exists y'\exists z'\exists w'[n(X_A)=x'\wedge n(X_B)=y'\wedge x'>y'\\ \wedge n(S_A)=z'\wedge n(S_B)=w'\wedge z'>w'\\ \wedge n(Y_A)=z'-x'\wedge n(Y_B)=w'-y'\wedge z'-x'>w'-y'\\ \wedge P(X_A)=x'/z'\wedge P(X_B)=y'/w'\wedge x'/z'>y'/w'])\\ \wedge(\exists x''\exists y''\exists z''\exists w''[n(X_A)=x''\wedge n(X_B)=y''\wedge x''>y''\\ \wedge n(S_A)=z''\wedge n(S_B)=w''\wedge z''>w''\\ \wedge n(Y_A)=z''-x''\wedge n(Y_B)=w''-y''\wedge z''-x''>w''-y''\\ \wedge P(X_A)=x''/z''\wedge P(X_B)=y''/w''\wedge x''/z''<y''/w''])\end{array}}\ (\wedge\wedge\, Int)$$

This proves that there exist $n(X_A), n(X_B), n(S_A)$, and $n(S_B)$ that satisfy $\neg A_1, B_2, C_2$, and D_1; $\neg A_1, B_2, C_2$, and D_2; and $\neg A_1, B_2, C_2$, and D_3, respectively.

2) Ratio-related procedural knowledge in the ratio context
Question 1

Supposition	$n(X_A) = 3, n(X_B) = 1, n(S_A) = 5, n(S_B) = 5$
Correct conclusion	D_2

$$\frac{Def : P(Z) = n(Z) \div n(S) \quad n(Z) = n(X_A) \;\; n(S) = n(S_A) \;\; P(Z) = P(X_A)}{P(X_A) = n(X_A) \div n(S_A)} \; (= Sub)$$

$$\cfrac{P(X_A) = n(X_A) \div n(S_A) \quad n(X_A) = 3 \;\; n(S_A) = 5}{\cfrac{P(X_A) = 3 \div 5}{P(X_A) = 3/5} \; (OI)} \; (= Sub)$$

$$\frac{Def : P(Z) = n(Z) \div n(S) \quad n(Z) = n(X_B) \;\; n(S) = n(S_B) \;\; P(Z) = P(X_B)}{P(X_B) = n(X_B) \div n(S_B)} \; (= Sub)$$

$$\cfrac{P(X_B) = n(X_B) \div n(S_B) \quad n(X_B) = 1 \;\; n(S_B) = 5}{\cfrac{P(X_B) = 1 \div 5}{P(X_B) = 1/5} \; (OI)} \; (= Sub)$$

$$\cfrac{\cfrac{P(X_A) = 3/5 \quad 3/5 > 1/5}{P(X_A) > 1/5} \; (TL) \qquad \cfrac{P(X_B) = 1/5}{1/5 = P(X_B)} \; (SL)}{D_2 : P(X_A) > P(X_B)} \; (TL)$$

This proves D_2 as the correct conclusion.

We similarly proved all of the test problems by mathematically deriving the correct answers from the suppositions, using propositional or predicate logic. The results showed all of the test problems to be free from contradiction and demonstrated their correct inference processes. By similarly representing the inference processes performed by the children, it was then possible to obtain a clear comparison between the correct reasoning based on probability definitions and the children's reasoning based on theorems of their own making.

(4) Children tested The tests were administered to children in the fifth and sixth grades of elementary schools. The sixth graders had been schooled in unit-element ratios and the fifth graders had not. The number of children in each test category was as follows.

Ratio-related conceptual knowledge in the ratio context
125 5^{th} graders, 129 6^{th} graders, 254 total
Ratio-related conceptual knowledge in the comparative-quantity context
117 5^{th} graders, 114 6^{th} graders, 231 total
Ratio-related conceptual knowledge in the base-quantity context
144 5^{th} graders, 139 6^{th} graders, 283 total
Ratio-related procedural knowledge in the ratio context
214 5^{th} graders, 229 6^{th} graders, 443 tota
Ratio-related procedural knowledge in the comparative-quantity context
188 5^{th} graders, 203 6^{th} grader, 391 total
Ratio-related procedural knowledge in the base-quantity context
207 5^{th} graders, 220 6^{th} graders, 427 total

3 Analysis of test results

(1) Mathematical explication of children's inference processes We listed the test problems in order from high to low correct-answer rate and analyzed the children's protocols. As a result, we found that the children's manner of reasoning was characteristic and that because they consistently used the same manner of reasoning it tended to be applicable only to specific problems. As shown in Table 13, we therefore added symbols relating to determinations based on half (1/2) as the basis/standard and then performed the symbolization of inferences seen in classic child protocols to obtain a mathematical explication of the children's manner of reasoning. We also performed level and stage categorization, with structural and qualitative changes in the children's manner of reasoning taken as changes of level and changes of stages within levels, respectively. For integrated analysis relating to the two types of ratio-related knowledge and the three contexts, we extracted the children-specific manner of reasoning as reasoning that is central to the reasoning of children.

In the following, we show typical examples of our symbolization of inferences made by the children and the related level and stage categories for several test problems on ratio-related conceptual knowledge in the ratio context. In these examples, we refer to correct conclusions derived from the suppositions as "correct answers" and answers derived by the children simply as "conclusions", and highlight the children-specific reasoning in inference schemata.

$W(z)$	$P(z) > 1/2$
$L(z)$	$P(z) < 1/2$
$H(z)$	$P(z) = 1/2$

Table 13: Determinations from base 1/2

1) Level 0

Question 2	
Supposition	A_1, B_1, C_1
Correct answer	D_1
Conclusion	D_3
Correct or Incorrect	Incorrect

2) Level 1, Stage 1A

Question 2	
Supposition	A_1, B_1, C_1
Correct answer	D_1
Conclusion	D_1
Correct or Incorrect	correct
$\dfrac{B_1 : n(X_A) = n(X_B) \quad \boldsymbol{n(X_A) = n(X_B) \rightarrow P(X_A) = P(X_B)}}{D_1 : P(X_A) = P(X_B)} \ (\rightarrow Rem)$	
Question 6	
Supposition	$\neg A_1, B_1, C_2$
Correct answer	D_3

Conclusion	D_1
Correct or Incorrect	Incorrect

$$\frac{B_1 : n(X_A) = n(X_B) \qquad \boldsymbol{n(X_A) = n(X_B) \rightarrow P(X_A) = P(X_B)}}{D_1 : P(X_A) = P(X_B)} (\rightarrow Rem)$$

3) Level 1, Stage 1B

Question 6	
Supposition	$\neg A_1, B_1, C_2$
Correct answer	D_3
Conclusion	D_3
Correct or Incorrect	correct

$$\frac{\dfrac{\dfrac{B_1 \quad C_2}{B_1 \wedge C_2}(\wedge Int)}{C_2 : n(Y_A) > n(Y_B)}(\wedge Rem) \qquad \boldsymbol{n(Y_A) > n(Y_B) \rightarrow P(X_A) < P(X_B)}}{D_3 : P(X_A) < P(X_B)} (\rightarrow Rem)$$

Question 4	
Supposition	$\neg A_1, B_2, C_2$
Correct answer	D_1, D_2, D_3
Conclusion	D_2
Correct or Incorrect	Incorrect

$$\frac{\dfrac{\dfrac{B_2 \quad C_2}{B_2 \wedge C_2}(\wedge Int)}{B_2 : n(X_A) > n(X_B)}(\wedge Rem) \qquad \boldsymbol{n(X_A) > n(X_B) \rightarrow P(X_A) > P(X_B)}}{D_2 : P(X_A) > P(X_B)} (\rightarrow Rem)$$

4) Level 2

Question 4	
Supposition	$\neg A_1, B_2, C_2$
Correct answer	D_1, D_2, D_3
Conclusion	D_1, D_2, D_3
Correct or Incorrect	correct

$n(X_A) = 3, n(X_B) = 2, n(Y_A) = 3, n(Y_B) = 2 \cdots (1)$

In the following inference schemata, [1] should be replaced with (1), excluding the commas.

$$\frac{[1] \quad 3 > 2 \quad 3 > 2 \quad 3/(3+3) = 2/(2+2)}{\begin{array}{l}\exists x \exists y \exists z \exists w [n(X_A) = x \wedge n(X_B) = y \wedge x > y \\ \quad \wedge n(Y_A) = z \wedge n(Y_B) = w \wedge z > w \\ \quad \wedge P(X_A) = x/(x+z) \wedge P(X_B) = y/(y+w) \wedge x/(x+z) = y/(y+w)] \cdots (a)\end{array}} (\exists\exists Int)$$

$n(X_A) = 6, n(X_B) = 2, n(Y_A) = 3, n(Y_B) = 2 \cdots (2)$

In the following inference schemata, [2] should be replaced with (2), excluding the commas.

$$\frac{[2] \quad 6>2 \quad 3>2 \quad 6/(6+3)>2/(2+2)}{\begin{array}{l}\exists x'\exists y'\exists z'\exists w'[n(X_A)=x' \wedge n(X_B)=y' \wedge x'>y' \\ \quad \wedge n(Y_A)=z' \wedge n(Y_B)=w' \wedge z'>w' \\ \quad \wedge P(X_A)=x'/(x'+z') \wedge P(X_B)=y'/(y'+w') \wedge x'/(x'+z')>y'/(y'+w')] \cdots (b)\end{array}} (\exists\exists Int)$$

$n(X_A)=6, n(X_B)=4, n(Y_A)=6, n(Y_B)=2 \cdots (3)$
In the following inference schemata, [3] should be replaced with (3), excluding the commas.

$$\frac{[3] \quad 6>4 \quad 6>2 \quad 6/(6+6)<4/(4+2)}{\begin{array}{l}\exists x''\exists y''\exists z''\exists w''[n(X_A)=x'' \wedge n(X_B)=y'' \wedge x''>y'' \\ \quad \wedge n(Y_A)=z'' \wedge n(Y_B)=w'' \wedge z''>w'' \\ \quad \wedge P(X_A)=x''/(x''+z'') \wedge P(X_B)=y''/(y''+w'') \wedge x''/(x''+z'')<y''/(y''+w'')] \cdots (c)\end{array}} (\exists\exists Int)$$

$$\frac{(a) \quad (b) \quad (c)}{\begin{array}{l}(\exists x\exists y\exists z\exists w[n(X_A)=x \wedge n(X_B)=y \wedge x>y \\ \quad \wedge n(Y_A)=z \wedge n(Y_B)=w \wedge z>w \\ \quad \wedge P(X_A)=x/(x+z) \wedge P(X_B)=y/(y+w) \wedge x/(x+z)=y/(y+w)]) \\ \wedge(\exists x'\exists y'\exists z'\exists w'[n(X_A)=x' \wedge n(X_B)=y' \wedge x'>y' \\ \quad \wedge n(Y_A)=z' \wedge n(Y_B)=w' \wedge z'>w' \\ \quad \wedge P(X_A)=x'/(x'+z') \wedge P(X_B)=y'/(y'+w') \wedge x'/(x'+z')>y'/(y'+w')]) \\ \wedge(\exists x''\exists y''\exists z''\exists w''[n(X_A)=x'' \wedge n(X_B)=y'' \wedge x''>y'' \\ \quad \wedge n(Y_A)=z'' \wedge n(Y_B)=w'' \wedge z''>w'' \\ \quad \wedge P(X_A)=x''/(x''+z'') \wedge P(X_B)=y''/(y''+w'') \wedge x''/(x''+z'')<y''/(y''+w'')])\end{array}} (\wedge\wedge\, Int)$$

The processes of inference in children unschooled in probability are not based on an explicit definition of probability. In their inference processes, leaps therefore tend to occur due to children-specific reasoning. The children's reasoning sequences $\boldsymbol{n(X_A)=n(X_B) \rightarrow P(X_A)=P(X_B)}$ in Level 1 Stage 1A and $\boldsymbol{n(Y_A)>n(Y_B) \rightarrow P(X_A)<P(X_B)}$ in Level 1 Stage 1B generally hold in cases where $n(S_A)=n(S_B)$, but it appears that they were also excessively applied in cases where $n(S_A) \neq n(S_B)$. The children at Level 2 apparently focused on $n(X_A), n(X_B), n(Y_A)$, and $n(Y_B)$, and derived inference schema conclusion (a) based on the following manner of reasoning.

$n(X_A)=3, n(X_B)=2, n(Y_A)=3, n(Y_B)=2$

$$\frac{\dfrac{n(X_A)=3 \quad 3>2}{n(X_A)>2}(TL) \qquad \dfrac{n(X_B)=2}{2=n(X_B)}(SL)}{B_2: n(X_A)>n(X_B)}(TL)$$

$$\frac{\dfrac{n(Y_A)=3 \quad 3>2}{n(Y_A)>2}(TL) \qquad \dfrac{n(Y_B)=2}{2=n(Y_B)}(SL)}{C_2: n(Y_A)>n(Y_B)}(TL)$$

$$\frac{\dfrac{\dfrac{n(X_A)=3 \quad n(Y_A)=3}{n(X_A)\div n(Y_A)=3\div 3}(==)}{n(X_A)\div n(Y_A)=1}(OI) \qquad \dfrac{\dfrac{\dfrac{n(X_B)=2 \quad n(Y_B)=2}{n(X_B)\div n(Y_B)=2\div 2}(==)}{n(X_B)\div n(Y_B)=1}(OI)}{1=n(X_B)\div n(Y_B)}(SL)}{n(X_A)\div n(Y_A)=n(X_B)\div n(Y_B)}(TL)$$

$$\dfrac{n(X_A)\div n(Y_A)=n(X_B)\div n(Y_B) \qquad \boldsymbol{n(X_A)\div n(Y_A)=n(X_B)\div n(Y_B)\to P(X_A)=P(X_B)}}{D_1 : P(X_A)=P(X_B)}\ (\to Rem)$$

The Level-2 children's reasoning, $\boldsymbol{n(X_A)\div n(Y_A)=n(X_B)\div n(Y_B)\to P(X_A)=P(X_B)}$, is not correct in terms of probability. It does have a certain generality, as in this reasoning the ratio $n(X_A)$ to $n(Y_A)$ extended to the ratio $n(X_A)$ to $n(S_A)$ and the ratio $n(X_B)$ to $n(Y_B)$ extended to the ratio $n(X_B)$ to $n(S_B)$. It is accordingly a mathematically correct concept in special cases, but its generality is not guaranteed. In the following, we show in terms of propositional logic the process of obtaining $\boldsymbol{n(X_A)\div n(Y_A)=n(X_B)\div n(Y_B)\to P(X_A)=P(X_B)}$.

$(OI)^{*}1$	$1\div(n(X_A)\div n(Y_A)) = 1\times n(Y_A)\div(n(X_A)\div n(Y_A)\times n(Y_A))$ $= n(Y_A)\div n(X_A)$
$(OI)^{*}2$	$1+n(Y_A)\div n(X_A) = n(X_A)\div n(X_A)+n(Y_A)\div n(X_A)$ $= (n(X_A)+n(Y_A))\div n(X_A)$
$(OI)^{*}3$	$n(X_A)\times(n(X_A)+n(Y_A))\div n(X_A) = (n(X_A)+n(Y_A))\times n(X_A)\div n(X_A)$ $= (n(X_A)+n(Y_A))\times 1$ $= n(X_A)+n(Y_A)$

$$\dfrac{\dfrac{\dfrac{1=1 \qquad \dfrac{\dfrac{n(X_A)=3 \quad n(Y_A)=3}{n(X_A)\div n(Y_A)=3\div 3}\ (==)}{n(X_A)\div n(Y_A)=1}\ (OI)}{1\div(n(X_A)\div n(Y_A))=1\div 1}\ (==)}{1\div(n(X_A)\div n(Y_A))=1}\ (OI)}{n(Y_A)\div n(X_A)=1}\ (OI)^{*}1$$

$$\dfrac{\dfrac{1=1 \quad n(Y_A)\div n(X_A)=1}{1+n(Y_A)\div n(X_A)=1+1}\ (==)}{(n(X_A)+n(Y_A))\div n(X_A)=1+1}\ (OI)^{*}2$$

$$\dfrac{\dfrac{\dfrac{n(X_A)=3 \quad (n(X_A)+n(Y_A))\div n(X_A)=1+1}{n(X_A)\times(n(X_A)+n(Y_A))\div n(X_A)=3\times(1+1)}\ (==)}{n(X_A)\times(n(X_A)+n(Y_A))\div n(X_A)=3+3}\ (OI)}{n(X_A)+n(Y_A)=3+3}\ (OI)^{*}3$$

$$\dfrac{Thm5 : n(S)=n(X)+n(Y) \quad n(X)=n(X_A) \quad n(Y)=n(Y_A) \quad n(S)=n(S_A)}{n(S_A)=n(X_A)+n(Y_A)}\ (=Sub)$$

$$\dfrac{n(X_A)=3 \qquad \dfrac{n(X_A)+n(Y_A)=3+3 \quad \dfrac{n(S_A)=n(X_A)+n(Y_A)}{n(X_A)+n(Y_A)=n(S_A)}\ (SL)}{n(S_A)=3+3}\ (=Sub)}{n(X_A)\div n(S_A)=3\div(3+3)}\ (==)$$

$$\dfrac{Def : P(Z)=n(Z)\div n(S) \quad n(Z)=n(X_A) \quad n(S)=n(S_A) \quad P(Z)=P(X_A)}{P(X_A)=n(X_A)\div n(S_A)}\ (=Sub)$$

$$\dfrac{\dfrac{n(X_A)\div n(S_A)=3\div(3+3) \quad \dfrac{P(X_A)=n(X_A)\div n(S_A)}{n(X_A)\div n(S_A)=P(X_A)}\,(SL)}{P(X_A)=3\div(3+3)}\,(=Sub)}{P(X_A)=1/2}\,(OI)$$

$$\dfrac{\dfrac{\dfrac{1=1 \quad \dfrac{\dfrac{n(X_B)=2 \quad n(Y_B)=2}{n(X_B)\div n(Y_B)=2\div 2}\,(==)}{n(X_B)\div n(Y_B)=1}\,(OI)}{1\div(n(X_B)\div n(Y_B))=1\div 1}\,(==)}{1\div(n(X_B)\div n(Y_B))=1}\,(OI)}{n(Y_B)\div n(X_B)=1}\,(OI)^{*}1$$

$$\dfrac{\dfrac{1=1 \quad n(Y_B)\div n(X_B)=1}{1+n(Y_B)\div n(X_B)=1+1}\,(==)}{(n(X_B)+n(Y_B))\div n(X_B)=1+1}\,(OI)^{*}2$$

$$\dfrac{\dfrac{\dfrac{n(X_B)=2 \quad (n(X_B)+n(Y_B))\div n(X_B)=1+1}{n(X_B)\times(n(X_B)+n(Y_B))\div n(X_B)=2\times(1+1)}\,(==)}{n(X_B)\times(n(X_B)+n(Y_B))\div n(X_B)=2+2}\,(OI)}{n(X_B)+n(Y_B)=2+2}\,(OI)^{*}3$$

$$\dfrac{Thm5: n(S)=n(X)+n(Y) \quad n(X)=n(X_B) \quad n(Y)=n(Y_B) \quad n(S)=n(S_B)}{n(S_B)=n(X_B)+n(Y_B)}\,(=Sub)$$

$$\dfrac{n(X_B)=2 \quad \dfrac{n(X_B)+n(Y_B)=2+2 \quad \dfrac{n(S_B)=n(X_B)+n(Y_B)}{n(X_B)+n(Y_B)=n(S_B)}\,(SL)}{n(S_B)=2+2}\,(=Sub)}{n(X_B)\div n(S_B)=2\div(2+2)}\,(==)$$

$$\dfrac{Def: P(Z)=n(Z)\div n(S) \quad n(Z)=n(X_B) \quad n(S)=n(S_B) \quad P(Z)=P(X_B)}{P(X_B)=n(X_B)\div n(S_B)}\,(=Sub)$$

$$\dfrac{\dfrac{n(X_B)\div n(S_B)=2\div(2+2) \quad \dfrac{P(X_B)=n(X_B)\div n(S_B)}{n(X_B)\div n(S_B)=P(X_B)}\,(SL)}{P(X_B)=2\div(2+2)}\,(=Sub)}{P(X_B)=1/2}\,(OI)$$

$$\dfrac{P(X_A)=1/2 \quad \dfrac{P(X_B)=1/2}{1/2=P(X_B)}\,(SL)}{D1: P(X_A)=P(X_B)}\,(TL)$$

(2) Children-specific reasoning As a result of the symbolization of the inferences performed by the children for all of the test problems and their level and stage classification as shown in Tables 14 and 15, we found children-specific reasoning to be present in all levels and stages. Tables 16 and 17 provide a summary of the children-specific reasoning extracted from the children's manner of reasoning at each level and stage, and the correct answers based on probabilistic definitions.

Level	Stage	Ratio	Comparative quantity	Base quantity
0				
1	1A	Question 2 Question 1 Question 3 Question 5	Question 2 Question 1	Question 2 Question 1
	1B	Question 6	Question 6 Question 4	Question 6 Question 3
2		Question 4	Question 5 Question 3	Question 5 Question 4

Table 14: Levels and stages of ratio-related conceptual knowledge

Level	Stage	Ratio	Comparative quantity	Base quantity
0				
1	1A	Question 1 Question 4	Question 1 Question 4 Question 11 Question 9 Question 10 Question 8 Question 7	Question 2 Question 5 Question 7 Question 11 Question 8 Question 9 Question 10 Question 12
	1B	Question 6 Question 3 Question 12	Question 2 Question 5	Question 3 Question 6
	1C	Question 2 Question 8 Question 9 Question 7		
2		Question 5 Question 10 Question 11	Question 6 Question 12 Question 3	Question 1 Question 4

Table 15: Levels and stages of ratio-related procedural knowledge

Level	Stage	Ratio	Comparative quantity	Base quantity
0				
1	1A	$\cdot n(X_A) = n(X_B) \rightarrow$ $P(X_A) = P(X_B)$ $\cdot n(X_A) > n(X_B) \rightarrow$ $P(X_A) > P(X_B)$	$\cdot n(Y_A) = n(Y_B) \rightarrow$ $n(X_A) = n(X_B)$ $\cdot n(Y_A) < n(Y_B) \rightarrow$ $n(X_A) > n(X_B)$	$\cdot n(X_A) = n(X_B) \rightarrow$ $n(Y_A) = n(Y_B)$ $\cdot n(X_A) > n(X_B) \rightarrow$ $n(Y_A) < n(Y_B)$
	1B	$\cdot n(X_A) = n(X_B) \rightarrow$ $n(Y_A) > n(Y_B) \rightarrow$ $P(X_A) < P(X_B)$	$\cdot n(Y_A) = n(Y_B) \rightarrow$ $P(X_A) > P(X_B) \rightarrow$ $n(X_A) > n(X_B)$	$\cdot n(X_A) = n(X_B) \rightarrow$ $P(X_A) < P(X_B) \rightarrow$ $n(Y_A) > n(Y_B)$

2		$\cdot n(X)=n(Y)\rightarrow H(X)$ $\cdot n(X)>n(Y)\rightarrow W(X)$ $\cdot n(X)<n(Y)\rightarrow L(X)$ $\cdot H(X_A)\wedge H(X_B)\rightarrow$ $P(X_A)=P(X_B)$ $\cdot W(X_A)\wedge L(X_B)\rightarrow$ $P(X_A)>P(X_B)$ $\cdot L(X_A)\wedge W(X_B)\rightarrow$ $P(X_A)<P(X_B)$	$\cdot n(X)=n(Y)\rightarrow H(X)$ $\cdot n(X)>n(Y)\rightarrow W(X)$ $\cdot n(X)<n(Y)\rightarrow L(X)$ $\cdot H(X_A)\wedge H(X_B)\rightarrow$ $P(X_A)=P(X_B)$ $\cdot L(X_A)\wedge W(X_B)\rightarrow$ $P(X_A)<P(X_B)$	$\cdot n(X)=n(Y)\rightarrow H(X)$ $\cdot n(X)>n(Y)\rightarrow W(X)$ $\cdot n(X)<n(Y)\rightarrow L(X)$ $\cdot H(X_A)\wedge H(X_B)\rightarrow$ $P(X_A)=P(X_B)$ $\cdot W(X_A)\wedge L(X_B)\rightarrow$ $P(X_A)>P(X_B)$
		$\cdot n(X_A)\div n(Y_A)=$ $n(X_B)\div n(Y_B)\rightarrow$ $P(X_A)=P(X_B)$ $\cdot n(X_A)\div n(Y_A)>$ $n(X_B)\div n(Y_B)\rightarrow$ $P(X_A)>P(X_B)$ $\cdot n(X_A)\div n(Y_A)<$ $n(X_B)\div n(Y_B)\rightarrow$ $P(X_A)<P(X_B)$ $\cdot n(X_A)\div n(S_A)=$ $n(X_B)\div n(S_B)\rightarrow$ $P(X_A)=P(X_B)$ $\cdot n(X_A)\div n(S_A)>$ $n(X_B)\div n(S_B)\rightarrow$ $P(X_A)>P(X_B)$ $\cdot n(X_A)\div n(S_A)<$ $n(X_B)\div n(S_B)\rightarrow$ $P(X_A)<P(X_B)$	$\cdot n(X_A)\div n(Y_A)=$ $n(X_B)\div n(Y_B)\rightarrow$ $P(X_A)=P(X_B)$ $\cdot n(X_A)\div n(Y_A)<$ $n(X_B)\div n(Y_B)\rightarrow$ $P(X_A)<P(X_B)$ $\cdot n(S_A)\times P(X_A)>$ $n(S_B)\times P(X_B)\rightarrow$ $n(X_A)>n(X_B)$	$\cdot n(X_A)\div n(Y_A)=$ $n(X_B)\div n(Y_B)\rightarrow$ $P(X_A)=P(X_B)$ $\cdot n(X_A)\div n(Y_A)>$ $n(X_B)\div n(Y_B)\rightarrow$ $P(X_A)>P(X_B)$ $\cdot n(S_A)\times P(Y_A)>$ $n(S_B)\times P(Y_B)\rightarrow$ $n(Y_A)>n(Y_B)$

Table 16: Children's reasoning related to conceptual knowledge

Level	Stage	Ratio	Comparative quantity	Base quantity
0		$\cdot n(S_A)=n(S_B)\rightarrow$ $P(X_A)=P(X_B)$	$\cdot n(S_A)=n(S_B)\rightarrow$ $n(X_A)=n(X_B)$	$\cdot P(X_A)=P(X_B)\rightarrow$ $n(S_A)=n(S_B)$
1	1A	$\cdot n(X_A)=(nX_B)\rightarrow$ $P(X_A)=P(X_B)$ $\cdot n(X_A)>n(X_B)\rightarrow$ $P(X_A)>P(X_B)$	$\cdot P(X_A)=P(X_B)\rightarrow$ $n(X_A)=n(X_B)$ $\cdot P(X_A)>P(X_B)\rightarrow$ $n(X_A)>n(X_B)$ $\cdot P(X_A)<P(X_B)\rightarrow$ $n(X_A)<n(X_B)$	$\cdot n(X_A)=n(X_B)\rightarrow$ $n(S_A)=n(S_B)$ $\cdot n(X_A)>n(X_B)\rightarrow$ $n(S_A)>n(S_B)$ $\cdot n(X_A)<n(X_B)\rightarrow$ $n(S_A)<n(S_B)$
	1B	$\cdot n(X_A)=(nX_B)\rightarrow$ $n(S_A)>n(S_B)\rightarrow$ $P(X_A)<P(X_B)$ $\cdot n(X_A)=(nX_B)\rightarrow$ $n(S_A)<n(S_B)\rightarrow$ $P(X_A)>P(X_B)$	$\cdot P(X_A)=P(X_B)\rightarrow$ $n(S_A)<n(S_B)\rightarrow$ $n(X_A)<n(X_B)$	$\cdot n(X_A)=n(X_B)\rightarrow$ $P(X_A)>P(X_B)\rightarrow$ $n(S_A)<n(S_B)$

	1B	$\cdot n(Y_A) > n(Y_B) \rightarrow$ $P(X_A) < P(X_B)$ $\cdot n(Y_A) < n(Y_B) \rightarrow$ $P(X_A) > P(X_B)$		
	1C	$\cdot n(X) = n(Y) \rightarrow H(X)$ $\cdot n(X) > n(Y) \rightarrow W(X)$ $\cdot n(X) < n(Y) \rightarrow L(X)$ $\cdot H(X_A) \wedge H(X_B) \rightarrow$ $P(X_A) = P(X_B)$ $\cdot L(X_A) \wedge H(X_B) \rightarrow$ $P(X_A) < P(X_B)$ $\cdot H(X_A) \wedge W(X_B) \rightarrow$ $P(X_A) < P(X_B)$		
		$\cdot n(X_A) - n(Y_A) <$ $\cdot n(X_B) - n(Y_B) \rightarrow$ $P(X_A) < P(X_B)$		
2		$\cdot n(S_A) \times a = n(S_B) \times b$ $\rightarrow n(X_A) \times a = n(X_B) \times b$ $\rightarrow P(X_A) = P(X_B)$ $\cdot n(S_A) \times a = n(S_B) \times b$ $\rightarrow n(X_A) \times a < n(X_B) \times b$ $\rightarrow P(X_A) < P(X_B)$		
		$\cdot n(X_A) \div n(S_A) =$ $n(X_B) \div n(S_B) \rightarrow$ $P(X_A) = P(X_B)$ $\cdot n(X_A) \div n(S_A) <$ $n(X_B) \div n(S_B) \rightarrow$ $P(X_A) < P(X_B)$	$\cdot n(S_A) \times P(X_A) =$ $n(S_B) \times P(X_B) \rightarrow$ $n(X_A) = n(X_B)$ $\cdot n(S_A) \times P(X_A) >$ $n(S_B) \times P(X_B) \rightarrow$ $n(X_A) > n(X_B)$	$\cdot n(X_A) \div P(X_A) =$ $n(X_B) \div P(X_B) \rightarrow$ $n(S_A) = n(S_B)$

Table 17: Children's reasoning related to procedural knowledge

4 Discussion In reasoning, children consider relations between two sets and relations within a set, which we refer to here as "Between" and "Within" relations, respectively. For Between relations, such as that of $n(X_A)$ and $n(X_B)$, they consider the relation between two quantities with each occurring in a different set. For Within relations, such as that of $n(X_A)$ and $n(Y_A)$, they consider the relation between two quantities occurring in the same set.

In comparing the children's reasoning processes, as shown in Tables 16 and 17, we found that additive reasoning (including size comparison) for Between relations and multiplicative reasoning for Within relations occur in relation to both conceptual knowledge and procedural knowledge in all three of the contexts, and that additive reasoning for the Between relation precedes multiplicative reasoning for the Within relation. We found additive reasoning (including size comparison) for the Within relation to occur consistently in relation to ratio-related conceptual knowledge in all three contexts. In relation to ratio-related procedural knowledge in the ratio context, we found additive reasoning (including size comparison) for the Within relation and multiplicative reasoning for the Between relation, again with additive reasoning for the Within relation preceding multiplicative reasoning for the Between relation. These findings indicate that the Between relation is easier for children to recognize than the Within relation, and that additive reasoning is easier than multiplicative reasoning. They also indicate that the transitions in reasoning proceed from additive reasoning for the Between relation to additive reasoning for the Within relation, to multiplicative reasoning for the Between relation, and finally to multiplicative reasoning for the Within relation. Additive reasoning for the Within relation was found to involve the use of half as a basis strategy. This is in accord with the findings in studies made to present on the stages of children's knowledge and development in proportional reasoning.

In cases where the number of winning lots and total number of lots in two sets were in a double-half (1/2) relation, some of the children considered the related numbers and performed inferences based on multiplicative reasoning for the Between relation. Even in problems containing no explicit numbers, some of the children on their own initiative set up actual numbers that were in the double-half (1/2) relation, e.g., (4, 2), (6, 3), (8, 4), and (10, 5), for the number of winning and losing lots and performed their inferences based on multiplicative reasoning for the Within relation. In these cases, they used half as a ratio rather than as a basis strategy. Their unprompted introduction of the half concept, in any case, clearly suggests that it holds a key role as a prime mover in the transition from additive reasoning in the Within relation to multiplicative reasoning in the Between relation and to multiplicative reasoning in the Within relation.

The occurrence of additive reasoning relating to ratio-related conceptual and procedural knowledge for Between relations and multiplicative reasoning for Within relations in all three contexts indicates that in each of the contexts an association is formed between ratio-related conceptual and procedural knowledge under additive reasoning and the structural change in the manner of thinking then leads to a formation of a new association under multiplicative reasoning. The structural change is a basic change from an additive to a multiplicative algebraic structure that is the foundation of the children's manner of reasoning and corresponds to a structural change in level. The emergence of Additive reasoning for Within relations can also be regarded as a qualitative change from the Between relation to the Within relation in additive reasoning, and the emergence of multiplicative reasoning for Between relations can be regarded as a qualitative change from the Between relation to the Within relation in multiplicative reasoning.

We also found an increase from one to two in the number of events considered in additive reasoning for the Between relation, with the proviso that although two events were considered in all three contexts for ratio-related conceptual and procedural knowledge, in those cases where equality was established for one event there was a tendency to perform the determination based only on the other event.

These qualitative changes signify a change in the children's mode of consideration from one event to two and from the Between relation to the Within relation, and correspond to a change in stage. Until the structural change from additive to multiplicative reasoning occurs, children consistently perform inferences based on additive reasoning. In summary, the findings indicate that the three contexts do not become integrated in terms of additive reasoning until after ratio-related conceptual and procedural knowledge become linked in additive reasoning in each of the three.

Additional note

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Appendix 1

Example test problem for ratio-related conceptual knowledge in the comparative quantity context

Sample question

In this lot drawing, some of the lots are winning lots and some of them are losing lots. There are two groups of lots. Lots from one group are called "A lots" and lots from the other group are called "B lots". Both groups include winning lots and losing lots. The "total number of lots" in one group means all the winning and losing lots in that group. If a winning lot is easy to draw, we call the group an "easy winner".

The total number of A lots is the same as the total number of B lots.
There are more losing B lots than losing A lots.
If just one lot is drawn, it is easier to win with an A lot than with a B lot. (Supposition)

Which of the A lots or the B lots have a larger number of winning lots, or is it the same for the A lots and B lots? Draw a circle in the box above any of the following answers that you think may be correct. Note that in some questions, a circle can be drawn in all of the boxes.

☐	☐	☐	
There are more winning A lots.	No difference between the A lots and B lots.	There are more winning B lots.	(Conclusion)

Test problem suppositions and correct conclusions for ratio-related conceptual knowledge in the comparative quantity context

	Supposition	Correct conclusion
Question 1	A_1, C_3, D_2	B_2
Question 2	A_1, C_1, D_1	B_1
Question 3	$\neg A_1, C_2, D_3$	B_1, B_2, B_3
Question 4	$\neg A_1, C_2, D_2$	B_2
Question 5	$\neg A_1, C_2, D_1$	B_2
Question 6	$\neg A_1, C_1, D_2$	B_2

Appendix 2

Example test problem for ratio-related conceptual knowledge in the base quantity context

Sample question

In this lot drawing, some of the lots are winning lots and some of them are losing lots. There are two groups of lots. Lots from one group are called “ A lots ” and lots from the other group are called “ B lots ”. Both groups include winning lots and losing lots. The “ total number of lots ” in one group means all the winning and losing lots in that group. If a winning lot is easy to draw, we call the group an “ easy winner ”.

The total number of A lots is the same as the total number of B lots.
There are more winning A lots than winning B lots.
If just one lot is drawn, it is easier to win with an A lot than with a B lot. (Supposition)

Which of the A lots or the B lots have a larger number of losing lots, or is it the same for the A lots and B lots? Draw a circle in the box above any of the following answers that you think may be correct. Note that in some questions, a circle can be drawn in all of the boxes.

☐	☐	☐	
There are more losing A lots.	No difference between the A lots and B lots.	There are more losing B lots.	(Conclusion)

Test problem suppositions and correct conclusions for ratio-related conceptual knowledge in the base quantity context

	Supposition	Correct conclusion
Question 1	A_1, B_2, D_2	C_3
Question 2	A_1, B_1, D_1	C_1
Question 3	$\neg A_1, B_2, D_3$	C_2
Question 4	$\neg A_1, B_2, D_2$	C_1, C_2, C_3
Question 5	$\neg A_1, B_2, D_1$	C_2
Question 6	$\neg A_1, B_1, D_3$	C_2

Appendix 3
Example problem for ratio-related procedural knowledge in the comparative quantity context

Sample question

In this lot drawing, some of the lots are winning lots and some of them are losing lots. There are two groups of lots. Lots from one group are called " A lots " and lots from the other group are called " B lots ". Both groups include winning lots and losing lots. The " total number of lots " in one group means all the winning and losing lots in that group. We call how easy it is to draw a winning lot " chance of winning ". If chance of winning is high, we call the group an " easy winner ".

The total number of A lots is 5, and chance of winning is 0.6.
The total number of B lots is 5, and chance of winning is 0.2.

(Supposition)

Which of the A lots or the B lots have a larger number of winning lots, or is it the same for the A lots and B lots? Draw a circle in the box above any of the following answers that you think may be correct.

☐	☐	☐
There are more winning A lots.	No difference between the A lots and B lots.	There are more winning B lots.

(Conclusion)

Test problem suppositions and correct conclusions for ratio-related procedural knowledge in the comparative quantity context

	Supposition	Correct conclusion
Question 1	$n(S_A) = 5, n(S_B) = 5, P(X_A) = 0.6, P(X_B) = 0.2$	B_2
Question 2	$n(S_A) = 2, n(S_B) = 6, P(X_A) = 0.5, P(X_B) = 0.5$	B_3
Question 3	$n(S_A) = 4, n(S_B) = 5, P(X_A) = 0.75, P(X_B) = 0.6$	B_1
Question 4	$n(S_A) = 4, n(S_B) = 4, P(X_A) = 0.25, P(X_B) = 0.75$	B_3
Question 5	$n(S_A) = 4, n(S_B) = 8, P(X_A) = 0.75, P(X_B) = 0.75$	B_3
Question 6	$n(S_A) = 4, n(S_B) = 5, P(X_A) = 0.5, P(X_B) = 0.4$	B_1
Question 7	$n(S_A) = 2, n(S_B) = 5, P(X_A) = 0.5, P(X_B) = 0.8$	B_3
Question 8	$n(S_A) = 4, n(S_B) = 6, P(X_A) = 0.25, P(X_B) = 0.5$	B_3
Question 9	$n(S_A) = 4, n(S_B) = 5, P(X_A) = 0.5, P(X_B) = 0.6$	B_3
Question 10	$n(S_A) = 8, n(S_B) = 10, P(X_A) = 0.25, P(X_B) = 0.3$	B_3
Question 11	$n(S_A) = 4, n(S_B) = 5, P(X_A) = 0.75, P(X_B) = 0.8$	B_3
Question 12	$n(S_A) = 10, n(S_B) = 6, P(X_A) = 0.4, P(X_B) = 0.5$	B_2

Appendix 4
Example problem for ratio-related procedural knowledge in the base quantity context

Sample question

In this lot drawing, some of the lots are winning lots and some of them are losing lots. There are two groups of lots. Lots from one group are called " A lots " and lots from the other group are called " B lots ". Both groups include winning lots and losing lots. The " total number of lots " in one group means all the winning and losing lots in that group. We call how easy it is to draw a winning lot " chance of winning ". If chance of winning is high, we call the group an " easy winner ".

Chance of winning of an A lot is 0.6, and the number of winning lots is 3.
Chance of winning of a B lot is 0.2, and the number of winning lots is 1. (Supposition)

Which of the A lots or the B lots have a larger total number of lots, or is it the same for the A lots and B lots? Draw a circle in the box above any of the following answers that you think may be correct.

☐	☐	☐
The total number of A lots is larger.	No difference between the A lots and B lots.	The total number of B lots is larger.

(Conclusion)

Test problem suppositions and correct conclusions for ratio-related procedural knowledge in the base quantity context

	Supposition	Correct conclusion
Question 1	$P(X_A) = 0.6, P(X_B) = 0.2, n(X_A) = 3, n(X_B) = 1$	A_1
Question 2	$P(X_A) = 0.5, P(X_B) = 0.5, n(X_A) = 1, n(X_B) = 3$	A_3
Question 3	$P(X_A) = 0.75, P(X_B) = 0.6, n(X_A) = 3, n(X_B) = 3$	A_3
Question 4	$P(X_A) = 0.25, P(X_B) = 0.75, n(X_A) = 1, n(X_B) = 3$	A_1
Question 5	$P(X_A) = 0.75, P(X_B) = 0.75, n(X_A) = 3, n(X_B) = 6$	A_3
Question 6	$P(X_A) = 0.5, P(X_B) = 0.4, n(X_A) = 2, n(X_B) = 2$	A_3
Question 7	$P(X_A) = 0.5, P(X_B) = 0.8, n(X_A) = 1, n(X_B) = 4$	A_3
Question 8	$P(X_A) = 0.25, P(X_B) = 0.5, n(X_A) = 1, n(X_B) = 3$	A_3
Question 9	$P(X_A) = 0.5, P(X_B) = 0.6, n(X_A) = 2, n(X_B) = 3$	A_3
Question 10	$P(X_A) = 0.25, P(X_B) = 0.3, n(X_A) = 2, n(X_B) = 3$	A_3
Question 11	$P(X_A) = 0.75, P(X_B) = 0.8, n(X_A) = 3, n(X_B) = 4$	A_3
Question 12	$P(X_A) = 0.4, P(X_B) = 0.5, n(X_A) = 4, n(X_B) = 3$	A_2

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