FUNCTION SPACES WITH VARIABLE EXPONENTS – AN INTRODUCTION –

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Abstract. This paper is oriented to an elementary introduction to function spaces with variable exponents and a survey of related function spaces. After providing basic and elementary properties of generalized Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}^n)$ with variable exponents, we give rearranged proofs of the theorems by Diening (2004), Cruz-Uribe, Fiorenza and Neugebauer (2003, 2004), Nekvinda (2004) and Lerner (2005). They are maybe simpler than the originals. Moreover, we deal with topics related to $L^{p(\cdot)}(\mathbb{R}^n)$. For example, we present an alternative proof for Lerner's theorem on the modular inequality and a detailed proof of the density in Sobolev spaces with variable exponents. Furthermore, we will describe the recent results of fractional integral operators and Calderón-Zygmund operators on $L^{p(\cdot)}(\mathbb{R}^n)$. Finally, we survey recent results (without proofs) on several function spaces with variable exponents, for example, generalized Morrey and Campanato spaces with variable growth condition, Hardy spaces $H^{p(\cdot)}(\mathbb{R}^n)$, Besov spaces $B^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ and Triebel-Lizorkin spaces $F^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$, etc.

Preface Recently, in harmonic analysis, partial differential equations, potential theory and applied mathematics, many authors investigate function spaces with variable exponents. In particular, function spaces with variable exponents are necessary in the field of electronic fluid mechanics [177] and the applications to the image restoration [17, 65, 108]. Kovacik and Rakosnik [101] gave an application of generalized Lebesgue spaces with variable exponents to Dirichlet boundary value problems for nonlinear partial differential equations with coefficients of a variable growth. Another simple example of the application to differential equations can be found in [49, p. 438, Example], where Fan and Zhao implicitly showed that the variable Lebesgue spaces can be used to control the non-linear term of differential equations.

The theory of Lebesgue spaces with variable exponents dates back to Orlicz's paper [163] (1931) and Nakano's books [158, 159] (1950, 1951). In particular, the definition of so-called Musielak-Orlicz spaces is clearly written in [158, Section 89], while it seems that Orlicz is mainly interested in completeness of function spaces. Later, Sharapudinov [208] (1979) and Kováčik and Rákosník [101] (1991) clarified fundamental properties of Lebesgue spaces with variable exponents and Sobolev spaces with variable exponents. This important achievement nowadays leads to the hot discussion of function spaces with variable exponents. A noteworthy fact is that Fan and Zhao independently investigated Lebesgue spaces with variable exponents and Sobolev spaces with variable exponents.

One of the important problems in this field is to prove the boundedness of the Hardy-Littlewood maximal operator M on generalized Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}^n)$ with variable exponents. Once this is established, our experience makes us feel that this boundedness can

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be applied to many parts of analysis. Actually, many authors tackled this hard problem. The paper [36] (2004) by Diening is a pioneering one. Based upon the paper [36], Cruz-Uribe, Fiorenza and Neugebauer [26, 27] (2003, 2004) have given sufficient conditions for M to be bounded on Lebesgue spaces with variable exponents and the condition is referred to as the log-Hölder condition.

Due to the extrapolation theorem with weighted norm inequalities by Cruz-Uribe, Fiorenza, Martell and Pérez [25] (2006) about Lebesgue spaces with variable exponents, we can prove the boundedness of singular integral operators of Calderón-Zygmund type, the boundedness of commutators generated by BMO functions and singular integral operators and the Fourier multiplier results.

Moreover, Hardy spaces $H^{p(\cdot)}(\mathbb{R}^n)$ with variable exponents (Nakai and Sawano [154] (2012), Sawano [203] (2013) and Cruz-Uribe and Wang [30]) and inhomogeneous Besov spaces $B^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ and inhomogeneous Triebel-Lizorkin spaces $F^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ with three variable exponents (Diening, Hästö and Roudenko [42] (2009) and Almeida and Hästö [3] (2010)) are also investigated. Remark that much was done by Xu [219, 220, 221, 222] (2008, 2008, 2009, 2012) when $q(\cdot) = q$ is a constant.

In Part I, we first state basic properties on the classical Lebesgue spaces $L^p(\Omega)$, the Hardy-Littlewood maximal operator M, A_p-weight and $\mathrm{BMO}(\mathbb{R}^n)$. Next we prove elementary properties on generalized Lebesgue spaces $L^{p(\cdot)}(\Omega)$ with variable exponents in Part II. Then we discuss the boundedness of the operator M on $L^{p(\cdot)}(\mathbb{R}^n)$ in Part III. We give rearranged proofs of the theorems by Diening [36], Cruz-Uribe, Fiorenza and Neugebauer [26, 27], Nekvinda [161] (2004) and Lerner [102] (2005). They are maybe simpler than the originals. In Part IV we deal with topics related to $L^{p(\cdot)}(\mathbb{R}^n)$. For example, we present an alternative proof for Lerner's theorem on the modular inequality and a detailed proof of the density in Sobolev spaces with variable exponents. Moreover, we will describe the recent results of fractional integral operators and Calderón-Zygmund operators on $L^{p(\cdot)}(\mathbb{R}^n)$.

Finally, in Part V we give recent results (without proofs) on several function spaces with variable exponents, for example, generalized Morrey and Campanato spaces with variable growth condition, Hardy spaces $H^{p(\cdot)}(\mathbb{R}^n)$, Besov spaces $B^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ and Triebel-Lizorkin spaces $F^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$, etc.

The feature of this paper is as follows:

(i) A presentation of the Lebesgue spaces with variable exponents is performed in comparison with the classical Lebesgue spaces in Parts I and II. This will supplement the introductory part of the book [40] (2011), while we referred to [40] for the structure of the Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}^n)$, but also for other topics.

(ii) We recall a recent technique of the proof of the boundedness of the Hardy-Littlewood maximal operator in Part III and IV. Here, by polishing the earlier results, we obtained some new results.

(iii) In Part V, we define and compare several function spaces with variable exponents, which may be of importance for further research.

This paper is based on the following:

(i) Two manuscripts [77, 78] by Mitsuo Izuki, Eiichi Nakai and Yoshihiro Sawano, for RIMS Kôkyûroku Bessatsu of the workshop "Harmonic Analysis and Nonlinear Partial Differential Equations" held on July 2–4, 2012.

(ii) Eight lectures by Mitsuo Izuki for master course students at Ibaraki University on July 9–11, 2012.

(iii) Three lectures by Eiichi Nakai at Chowa-Kaiseki (Harmonic Analysis) Seminar held at The University of Tokyo on December 25–27, 2012.

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Contents

Part I
Notation and basic properties

1 Notation In the whole paper we will use the following notation:

(i) Let $\mathbb{R}^n$ be the n-dimensional Euclidean space. We denote by $L^0(\mathbb{R}^n;T)$ be the set of all measurable functions from $\mathbb{R}^n$ to T, where $T \subset \mathbb{C}$ or $T \subset [0,\infty]$. If $T = \mathbb{C}$, then we abbreviate $L^0(\mathbb{R}^n;\mathbb{C})$ to $L^0(\mathbb{R}^n)$.

(ii) We denote by $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ the set of all locally integrable functions. We also denote by $L^p_{\mathrm{comp}}(\mathbb{R}^n)$ the set of all $f \in L^p(\mathbb{R}^n)$ with compact support. For $f \in L^p(\mathbb{R}^n)$, we write

$$\|f\|_{L^p} = \|f\|_{L^p(\mathbb{R}^n)} := \left(\int_{\mathbb{R}^n} |f(x)|^p \, dx\right)^{\frac{1}{p}}.$$

(iii) The set $C^\infty_{\mathrm{comp}}(\mathbb{R}^n)$ consists of all compactly supported and infinitely differentiable functions f defined on $\mathbb{R}^n$.

(iv) Given a measurable set $S \subset \mathbb{R}^n$, we denote the Lebesgue measure by $|S|$ and the characteristic function by χ_S.

(v) Given a function f defined on a set E and an interval I, we denote by $\{f \in I\}$ the level set given by

$$f^{-1}(I) = \{x \in E \, : \, f(x) \in I\}.$$

When we want to clarify the set on which f is defined, we write $\{x \in E \, : \, f(x) \in I\}$ instead of $\{f \in I\}$.

(vi) For a measurable set $G \subset \mathbb{R}^n$, $f \in L^0(\mathbb{R}^n)$ and $t \geq 0$, let

$$m(G,f,t) := |\{x \in G : |f(x)| > t\}|. \tag{1.1}$$

If $G = \mathbb{R}^n$, then we denote it by $m(f,t)$ simply.

(vii) Given a measurable set $S \subset \mathbb{R}^n$ with $|S| > 0$ and a function f on $\mathbb{R}^n$, we denote the mean value of f on S by f_S or $⨍_S f$, namely,

$$f_S = ⨍_S f = ⨍_S f(y)\, dy := \frac{1}{|S|}\int_S f(y)\, dy. \tag{1.2}$$

(viii) We define an open ball by

$$B(x,r) := \{y \in \mathbb{R}^n \, : \, |x-y| < r\},$$

where $x \in \mathbb{R}^n$ and $r > 0$.

(ix) An open cube $Q \subset \mathbb{R}^n$ is always assumed to have sides parallel to the coordinate axes. Namely, for any cube Q, we can write

$$Q = Q(x,r) := \prod_{\nu=1}^{n} (x_\nu - r/2 \, , \, x_\nu + r/2)$$

using $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $r > 0$. Let $\mathcal{Q}$ be the set of all open cubes $Q \subset \mathbb{R}^n$ with sides parallel to the coordinate axes.

(x) Given a positive number s, a cube $Q = Q(x, r)$ and an open ball $B = B(x, r)$, we define $sQ := Q(x, sr)$ and $sB := B(x, sr)$.

(xi) The set $\mathbb{N}_0$ consists of all non-negative integers.

(xii) Given a multi-index $\alpha = (\alpha_1, \dots, \alpha_n) \in {\mathbb{N}_0}^n$, we write

$$|\alpha| := \sum_{\nu=1}^{n} \alpha_\nu.$$

In addition the derivative of f is denoted by

$$D^\alpha f := \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}.$$

(xiii) We adopt the following definition of the Fourier transform and its inverse:

$$\mathcal{F}f(\xi) := \int_{\mathbb{R}^n} f(x) e^{-2\pi i x \cdot \xi}\, dx,\ \mathcal{F}^{-1}f(x) := \int_{\mathbb{R}^n} f(\xi) e^{2\pi i x \cdot \xi}\, d\xi$$

for $f \in L^1(\mathbb{R}^n)$.

(xiv) Using this definition of Fourier transform and its inverse, we also define

$$\varphi(D)f(x) := \mathcal{F}^{-1}[\varphi \cdot \mathcal{F}f](x) = \langle f, \mathcal{F}^{-1}\varphi(x - \cdot)\rangle \quad (x \in \mathbb{R}^n) \tag{1.3}$$

for $f \in \mathcal{S}'(\mathbb{R}^n)$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$.

(xv) For (quasi-)norm spaces E and F, let $B(E, F)$ be the set of all bounded operators from E to F. We denote $B(E, E)$ by $B(E)$.

(xvi) The set $\Omega \subset \mathbb{R}^n$ is measurable and satisfies $|\Omega| > 0$.

(xvii) By a weight on Ω we mean any non-negative locally integrable function defined on Ω. We exclude the possibility that a weight is zero on a set of positive measure. If $\Omega = \mathbb{R}^n$, we mean it by a weight simply.

(xviii) A symbol C always stands for a positive constant independent of the main parameters. Inequality $A \lesssim B$ means $A \le CB$ and inequality $A \gtrsim B$ means $A \ge CB$.

2 Some basic inequalities We use the following generalized inequality of arithmetic and geometric means:

Lemma 2.1. *If a, $b > 0$ and $0 < \alpha < 1$, then we have the inequality*

$$a^\alpha b^{1-\alpha} \le \alpha a + (1 - \alpha) b$$

with equality if and only if $a = b$.

The next inequality is a consequence of the convexity or Lemma 2.1.

Lemma 2.2. *If x, $y \in \mathbb{R}^n$ and $0 \le t \le 1 \le r < \infty$, then the following inequality holds:*

$$|tx + (1 - t)y|^r \le t|x|^r + (1 - t)|y|^r.$$

3 Lebesgue spaces In this section, we review classical Lebesgue spaces. We provide a detailed proof so that the proof motivates the argument about Lebesgue spaces with variable exponents.

3.1 Definition and norm We suppose here that the set $\Omega \subset \mathbb{R}^n$ is measurable and satisfies $|\Omega| > 0$. We recall the definition and the fundamental property of $L^p(\Omega)$.

Definition 3.1. Let $1 \le p \le \infty$. The Lebesgue space $L^p(\Omega)$ is the set of all complex-valued measurable functions f defined on Ω satisfying $\|f\|_{L^p(\Omega)} < \infty$, where

$$\|f\|_{L^p(\Omega)} := \begin{cases} \left(\int_\Omega |f(x)|^p \, dx\right)^{1/p} & (1 \le p < \infty), \\ \operatorname{ess\,sup}_{x\in\Omega} |f(x)| & (p = \infty). \end{cases}$$

Then $L^p(\Omega)$ is a complex vector space, since, by Lemma 2.2 we have

$$\int_\Omega |f(x) + g(x)|^p \, dx \le \int_\Omega 2^{p-1}(|f(x)|^p + |g(x)|^p) \, dx < \infty$$

for $f, g \in L^p(\Omega)$ if $1 \le p < \infty$. The case $p = \infty$ is easy.

Theorem 3.1 (Hölder's inequality)**.** *Let $1 \le p \le \infty$. Then, we have*

$$\int_\Omega |f(x)g(x)| \, dx \le \|f\|_{L^p(\Omega)} \|g\|_{L^{p'}(\Omega)}$$

for all $f \in L^p(\Omega)$ and all $g \in L^{p'}(\Omega)$.

Proof. The case $p = 1$ or $p = \infty$ is easy. We consider the case $1 < p < \infty$. We may assume that $\|f\|_{L^p(\Omega)} > 0$ and $\|g\|_{L^{p'}(\Omega)} > 0$. If we put

$$F := \left(\frac{|f|}{\|f\|_{L^p(\Omega)}}\right)^p, \quad G := \left(\frac{|g|}{\|g\|_{L^{p'}(\Omega)}}\right)^{p'},$$

then Lemma 2.1 to follow gives us

$$\frac{\int_\Omega |f(x)g(x)| \, dx}{\|f\|_{L^p(\Omega)} \|g\|_{L^{p'}(\Omega)}} = \int_\Omega F(x)^{1/p} G(x)^{1/p'} \, dx \le \int_\Omega \left(\frac{F(x)}{p} + \frac{G(x)}{p'}\right) dx = 1.$$

Thus, the proof is complete. □

Applying Hölder's inequality, we obtain the following:

Theorem 3.2 (Minkowski's inequality)**.** *Let $1 \le p \le \infty$. Then, we have*

$$\|f + g\|_{L^p(\Omega)} \le \|f\|_{L^p(\Omega)} + \|g\|_{L^p(\Omega)}$$

for all f, $g \in L^p(\Omega)$.

Proof. The case $p = 1$ or $p = \infty$ is easy. We consider the case $1 < p < \infty$. Let $f, g \in L^p(\Omega)$. We may assume that $\|f + g\|_{L^p(\Omega)} > 0$. Let $h = (|f + g|/\|f + g\|_{L^p(\Omega)})^{p-1}$. Then $\|h\|_{L^{p'}(\Omega)} = 1$, since $(p-1)p' = p$. Hence, by the triangle inequality and Hölder's inequality we have

$$\begin{aligned} \|f + g\|_{L^p(\Omega)} &= \int_\Omega \frac{|f(x) + g(x)|^p}{\|f + g\|_{L^p(\Omega)}^{p-1}} \, dx = \int_\Omega |f(x) + g(x)| \cdot |h(x)| \, dx \\ &\le \|f\|_{L^p(\Omega)} \|h\|_{L^{p'}(\Omega)} + \|g\|_{L^p(\Omega)} \|h\|_{L^{p'}(\Omega)} = \|f\|_{L^p(\Omega)} + \|g\|_{L^p(\Omega)}. \end{aligned}$$

□

Theorem 3.3. *If* $1 \le p \le \infty$*, then* $\|\cdot\|_{L^p(\Omega)}$ *is a norm.*

Proof. We have only to check the following conditions are true:

(i) $\|f\|_{L^p(\Omega)} \ge 0$,

(ii) $\|f\|_{L^p(\Omega)} = 0$ if and only if $f = 0$ a.e. Ω,

(iii) $\|\alpha f\|_{L^p(\Omega)} = |\alpha| \cdot \|f\|_{L^p(\Omega)}$,

(iv) $\|f+g\|_{L^p(\Omega)} \le \|f\|_{L^p(\Omega)} + \|g\|_{L^p(\Omega)}$,

for all f, $g \in L^p(\Omega)$ and all $\alpha \in \mathbb{C}$. We omit the detail. □

3.2 Weak type Lebesgue spaces For a measurable set $\Omega \subset \mathbb{R}^n$, $f \in L^0(\mathbb{R}^n)$ and $t \ge 0$, the distribution function of f over Ω is defined by;

$$m(\Omega, f, t) \equiv |\{x \in \Omega : |f(x)| > t\}|.$$

If $\Omega = \mathbb{R}^n$, then we denote it by $m(f, t)$ simply, see (1.1).

Definition 3.2. For $0 < p < \infty$, let $L^p_{\mathrm{weak}}(\Omega)$ be the set of all measurable functions f on Ω such that

$$\|f\|_{L^p_{\mathrm{weak}}(\Omega)} := \sup_{t>0} t m(\Omega, f, t)^{1/p} < \infty.$$

By the Chebychev inequality we have $L^p(\Omega) \subset L^p_{\mathrm{weak}}(\Omega)$ and

$$\|f\|_{L^p_{\mathrm{weak}}(\Omega)} \le \|f\|_{L^p(\Omega)}.$$

3.3 Weighted Lebesgue spaces Recall that "by a weight on Ω" we mean any non-negative locally integrable function defined on Ω. If $\Omega = \mathbb{R}^n$, we mean it by a weight simply.

Definition 3.3. For $1 \le p < \infty$ and a weight w on Ω, let $L^p_w(\Omega)$ be the set of all functions f in $L^0(\Omega)$ such that

$$\|f\|_{L^p_w(\Omega)} := \left(\int_\Omega |f(x)|^p w(x)\, dx \right)^{1/p} < \infty.$$

Of course if $w(x) \equiv 1$, then $L^p_w(\Omega)$ means the usual Lebesgue space $L^p(\Omega)$.

4 Maximal operator In this section we supply the proof of the boundedness of the Hardy-Littlewood maximal operator M.

Recall that, for a function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, the uncentered Hardy-Littlewood maximal operator $Mf(x)$ is defined by

$$Mf(x) := \sup_{B \ni x} \fint_B |f(y)|\, dy, \text{ where } \fint_B |f(y)|\, dy := \frac{1}{|B|} \int_B |f(y)|\, dy,$$

where the supremum is taken over all balls B containing x. See (1.2). Meanwhile, for a function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, the centered Hardy-Littlewood maximal operator $M_{\mathrm{centered}} f(x)$ is defined by

$$M_{\mathrm{centered}} f(x) := \sup_{r>0} \fint_{B(x,r)} |f(y)|\, dy. \tag{4.1}$$

Due to the estimate $M_{\mathrm{centered}} f(x) \le Mf(x) \le 2^n M_{\mathrm{centered}} f(x)$, most of the results for Mf carry over to those for $M_{\mathrm{centered}} f$. We do not allude to this point, unless there is not difference between Mf and $M_{\mathrm{centered}} f$.

4.1 Measurability of the Hardy-Littlewood maximal operator First, we check that Mf and $M_{\text{centered}}f$ are both measurable functions. Our proof is simpler than that in the textbook [113].

Proposition 4.1. *Let $\lambda > 0$ and $f \in L^0(\mathbb{R}^n)$. Then the sets $E_\lambda := \{x \in \mathbb{R}^n : Mf(x) > \lambda\}$ and $E'_\lambda := \{x \in \mathbb{R}^n : M_{\text{centered}}f(x) > \lambda\}$ are open.*

Proof. To prove this, we choose $x \in E_\lambda$ arbitrarily. Then by the definition of $Mf(x)$, we can find a ball B such that

$$x \in B, \ \frac{1}{|B|}\int_B |f(y)|\,dy > \lambda. \tag{4.2}$$

Then, by the definition of Mf, $B \subset E_\lambda$, and hence x is an interior point of E_λ. The point x being arbitrary, we see that E_λ is open.

We modify the above proof to obtain the proof for E'_λ. In view of the definition (4.1), the ball B in (4.2) must be centered at x, so that B assumes the form of $B = B(x, r)$ for some $r > 0$. By choosing κ slightly larger than 1, we have

$$\frac{1}{|B(x,\kappa r)|}\int_{B(x,r)} |f(y)|\,dy > \lambda.$$

Let $y \in B(x, (\kappa-1)r)$. Then a geometric observation shows that $B(y, \kappa r) \supset B(x, r)$. Thus, it follows that

$$\frac{1}{|B(y,\kappa r)|}\int_{B(y,\kappa r)} |f(y)|\,dy \geq \frac{1}{|B(x,\kappa r)|}\int_{B(x,r)} |f(y)|\,dy > \lambda.$$

Hence, $B(x, (\kappa-1)r) \subset E'_\lambda$. Since x is again arbitrary, it follows that E'_λ is an open set as well. □

4.2 Boundedness of the Hardy-Littlewood maximal operator on $L^p(\mathbb{R}^n)$ In this paper, we are mainly concerned with the extension of the following fundamental results on the $L^p(\mathbb{R}^n)$-boundedness of the Hardy-Littlewood maximal operator:

Theorem 4.2.

(1) *The Hardy-Littlewood maximal operator M is of weak type $(1,1)$, namely,*

$$|\{x \in \mathbb{R}^n : Mf(x) > \lambda\}| \leq C\,\lambda^{-1}\|f\|_{L^1(\mathbb{R}^n)}$$

holds for all $\lambda > 0$ and all $f \in L^1(\mathbb{R}^n)$.

(2) *If $1 < p \leq \infty$, then M is bounded on $L^p(\mathbb{R}^n)$, namely,*

$$\|Mf\|_{L^p(\mathbb{R}^n)} \leq C\,\|f\|_{L^p(\mathbb{R}^n)}$$

holds for all $f \in L^p(\mathbb{R}^n)$.

Before we proceed further, a couple of remarks may be in order.

Remark 4.1.

(1) If $p = \infty$, then Theorem 4.2 (2) with $C = 1$ is immediately proved by the definition of the norm $\|\cdot\|_{L^\infty(\mathbb{R}^n)}$.

(2) If $1 < p < \infty$, then M is of weak type (p, p), namely,

$$\|\chi_{\{x\in\mathbb{R}^n\,:\,Mf(x)>\lambda\}}\|_{L^p} = |\{x \in \mathbb{R}^n \,:\, Mf(x) > \lambda\}|^{1/p} \le C\,\lambda^{-1}\|f\|_{L^p(\mathbb{R}^n)} \tag{4.3}$$

holds for all $\lambda > 0$ and all $f \in L^p(\mathbb{R}^n)$. (4.3) is easily checked by the Chebychev inequality and Theorem 4.2 (2).

(3) One of the important reasons why we are led to the weak $(1, 1)$ inequality is that M always maps $L^1(\mathbb{R}^n)$ functions to non-integrable functions except the zero function. To explain why, let us place ourselves in the case of $n = 1$. Then, a simple computation shows that $M(\chi_{[-1,1]}) \notin L^1(\mathbb{R})$ but that $\chi_{[-1,1]} \in L^1(\mathbb{R})$. By a similar reason, even in $\mathbb{R}^n$, $Mf \notin L^1(\mathbb{R}^n)$ unless $f = 0$.

(4) The remark (3) above applies to the centered Hardy-Littlewood maximal operator.

Classically the boundedness of the Hardy-Littlewood maximal operator is shown as follows: In order to prove Theorem 4.2 we will use the following two lemmas:

Lemma 4.3 (Vitali's covering lemma). *Given a bounded set $E \subset \mathbb{R}^n$, we take a covering $\{B(x_j, r_j)\}_j$ of E. If $\{r_j\}_j$ is bounded, then there exists a disjoint subfamily $\{B(x_{j'}, r_{j'})\}$ such that $E \subset \bigcup_{j'} B(x_{j'}, 5r_{j'})$.*

In connection with covering lemmas, we introduce some Japanese books, for example, Igari [69], Mizuta [120] and Sawano [202] for further information on the covering lemma. In [202] a covering lemma is presented as Theorem 2.2.8 but the condition $\sup_{\lambda\in\Lambda} r_\lambda < \infty$ was indispensable.

The next lemma enables us to express the L^p-norm of a measurable function f in terms of distribution functions. However, in the variable setting, this expression is not effective.

Lemma 4.4. *If $1 \le p < \infty$ and $f \in L^p(\mathbb{R}^n)$, then we have*

$$\int_{\mathbb{R}^n} |f(x)|^p\,dx = p\int_0^\infty t^{p-1}\,|\{x \in \mathbb{R}^n \,:\, |f(x)| > t\}|\,dt.$$

Proof. If we define the set $A := \{(x, t) \in \mathbb{R}^n \times [0, \infty) : |f(x)| > t\}$, then we get by Fubini's theorem,

$$\begin{aligned}
\int_{\mathbb{R}^n} |f(x)|^p\,dx &= \int_{\mathbb{R}^n} p\left(\int_0^{|f(x)|} t^{p-1}\,dt\right)dx \\
&= \int_{\mathbb{R}^n} p\left(\int_0^\infty t^{p-1}\chi_A(x,t)\,dt\right)dx \\
&= p\int_0^\infty t^{p-1}\left(\int_{\mathbb{R}^n} \chi_A(x,t)\,dx\right)dt \\
&= p\int_0^\infty t^{p-1}\,|\{x \in \mathbb{R}^n : |f(x)| > t\}|\,dt.
\end{aligned}$$

This is the desired result. □

Proof of Theorem 4.2. We first prove (1). For every $\lambda > 0$ and $N \in \mathbb{N}$, we write

$$E_\lambda := \{x \in \mathbb{R}^n \,:\, Mf(x) > \lambda\} \quad \text{and} \quad E_{\lambda,N} := E_\lambda \cap B(0, N).$$

By the definition of $Mf(x)$, for each $x \in E_\lambda$ there exists a ball B_x such that $x \in B_x$ and that

$$\frac{1}{|B_x|}\int_{B_x} |f(y)|\,dy > \lambda.$$

We remark that $\{B_x\}_{x\in E_\lambda}$ is a covering of a bounded set $E_{\lambda,N}$ and that the radius of B_x is bounded, since $|B_x| \le \|f\|_{L^1(\mathbb{R}^n)}/\lambda$. By virtue of Vitali's covering lemma, there exists a disjoint subfamily

$$\{B_j := B_{x_j}\}_j \subset \{B_x\}_{x\in E_\lambda}$$

such that

$$E_{\lambda,N} \subset \bigcup_j 5B_j \quad \text{and} \quad \frac{1}{|B_j|}\int_{B_j} |f(y)|\,dy > \lambda.$$

Since $\{B_j\}_j$ is disjoint, we obtain

$$|E_{\lambda,N}| \le \left|\bigcup_j 5B_j\right| \le 5^n \sum_j |B_j| \le 5^n \sum_j \left(\lambda^{-1}\int_{B_j} |f(y)|\,dy\right) \le 5^n\lambda^{-1}\|f\|_{L^1(\mathbb{R}^n)}.$$

Moreover by $E_{\lambda,N} \subset E_{\lambda,N+1} \subset \cdots$ and $\bigcup_{N=1}^\infty E_{\lambda,N} = E_\lambda$, we have

$$|\{x \in \mathbb{R}^n \,:\, Mf(x) > \lambda\}| = |E_\lambda| = \lim_{N\to\infty} |E_{\lambda,N}| \le 5^n\lambda^{-1}\|f\|_{L^1(\mathbb{R}^n)}.$$

Next we prove (2). Let $1 < p < \infty$. Take $a > 0$ arbitrarily and define

$$f^a(x) := \begin{cases} f(x) & (|f(x)| > a/2), \\ 0 & (|f(x)| \le a/2), \end{cases} \qquad f_a(x) := f(x) - f^a(x) \quad (x \in \mathbb{R}^n).$$

Since

$$Mf(x) \le M(f^a)(x) + M(f_a)(x) \le M(f^a)(x) + \frac{a}{2} \quad (x \in \mathbb{R}^n),$$

we have

$$\{x \in \mathbb{R}^n \,:\, Mf(x) > a\} \subset \{x \in \mathbb{R}^n \,:\, M(f^a)(x) > a/2\}.$$

The weak $(1,1)$ inequality gives us

$$|\{x \in \mathbb{R}^n : Mf(x) > a\}| \le |\{x \in \mathbb{R}^n : M(f^a)(x) > a/2\}| \le C\cdot\frac{2}{a}\cdot\|f^a\|_{L^1(\mathbb{R}^n)}.$$

By virtue of Lemma 4.4 we get

$$\begin{aligned} \int_{\mathbb{R}^n} (Mf(x))^p\,dx &= p\int_0^\infty a^{p-1}\,|\{x \in \mathbb{R}^n \,:\, Mf(x) > a\}|\,da \\ &\le C_p \int_0^\infty a^{p-2}\|f^a\|_{L^1(\mathbb{R}^n)}\,da \\ &= C_p \int_{\mathbb{R}^n}\left(\int_0^\infty a^{p-2}|f^a(y)|\,da\right)dy. \end{aligned}$$

An arithmetic shows that

$$\int_0^\infty a^{p-2}|f^a(y)|\,da = \int_0^{2|f(y)|} a^{p-2}|f(y)|\,da = \frac{1}{p-1}(2|f(y)|)^{p-1}|f(y)| = \frac{2^{p-1}}{p-1}|f(y)|^p.$$

Consequently we have

$$\int_{\mathbb{R}^n} (Mf(x))^p\,dx \le \frac{C2^{p-1}p}{p-1}\int_{\mathbb{R}^n} |f(y)|^p\,dy.$$

Thus, the proof is therefore complete. □

4.3 Inequality for the convolution Recall that M_{centered} is the centered Hardy -Littlewood maximal operator generated by balls. Here and below we write $M_{\text{centered,balls}}$ for definiteness. The following result is known about the mollifier:

Lemma 4.5 ([48, Proposition 2.7], [213, p. 63])**.** *Let $\psi \in L^1(\mathbb{R}^n)$ be a radial decreasing function. Define*

$$\psi_t := t^{-n}\psi(t^{-1}\cdot) \quad (t > 0). \tag{4.4}$$

Then we have that for all $t > 0$ and all $f \in L^1_{\text{loc}}(\mathbb{R}^n)$,

$$|\psi_t * f(x)| \le \|\psi\|_{L^1} M_{\text{centered,balls}} f(x). \tag{4.5}$$

The collection $\{\psi_t\}_{t>0}$ is often called a mollifier.

Proof. We have only to prove the case $t = 1$, since $\|\psi_t\|_{L^1} = \|\psi\|_{L^1}$ for all $t > 0$. Take simple functions $\{\phi_n\}_{n=1}^\infty$ of the form

$$\phi_n = \sum_{j=1}^n c_{n,j}\chi_{B(0,r_{n,j})}, \quad c_{n,j} \ge 0,\ r_{n,1} > r_{n,2} > \cdots > r_{n,n} > 0,$$

which satisfy $\phi_n \le \psi$ and $\phi_n \to \psi$ a.e. as $n \to \infty$. Then

$$\begin{aligned}
|\phi_n * f(x)| &\le \int_{\mathbb{R}^n} \sum_{j=1}^n c_{n,j}\chi_{B(0,r_{n,j})}(x-y)|f(y)|\,dy \\
&= \sum_{j=1}^n c_{n,j} \int_{B(x,r_{n,j})} |f(y)|\,dy \\
&\le \sum_{j=1}^n c_{n,j}|B(0,r_{n,j})| M_{\text{centered,balls}} f(x) \\
&\le \|\psi\|_{L^1} M_{\text{centered,balls}} f(x).
\end{aligned}$$

As $n \to \infty$, we have the conclusion. □

Let $M_{\text{centered,cubes}}$ be the centered Hardy-Littlewood maximal operator generated by cubes. Since the volume of unit ball is $\pi^{n/2}/\Gamma(1+n/2)$, we have

$$M_{\text{centered,balls}} f(x) \le \Gamma(1+n/2)2^n\pi^{-n/2} M_{\text{centered,cubes}} f(x).$$

Thus, if we use Lemma 4.5, then we obtain

$$|\psi_t * f(x)| \le \|\psi\|_{L^1} M_{\text{centered,balls}} f(x) \le \Gamma(1+n/2)2^n\pi^{-n/2}\|\psi\|_{L^1} M_{\text{centered,cubes}} f(x).$$

4.4 Rearrangement The nonincreasing rearrangement of $f \in L^0(\mathbb{R}^n)$ is defined by

$$f^*(t) := \inf\{\lambda > 0 : m(f,\lambda) \le t\} \quad (0 < t < \infty).$$

The average function f^{**} of f is defined by

$$f^{**}(t) := \frac{1}{t}\int_0^t f^*(s)\,ds \quad (0 < t < \infty).$$

It is known that (see, for example, [10, p. 122]) for any $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$,

$$(Mf)^*(t) \le \nu_n f^{**}(t) \quad (t > 0), \tag{4.6}$$

where ν_n is a positive constant depending only on n. We will also need the fact that

$$tf^{**}(t) = \int_0^t f^*(s)\,ds = \sup_{|E|=t} \int_E |f(x)|\,dx, \tag{4.7}$$

where the supremum is taken over all measurable set E with $|E| = t$.

5 A_p-weights and BMO($\mathbb{R}^n$) The theory of A_p-weights dates back to the work by Muckenhoupt and Wheeden in 1972 [137], while the space BMO($\mathbb{R}^n$) including the John-Nirenberg inequality is investigated in 1961 [82]. Both theories became more and more important not only in harmonic analysis but also in PDEs. The two things seemingly are independent topics, but, some relations between A_p-weights and BMO($\mathbb{R}^n$) are known, see Theorem 5.3 for example. Later, we shall see that these relations are used to prove some non-trivial property in the theory of Lebesgue spaces with variable exponent, see Subsection 16.3, the proof of Lerner's theorem (Theorem 15.4).

We recall that $\mathcal{Q}$ is the set of all open cubes $Q \subset \mathbb{R}^n$ with sides parallel to the coordinate axes.

5.1 A_p-weights In this paper, the weight will play a key role for the boundedness of the Hardy-Littlewood maximal operator on generalized Lebesgue spaces with variable exponents.

Definition 5.1. A weight w is said to satisfy the Muckenhoupt A_p condition, $1 \le p < \infty$, if

$$[w]_{A_p} := \sup_{Q \in \mathcal{Q}} \left(⨍_Q w(x)\,dx \right) \left(⨍_Q w(x)^{-1/(p-1)}\,dx \right)^{p-1} < \infty, \quad 1 < p < \infty,$$

and

$$[w]_{A_1} := \sup_{Q \in \mathcal{Q}} \left(⨍_Q w(x)\,dx \right) \left(\operatorname*{ess\,sup}_{x \in Q} \frac{1}{w(x)} \right) < \infty.$$

Let A_p be the set of all weights satisfying the Muckenhoupt A_p condition. The quantity $[w]_{A_p}, 1 \le p < \infty$ is referred to as the A_p-constant or the A_p-norm of w.

Here we content ourselves with recalling the most elementary fact in the class A_p.

Theorem 5.1 (Muckenhoupt [137] (1972))**.** *Let $1 < p < \infty$. Then $M \in B(L^p_w(\mathbb{R}^n))$ if and only if $w \in A_p$. Moreover, the operator norm of M is bounded by a constant depending only on n, p and $[w]_{A_p}$.*

See Section 20 for further properties.

5.2 BMO($\mathbb{R}^n$) Having set down the definition of weights and the fundamental properties, we now recall the definition of BMO($\mathbb{R}^n$). Recall that f_Q is the average of a locally integrable function f over a cube Q, see (1.2).

Definition 5.2. Let BMO($\mathbb{R}^n$) be the set of all measurable functions f on $\mathbb{R}^n$ such that

$$\|f\|_{\mathrm{BMO}} := \sup_{Q \in \mathcal{Q}} ⨍_Q |f(x) - f_Q|\,dx < \infty.$$

We collect important properties used in this paper.

Theorem 5.2 (John and Nirenberg [82] (1961))**.** *There exist positive constants B and b, depending only on n, such that, for $f \in \mathrm{BMO}(\mathbb{R}^n)$, $Q \in \mathcal{Q}$ and $\sigma > 0$,*

$$|\{x \in Q : |f(x) - f_Q| > \sigma\}| \le B|Q|e^{-b\sigma/\|f\|_{\mathrm{BMO}}}.$$

One can take $B = e^{1/(e-1)} < 2$ and $b = 1/(2^n e)$.

The following is a consequence of Theorem 5.2, see [54, pp. 407–409]:

Theorem 5.3. *Let $1 < p < \infty$. If $\omega \in A_p$, then $\log \omega \in \mathrm{BMO}(\mathbb{R}^n)$, and conversely, if $\varphi \in \mathrm{BMO}(\mathbb{R}^n)$, then $e^{\varepsilon\varphi} \in A_p$ for some $\varepsilon > 0$.*

Theorem 5.4 (Coifman and Rochberg [19] (1980))**.** *There exists a constant γ_n (which depends only on n) such that if α and β are positive constants, g and h are nonnegative locally integrable functions with $Mg < \infty$ and $Mh < \infty$ a.e., and b is any bounded measurable function then the function*

$$f(x) = \alpha \log Mg(x) - \beta \log Mh(x) + b(x) \tag{5.1}$$

is in $\mathrm{BMO}(\mathbb{R}^n)$ and

$$\|f\|_{\mathrm{BMO}} \le \gamma_n(\alpha + \beta + \|b\|_{L^\infty}).$$

Conversely, if f is any function in $\mathrm{BMO}(\mathbb{R}^n)$ then f can be written in the form (5.1) with

$$\alpha + \beta + \|b\|_{L^\infty} \le \gamma_n \|f\|_{\mathrm{BMO}}.$$

Corollary 5.5. *Let $Mf < \infty$ a.e. Then $\log(Mf) \in \mathrm{BMO}(\mathbb{R}^n)$ and*

$$\|\log(Mf)\|_{\mathrm{BMO}} \le \gamma_n.$$

5.3 Sharp maximal operator To prove the boundedness of Calderón-Zygmund operators in particular, the Riesz transform on generalized Lebesgue spaces with variable exponents, we rely upon the control by the sharp maximal operator.

Definition 5.3. The Fefferman-Stein sharp maximal operator is defined as

$$f^\sharp(x) := M^\sharp f(x) = \sup_{x \in Q \in \mathcal{Q}} \fint_Q |f(y) - f_Q|\, dy \quad (x \in \mathbb{R}^n)$$

for $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, where f_Q denotes the average of f over Q, see (1.2). More generally, let $0 < \delta < \infty$. Define

$$M^\sharp_\delta f(x) := (M^\sharp(|f|^\delta)(x))^{1/\delta} \quad (x \in \mathbb{R}^n)$$

for $f \in L^\delta_{\mathrm{loc}}(\mathbb{R}^n)$.

By the definition we have $\|f\|_{\mathrm{BMO}} = \|f^\sharp\|_{L^\infty}$. On the other hand, if $1 < p < \infty$, then $\|f\|_{L^p} \sim \|f^\sharp\|_{L^p}$ for all $f \in L^p(\mathbb{R}^n)$. Actually, from the boundedness of M it follows that

$$\|f^\sharp\|_{L^p} \lesssim \|f\|_{L^p}.$$

Moreover, we have the following:

Theorem 5.6 (Fefferman and Stein [51] (1972))**.** *Let $0 < p_0 < \infty$. For any $p_0 \le p < \infty$ and for all $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ with $Mf \in L^{p_0}(\mathbb{R}^n)$,*

$$\|f\|_{L^p} \lesssim \|f^\sharp\|_{L^p}.$$

5.4 Pointwise multipliers on BMO Let $E \subset L^0(\mathbb{R}^n)$ be a normed function space. We say that a function $g \in L^0(\mathbb{R}^n)$ is a pointwise multiplier on E, if the pointwise multiplication fg is in E for any $f \in E$. We denote by $\mathrm{PWM}(E)$ the set of all pointwise multipliers on E. If E is a Banach space and has the following property (5.2), then $\mathrm{PWM}(E) \subset B(E)$:

(5.2) $f_n \to f$ in $E \quad \Longrightarrow \quad$ there exists a subsequence $\{n(j)\}_{j=1}^{\infty}$ such that $f_{n(j)} \to f$ a.e.

Actually, from (5.2) we see that each pointwise multiplier is a closed operator. Hence it is a bounded operator by the closed graph theorem. Note that

$$\|f\|_{\mathrm{BMO}} + |f_{Q(0,1)}|$$

is a norm on the function space $\mathrm{BMO}(\mathbb{R}^n)$ and thereby $\mathrm{BMO}(\mathbb{R}^n)$ is a Banach space with the property (5.2). For $g \in \mathrm{PWM}(\mathrm{BMO}(\mathbb{R}^n))$, let us define its operator norm $\|g\|_{Op}$ by

$$\|g\|_{Op} = \sup_{f \not\equiv 0} \frac{\|fg\|_{\mathrm{BMO}} + |(fg)_{Q(0,1)}|}{\|f\|_{\mathrm{BMO}} + |f_{Q(0,1)}|}.$$

For a function $\phi : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$, let

$$\|f\|_{\mathrm{BMO}_\phi} = \sup_{Q(x,r) \in \mathcal{Q}} \frac{1}{\phi(x,r)} ⨍_{Q(x,r)} |f(y) - f_{Q(x,r)}| \, dy.$$

The following result is a basic result that will be used in this paper:

Theorem 5.7 (Nakai and Yabuta [157]). *Let*

$$\phi(x,r) = \frac{1}{\log(r + 1/r + |x|)}, \quad x \in \mathbb{R}^n, \ r > 0.$$

Then $g \in \mathrm{PWM}(\mathrm{BMO}(\mathbb{R}^n))$ *if and only if* $\|g\|_{L^\infty} + \|g\|_{\mathrm{BMO}_\phi} < \infty$. *Moreover,*

$$\|g\|_{Op} \sim \|g\|_{L^\infty} + \|g\|_{\mathrm{BMO}_\phi}.$$

For example,

$$g_1(x) := \sin\left(\chi_{B(0,1/e)}(x) \log\log(|x|^{-1})\right), \tag{5.3}$$

and

$$g_2(x) := \sin\left(\chi_{B(0,e)}(x) \log\log|x|\right) \tag{5.4}$$

are pointwise multipliers on $\mathrm{BMO}(\mathbb{R}^n)$. For the example (5.3), see Janson [81] (1976) and Stegenga [212] (1976). For the example (5.4), see Nakai and Yabuta [157] (1985).

6 Banach function spaces Lebesgue spaces with variable exponents were hard to handle. Despite a concrete expression as we shall give in Definition 8.1 below, the effective techniques had been scarce until the advent of the paper by Diening [36]. Looking back on the proof and the history, we are led to a generalized setting. Banach function spaces generalize many other function spaces including Lebesgue spaces with variable exponents.

In this section we outline the definition of Banach function spaces and the Fatou lemma. For further information we refer to Bennett and Sharpley [10].

Let $L^0(\Omega)$ be the set of all complex-valued measurable functions on Ω as before.

Definition 6.1. A linear space $X \subset L^0(\Omega)$ is said to be a Banach function space if X is equipped with a functional $\|\cdot\|_X : L^0(\Omega) \to [0,\infty]$ enjoying the following properties: Let f, g, $f_j \in L^0(\Omega)$ $(j = 1, 2, \ldots)$ and $\lambda \in \mathbb{C}$.

(1) $f \in X$ holds if and only if $\|f\|_X < \infty$.

(2) (Norm property):

(A1) (Positivity): $\|f\|_X \geq 0$.

(A2) (Strict positivity) $\|f\|_X = 0$ if and only if $f = 0$ a.e..

(B) (Homogeneity): $\|\lambda f\|_X = |\lambda| \cdot \|f\|_X$.

(C) (Triangle inequality): $\|f + g\|_X \leq \|f\|_X + \|g\|_X$.

(3) (Symmetry): $\|f\|_X = \|\,|f|\,\|_X$.

(4) (Lattice property): If $0 \leq g \leq f$ a.e., then $\|g\|_X \leq \|f\|_X$.

(5) (Fatou property): If $0 \leq f_1 \leq f_2 \leq \cdots$ and $\lim\limits_{j\to\infty} f_j = f$, then $\lim\limits_{j\to\infty} \|f_j\|_X = \|f\|_X$.

(6) For all measurable sets F with $|F| < \infty$, we have $\|\chi_F\|_X < \infty$.

(7) For all measurable sets F with $|F| < \infty$, there exists a constant $C_F > 0$ such that $\displaystyle\int_F |f(x)|\,dx \leq C_F\,\|f\|_X$.

Example 6.1. Both the usual Lebesgue spaces $L^p(\Omega)$ with constant exponent $1 \leq p \leq \infty$ and the Lebesgue spaces $L^{p(\cdot)}(\Omega)$ with variable exponents $p(\cdot) : \Omega \to [1,\infty]$ are Banach function spaces (see Theorems 3.3 and 8.3).

In this generalized setting, we can formulate an inequality of Fatou type as follows:

Lemma 6.1 (The Fatou lemma)**.** *Let X be a Banach function space and $f_j \in X$ $(j = 1, 2, \ldots)$. If f_j converges to a function f a.e. on Ω and $\liminf_{j\to\infty} \|f_j\|_X < \infty$, then we have $f \in X$ and $\|f\|_X \leq \liminf_{j\to\infty} \|f_j\|_X$.*

Proof. If we put $h_l(x) := \inf_{m\geq l} |f_m(x)|$ $(l = 1, 2, \ldots)$, then we have

$$0 \leq h_1 \leq h_2 \leq \cdots \leq h_l \leq h_{l+1} \leq \cdots \to |f| \quad \text{a.e. } \Omega.$$

Thus by virtue of the Fatou property, we obtain $\|f\|_X = \lim_{l\to\infty} \|h_l\|_X$. Note that $h_l \leq |f_m|$ a.e. if $m \geq l$. Hence by the lattice property we get $\|h_l\|_X \leq \|f_m\|_X$, that is $\|h_l\|_X \leq \inf_{m\geq l} \|f_m\|_X$. Therefore, we have

$$\|f\|_X \leq \lim_{l\to\infty} \left(\inf_{m\geq l} \|f_m\|_X \right) = \liminf_{l\to\infty} \|f_l\|_X < \infty.$$

Thus, the proof is complete. □

Remark 6.1. In the proof of Lemma 8.2 we have used the Fatou lemma with $X = L^1(\{p(x) < \infty\})$, $L^\infty(\Omega_\infty)$.

7 Density Let E be a subspace of $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ equipped with a norm or quasi-norm $\|\cdot\|_E$. Let E_k be the space of all functions $f \in E$ such that $\dfrac{\partial^\alpha f}{\partial x^\alpha}$ exist in the weak sense and $\dfrac{\partial^\alpha f}{\partial x^\alpha} \in E$ whenever $|\alpha| \le k$. Then the space E_k is a normed space or a quasi-normed space with

$$\|f\|_{E_k} := \sum_{|\alpha|\le k} \left\| \frac{\partial^\alpha f}{\partial x^\alpha} \right\|_E, \quad \left(\frac{\partial^0 f}{\partial x^0} = f \right).$$

We invoke the following criteria for a density result.

Theorem 7.1 (Nakai, Tomita and Yabuta [156] (2004))**.** *Let k be a non-negative integer and let E enjoy the following properties:*

(i) *The characteristic functions of all balls in $\mathbb{R}^n$ are in E.*

(ii) *If $g \in E$ and $|f(x)| \le |g(x)|$ a.e., then $f \in E$.*

(iii) *If $g \in E$, $|f_j(x)| \le |g(x)|$ a.e. $(j = 1, 2, \ldots)$ and $f_j(x) \to 0$ $(j \to \infty)$ a.e., then $f_j \to 0$ $(j \to \infty)$ in E.*

If the Hardy-Littlewood maximal operator M is bounded on E, then $C^\infty_{\mathrm{comp}}(\mathbb{R}^n)$ is dense in E_k.

Let $E = L^p(\mathbb{R}^n)$ or $E = L^p_w(\mathbb{R}^n)$ with $w \in A_p$. Then E satisfies the assumption in Theorem 7.1, if $1 < p < \infty$. Moreover, for $p = 1$, the same conclusion still holds; see [156, Theorem 1.1].

Part II

Lebesgue spaces with variable exponents

Let $\Omega \subset \mathbb{R}^n$. We recall that $L^p(\Omega)$ is the set of all measurable functions for which the norm

$$\|f\|_{L^p} = \left(\int_\Omega |f(x)|^p \, dx \right)^{\frac{1}{p}}$$

is finite. Here and below we consider Lebesgue spaces with variable exponent, which is the heart of this paper. We are placing ourselves in the setting where the value of p above varies according to the position of $x \in \Omega$. The simplest case is as follows: Suppose we are given a measurable partition $\Omega = \Omega_1 \cup \Omega_2$ of Ω. Consider the norm $\|f\|_Z$ given by

$$\|f\|_Z = \left(\int_{\Omega_1} |f(x)|^{p_1} \, dx \right)^{\frac{1}{p_1}} + \left(\int_{\Omega_2} |f(x)|^{p_2} \, dx \right)^{\frac{1}{p_2}}.$$

So, if we set

$$p(\cdot) := p_1 \chi_{\Omega_1} + p_2 \chi_{\Omega_2},$$

then we are led to the space $L^{p(\cdot)}(\Omega)$. What happens if the measurable function $p(\cdot)$ assumes infinitely many different values ? The answer can be given by way of modulars.

Lebesgue spaces with variable exponents have been studied intensively for these two decades right after some basic properties were established by Kováčik and Rákosník [101]. We refer to surveys [63, 66, 121, 189] and a book [40] for recent developments. In this part we state and recall some known basic properties and their proofs.

The set $\Omega \subset \mathbb{R}^n$ is measurable and satisfies $|\Omega| > 0$ throughout this part. In this part, by "a variable exponent", we mean a measurable function $p(\cdot) : \Omega \to [1, \infty]$. The symbol "$(\cdot)$" emphasizes that the function p does not always mean a constant exponent $p \in [1, \infty]$.

8 Elementary properties Given a variable exponent $p(\cdot)$, we define the following:

(a) $p_- := \operatorname{ess\,inf}_{x\in\Omega} p(x) = \sup\{a : p(x) \geq a \text{ a.e. } x \in \Omega\}$.

(b) $p_+ := \operatorname{ess\,sup}_{x\in\Omega} p(x) = \inf\{a : p(x) \leq a \text{ a.e. } x \in \Omega\}$.

(c) $\Omega_0 := \{x \in \Omega : 1 < p(x) < \infty\} = p^{-1}((1, \infty))$.

(d) $\Omega_1 := \{x \in \Omega : p(x) = 1\} = p^{-1}(1)$.

(e) $\Omega_\infty := \{x \in \Omega : p(x) = \infty\} = p^{-1}(\infty)$.

(f) the conjugate exponent $p'(\cdot)$:

$$p'(x) := \begin{cases} \infty & (x \in \Omega_1), \\ \frac{p(x)}{p(x)-1} & (x \in \Omega_0), \\ 1 & (x \in \Omega_\infty), \end{cases}$$

namely, $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$ always holds for a.e. $x \in \Omega$. In particular, if $p(\cdot)$ equals to a constant p, then of course $p'(\cdot) = p'$ is the usual conjugate exponent. By no means the function $p'(\cdot)$ stands for the derivative of $p(\cdot)$.

We define variable Lebesgue spaces in a modern fashion. We compare the definition we shall give here with the one by Nakano [159] later.

Definition 8.1. Let $L^0(\Omega)$ be the set of all complex-valued measurable functions defined on $\Omega \subset \mathbb{R}^n$. Given a measurable function $p(\cdot) : \Omega \to [1, \infty]$, define the Lebesgue space $L^{p(\cdot)}(\Omega)$ with variable exponents by;

$$L^{p(\cdot)}(\Omega) := \{f \in L^0(\Omega) : \rho_p(f/\lambda) < \infty \text{ for some } \lambda > 0\},$$

where

$$\rho_p(f) := \int_{\Omega\setminus\Omega_\infty} |f(x)|^{p(x)} dx + \|f\|_{L^\infty(\Omega_\infty)}.$$

Moreover, define

$$\|f\|_{L^{p(\cdot)}(\Omega)} := \inf\{\lambda > 0 : \rho_p(f/\lambda) \leq 1\}.$$

We sometimes use an equivalent norm to $\|f\|_{L^{p(\cdot)}(\Omega)}$, see Remark 8.2.

Remark 8.1. An arithmetic shows

$$L^{p(\cdot)}(\Omega) = L^{p_0}(\Omega) \quad \text{and} \quad \|f\|_{L^{p(\cdot)}(\Omega)} = \|f\|_{L^{p_0}(\Omega)},$$

if $p(\cdot)$ equals to a constant $p_0 \in [1, \infty]$.

We will prove that $\rho_p(\cdot)$ is a modular and that $\|\cdot\|_{L^{p(\cdot)}(\Omega)}$ is a norm in the above. The modular was first defined by Nakano [158] on vector lattices. For other definitions, see Musielak and Orlicz [138] and Maligranda [116]. We adopt a terminology in [40].

Definition 8.2. A functional $\rho : L^0(\Omega) \to [0, \infty]$ is said to be a semimodular if the following conditions are fulfilled:

(a) $\rho(0) = 0$.

(b) For all $f \in L^0(\Omega)$ and $\lambda \in \mathbb{C}$ with $|\lambda| = 1$, we have $\rho(\lambda f) = \rho(f)$.

(c) ρ is convex, namely, we have that, for all $f, g \in L^0(\mathbb{R}^n)$ and all $t \in [0, 1]$,

$$\rho(tf + (1-t)g) \leq t\rho(f) + (1-t)\rho(g).$$

(d) For every $f \in L^0(\mathbb{R}^n)$ such that $\rho(\lambda f) < \infty$ for any $\lambda \in [0, 1)$, we have that

$$\lim_{\lambda \to 1-0} \rho(\lambda f) = \rho(f).$$

(e) $\rho(\lambda f) = 0$ for all $\lambda > 0$ implies $f = 0$.

Note that, if $0 < s < t < \infty$, then $\rho(sf) \leq \rho(tf)$ by the properties (a) and (c). We call the property (d) the left-continuity, while $\rho_p(f)$ is allowed to assume infinity. A semimodular is called a modular if

(f) $\rho(f) = 0$ implies $f = 0$.

A semimodular ρ is called continuous if

(g) for every $f \in L^0(\Omega)$ such that $\rho(f) < \infty$, the mapping $\lambda \in [0, \infty) \mapsto \rho(\lambda f) \in [0, \infty]$ is continuous on $[0, \infty)$.

Since $[0, \infty)$ satisfies the first axiom of countability, we can rephrase (g) as follows; if $\{\lambda_j\}_{j=1}^\infty$ is a convergent positive sequence and $f \in L^0(\Omega)$, then

$$\lim_{j \to \infty} \rho(\lambda_j f) = \rho\left(\lim_{j \to \infty} \lambda_j f\right).$$

As is pointed out in [23, Theorem 1.4], the notions of modular and semimodular are different.

About the above notions and the Lebesgue spaces with variable exponents, we have the following:

Theorem 8.1. *Let $p(\cdot) : \Omega \to [1, \infty]$ be a variable exponent. Then $\rho_p(\cdot)$ is a modular. If $p(\cdot)$ additionally satisfies $\tilde{p}_+ := \operatorname{ess\,sup}_{x \in \Omega \setminus \Omega_\infty} p(x) < \infty$, then $\rho_p(\cdot)$ is a continuous modular.*

Proof. We can easily check conditions (a), (b), (e) and (f). We shall prove that (c), (d) and (g) are also true.

Using Lemma 2.2, we obtain that, for all f, $g \in L^{p(\cdot)}(\Omega)$ and all $t \in [0, 1]$,

$$\begin{aligned}
&\rho_p(tf + (1-t)g) \\
&\quad = \int_{\{p(x)<\infty\}} |tf(x) + (1-t)g(x)|^{p(x)}\, dx + \|tf + (1-t)g\|_{L^\infty(\Omega_\infty)} \\
&\quad \leq \int_{\{p(x)<\infty\}} \left(t|f(x)|^{p(x)} + (1-t)|g(x)|^{p(x)}\right) dx + t\|f\|_{L^\infty(\Omega_\infty)} + (1-t)\|g\|_{L^\infty(\Omega_\infty)} \\
&\quad = t\,\rho_p(f) + (1-t)\,\rho_p(g),
\end{aligned}$$

namely, (c) is true.

Next we check the left-continuity (d). We consider any positive increasing sequence $\{\lambda_j\}_{j=1}^\infty$ which converges to 1. Then we see that

$$|\lambda_1 f(x)|^{p(x)} \le |\lambda_2 f(x)|^{p(x)} \le \cdots, \quad |\lambda_j f(x)|^{p(x)} \uparrow |f(x)|^{p(x)} \text{ a.e. } x \in \Omega \setminus \Omega_\infty.$$

Hence by the monotone convergence theorem we obtain

$$\lim_{j\to\infty} \rho_p(\lambda_j f) = \lim_{j\to\infty} \left\{ \int_{\{p(x)<\infty\}} |\lambda_j f(x)|^{p(x)}\, dx + \|\lambda_j f\|_{L^\infty(\Omega_\infty)} \right\} = \rho_p(f).$$

This implies (d).

If we additionally suppose that $\tilde{p}_+ = \operatorname{ess\,sup}_{x\in\Omega\setminus\Omega_\infty} p(x) < \infty$ and take a positive sequence $\{\lambda_j\}_{j=1}^\infty$ which converges to λ, then letting

$$c_0 := \sup_{j\in\mathbb{N}} |\lambda_j|^{p_-} + \sup_{j\in\mathbb{N}} |\lambda_j|^{\tilde{p}_+},$$

we have the estimate

$$|\lambda_j f(x)|^{p(x)} \le \left(\sup_{k\in\mathbb{N}} |\lambda_k|^{p(x)} \right) |f(x)|^{p(x)} \le c_0 |f(x)|^{p(x)}$$

for a.e. $x \in \Omega\setminus\Omega_\infty$. Hence, by the argument similar to above using the Lebesgue dominated convergence theorem, we get (g). □

Lemma 8.2. *Assume that $f \in L^0(\Omega)$ satisfies $0 < \|f\|_{L^{p(\cdot)}(\Omega)} < \infty$.*

(1) $\rho_p\left(\frac{f}{\|f\|_{L^{p(\cdot)}(\Omega)}}\right) \le 1$.

(2) *If $\tilde{p}_+ := \operatorname{ess\,sup}_{x\in\Omega\setminus\Omega_\infty} p(x) < \infty$, then $\rho_p\left(\frac{f}{\|f\|_{L^{p(\cdot)}(\Omega)}}\right) = 1$ holds.*

Proof. Suppose $0 < \|f\|_{L^{p(\cdot)}(\Omega)} < \infty$. We first prove (1). Take a decreasing sequence $\{\gamma_j\}_{j=1}^\infty$ which converges to $\|f\|_{L^{p(\cdot)}(\Omega)}$. Using the left-continuity of the modular ρ_p or Lemma 6.1 and the definition of the modular ρ_p, we obtain

$$\rho_p\left(\frac{f}{\|f\|_{L^{p(\cdot)}(\Omega)}}\right) = \lim_{j\to\infty} \rho_p\left(\frac{f}{\gamma_j}\right) \le 1.$$

Thus, (1) is proved.

Next we suppose $\operatorname{ess\,sup}_{x\in\Omega\setminus\Omega_\infty} p(x) < \infty$ and let us prove (2). Define the function

$$\zeta(t) := \rho_p\left(\frac{f}{t}\right) \quad (0 < t < \infty).$$

Observe that ζ is strictly decreasing by the definition ρ_p. The result (1) and the Lebesgue dominated convergence theorem give us $\lim_{t\to\infty}\zeta(t) = 0$ and $\lim_{t\to 0}\zeta(t) = \infty$. Furthermore, by virtue of Theorem 8.1, we conclude ζ is continuous. Hence there exists a unique constant $0 < \Lambda < \infty$ such that

$$\zeta(\Lambda) = \rho_p\left(\frac{f}{\Lambda}\right) = 1. \tag{8.1}$$

By the definition of the norm

$$\|f\|_{L^{p(\cdot)}(\Omega)} = \Lambda. \tag{8.2}$$

If we combine (8.1) and (8.2), then we obtain (2). □

Theorem 8.3. *Let $p(\cdot) : \Omega \to [1, \infty]$ be a variable exponent. Then $\| \cdot \|_{L^{p(\cdot)}(\Omega)}$ is a norm.*

This norm is referred to as the Luxemberg-Nakano norm after L. Maligranda called so.

Note that an argument used in Theorem 3.1 no longer works because of "inf" in the definition of $L^{p(\cdot)}(\mathbb{R}^n)$.

Proof. The three conditions

(i) $\|f\|_{L^{p(\cdot)}(\Omega)} \geq 0$,

(ii) $\|f\|_{L^{p(\cdot)}(\Omega)} = 0$ holds if and only if $f = 0$ a.e. holds,

(iii) $\|\alpha f\|_{L^{p(\cdot)}(\Omega)} = |\alpha| \cdot \|f\|_{L^{p(\cdot)}(\Omega)}$

for all $f \in L^{p(\cdot)}(\Omega)$ and all $\alpha \in \mathbb{C}$ are clearly true. We have only to check the triangle inequality:

(iv) $\|f + g\|_{L^{p(\cdot)}(\Omega)} \leq \|f\|_{L^{p(\cdot)}(\Omega)} + \|g\|_{L^{p(\cdot)}(\Omega)}$ for all $f,\, g \in L^{p(\cdot)}(\Omega)$.

We may assume $\|f\|_{L^{p(\cdot)}(\Omega)} > 0$ and $\|g\|_{L^{p(\cdot)}(\Omega)} > 0$ without loss of generality. Otherwise we have $f \equiv 0$ or $g \equiv 0$ a.e.. We denote the normalized functions by

$$F := \frac{f}{\|f\|_{L^{p(\cdot)}(\Omega)}} \quad \text{and} \quad G := \frac{g}{\|g\|_{L^{p(\cdot)}(\Omega)}}.$$

By virtue of the convexity of the modular and Lemma 8.2(1), we have that for all $0 \leq t \leq 1$,

$$\rho_p(tF + (1-t)G) \leq t\rho_p(F) + (1-t)\rho_p(G) \leq 1.$$

Taking $t := \frac{\|f\|_{L^{p(\cdot)}(\Omega)}}{\|f\|_{L^{p(\cdot)}(\Omega)} + \|g\|_{L^{p(\cdot)}(\Omega)}}$, we obtain

$$\rho_p\left(\frac{f+g}{\|f\|_{L^{p(\cdot)}(\Omega)} + \|g\|_{L^{p(\cdot)}(\Omega)}}\right) = \rho_p(tF + (1-t)G) \leq 1.$$

This implies that the triangle inequality (iv) is true. □

Lemma 8.4. *Let $p(\cdot) : \Omega \to [1, \infty]$ be a variable exponent and $f \in L^0(\Omega)$.*

(1) *If $\|f\|_{L^{p(\cdot)}(\Omega)} \leq 1$, then we have $\rho_p(f) \leq \|f\|_{L^{p(\cdot)}(\Omega)} \leq 1$.*

(2) *Conversely if $\rho_p(f) \leq 1$, then $\|f\|_{L^{p(\cdot)}(\Omega)} \leq 1$ holds.*

(3) *Assume in addition that $1 \leq \tilde{p}_+ := \operatorname{ess.sup}_{x \in \Omega \setminus \Omega_\infty} p(x) < \infty$ and that $\rho_p(f) \leq 1$ holds. Then $\|f\|_{L^{p(\cdot)}} \leq \rho_p(f)^{1/\tilde{p}_+} \leq 1$.*

Proof. The definition of $\|f\|_{L^{p(\cdot)}(\Omega)}$ directly shows (2). We shall prove (3) in Theorem 10.1. It remains to prove (1) for $\|f\|_{L^{p(\cdot)}(\Omega)} > 0$. Lemma 8.2 (1) implies that $\rho_p\left(\frac{f}{\|f\|_{L^{p(\cdot)}(\Omega)}}\right) \leq 1$. Since $p(x) \geq 1$ and $\|f\|_{L^{p(\cdot)}(\Omega)} \leq 1$ we get

$$\begin{aligned} 1 \;&\geq\; \left(\int_{\{p(x)<\infty\}} \left|\frac{f(x)}{\|f\|_{L^{p(\cdot)}(\Omega)}}\right|^{p(x)} dx\right) + \left\|\frac{f}{\|f\|_{L^{p(\cdot)}(\Omega)}}\right\|_{L^\infty(\Omega_\infty)} \\ &\geq\; \frac{1}{\|f\|_{L^{p(\cdot)}(\Omega)}}\left(\int_{\{p(x)<\infty\}} |f(x)|^{p(x)}\, dx + \|f\|_{L^\infty(\Omega_\infty)}\right), \end{aligned}$$

namely, we have $\|f\|_{L^{p(\cdot)}(\Omega)} \geq \rho_p(f)$.

□

Remark 8.2. Let

$$\rho_p^{(0)}(f) = \int_\Omega |f(x)|^{p(x)}\,dx \quad \text{and} \quad \|f\|^{(0)}_{L^{p(\cdot)}(\Omega)} = \inf\left\{\lambda > 0 : \rho_p^{(0)}(f/\lambda) \le 1\right\},$$

where it is understood that

$$r^\infty = \begin{cases} 0, & 0 \le r \le 1, \\ \infty, & r > 1. \end{cases}$$

Then $\rho_p^{(0)}$ is a semimodular and $\|f\|^{(0)}_{L^{p(\cdot)}(\Omega)}$ is a norm. If $p_+ < \infty$, then $\rho_p^{(0)}$ clearly coincides with ρ_p and it is continuous. Theorems 8.1 and 8.3 are valid for $\rho_p^{(0)}$ and $\|f\|^{(0)}_{L^{p(\cdot)}(\Omega)}$ as well as Lemmas 8.2 and 8.4, if we replace $\tilde{p}_+$ with p_+. Moreover, $\|f\|^{(0)}_{L^{p(\cdot)}(\Omega)}$ is an equivalent norm to $\|f\|_{L^{p(\cdot)}(\Omega)}$ on the space $L^{p(\cdot)}(\Omega)$. Namely, we have the equivalence

$$\|f\|^{(0)}_{L^{p(\cdot)}(\Omega)} \le \|f\|_{L^{p(\cdot)}(\Omega)} \le 2\|f\|^{(0)}_{L^{p(\cdot)}(\Omega)}. \tag{8.3}$$

Actually, if $\lambda > \|f\|_{L^{p(\cdot)}(\Omega)} = \|f\|_{L^\infty(\Omega_\infty)} + \|f\|_{L^{p(\cdot)}(\Omega\setminus\Omega_\infty)}$, then $\|f\|_{L^\infty(\Omega_\infty)} < \lambda$. Hence, in this case $(|f(x)|/\lambda)^\infty = 0$ for a.e. $x \in \Omega_\infty$. Hence $\|f\|^{(0)}_{L^{p(\cdot)}(\Omega)} \le \lambda$. Conversely, if $\|f\|^{(0)}_{L^{p(\cdot)}(\Omega)} < \lambda$, then $(|f(x)|/\lambda)^\infty = 0$ for a.e. $x \in \Omega_\infty$ and

$$\int_{\Omega\setminus\Omega_\infty} \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} dx \le 1.$$

Hence, since $p_- \ge 1$, we have

$$\int_{\Omega\setminus\Omega_\infty} \left(\frac{|f(x)|}{2\lambda}\right)^{p(x)} dx + \left\|\frac{f}{2\lambda}\right\|_{L^\infty(\Omega_\infty)} \le \int_{\Omega\setminus\Omega_\infty} \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} \frac{dx}{2^{p_-}} + \frac{1}{2}\left\|\frac{f}{\lambda}\right\|_{L^\infty(\Omega_\infty)} \le 1.$$

That is, $\|f\|_{L^{p(\cdot)}(\Omega)} \le 2\lambda$. Therefore, we have (8.3).

In Part III and after, we use the norm $\|f\|^{(0)}_{L^{p(\cdot)}(\Omega)}$ as $\|f\|_{L^{p(\cdot)}(\Omega)}$.

9 Hölder's inequality and the associate space The aim of this section is to prove results related to duality. Recall that, for a measurable function $p(\cdot) : \Omega \to [1,\infty]$, the generalized Lebesgue space $L^{p(\cdot)}(\Omega)$ with variable exponents is defined by

$$L^{p(\cdot)}(\Omega) := \{f \,:\, \rho_p(f/\lambda) < \infty \text{ for some } \lambda > 0\},$$

where

$$\rho_p(f) := \int_{\{x\in\Omega\,:\,p(x)<\infty\}} |f(x)|^{p(x)}dx + \|f\|_{L^\infty(\{x\in\Omega\,:\,p(x)=\infty\})}.$$

Moreover,

$$\|f\|_{L^{p(\cdot)}(\Omega)} := \inf\{\lambda > 0 : \rho_p(f/\lambda) \le 1\}.$$

For $p(\cdot) : \Omega \to [1,\infty]$, we defined $p'(\cdot) : \Omega \to [1,\infty]$ as

$$1 = \frac{1}{p(x)} + \frac{1}{p'(x)} \quad (x \in \mathbb{R}^n).$$

We use Lemma 2.1 to prove the following theorem:

Theorem 9.1 (Generalized Hölder's inequality)**.** *Let $p(\cdot) : \Omega \to [1, \infty]$ be a variable exponent. Then, for all $f \in L^{p(\cdot)}(\Omega)$ and all $g \in L^{p'(\cdot)}(\Omega)$,*

$$\int_\Omega |f(x)g(x)|\, dx \le r_p \|f\|_{L^{p(\cdot)}(\Omega)} \|g\|_{L^{p'(\cdot)}(\Omega)}, \tag{9.1}$$

where

$$r_p := 1 + \frac{1}{p_-} - \frac{1}{p_+}. \tag{9.2}$$

Remark 9.1. If we use the norm $\|\cdot\|^{(0)}_{L^{p(\cdot)}(\Omega)}$ instead of $\|\cdot\|_{L^{p(\cdot)}(\Omega)}$, then we have the same inequality as (9.1). That is,

$$\int_\Omega |f(x)g(x)|\, dx \le r_p \|f\|^{(0)}_{L^{p(\cdot)}(\Omega)} \|g\|^{(0)}_{L^{p'(\cdot)}(\Omega)}.$$

Proof of Theorem 9.1. Recall that

$$\begin{aligned}
\Omega_1 &= \{x \in \Omega : p(x) = 1\} = \{x \in \Omega : p'(x) = \infty\},\\
\Omega_0 &= \{x \in \Omega : 1 < p(x) < \infty\} = \{x \in \Omega : 1 < p'(x) < \infty\},\\
\Omega_\infty &= \{x \in \Omega : p(x) = \infty\} = \{x \in \Omega : p'(x) = 1\}.
\end{aligned}$$

We may assume that $\|f\|_{L^{p(\cdot)}(\Omega)} = \|g\|_{L^{p'(\cdot)}(\Omega)} = 1$.

If $\Omega_0 \ne \emptyset$, then, by Lemma 2.1 we have

$$|f(x)g(x)| \le \frac{|f(x)|^{p(x)}}{p(x)} + \frac{|g(x)|^{p'(x)}}{p'(x)} \le \frac{|f(x)|^{p(x)}}{p_-} + \frac{|g(x)|^{p'(x)}}{p'_-} \quad \text{for a.e. } x \in \Omega_0.$$

If $\Omega_1 \ne \emptyset$, then $p_- = 1$, $p'_+ = \infty$ and

$$|f(x)g(x)| \le |f(x)| \|g\|_{L^\infty(\Omega_1)} \le |f(x)| = \frac{|f(x)|^{p(x)}}{p_-} \quad \text{for a.e. } x \in \Omega_1.$$

If $\Omega_\infty \ne \emptyset$, then $p_+ = \infty$, $p'_- = 1$ and

$$|f(x)g(x)| \le \|f\|_{L^\infty(\Omega_\infty)} \cdot |g(x)| \le |g(x)| = \frac{|g(x)|^{p'(x)}}{p'_-} \quad \text{for a.e. } x \in \Omega_\infty.$$

Therefore, we have

$$\int_\Omega |f(x)g(x)|\, dx \le \int_{\Omega_0 \cup \Omega_1} \frac{|f(x)|^{p(x)}}{p_-}\, dx + \int_{\Omega_0 \cup \Omega_\infty} \frac{|g(x)|^{p'(x)}}{p'_-}\, dx \le \frac{1}{p_-} + \frac{1}{p'_-} = r_p.$$

This shows the conclusion. □

It is well known that $L^p(\Omega)\,(1 \le p < \infty)$ has $L^{p'}(\Omega)$ as its dual. This is not the case when $p = \infty$. The notion of associated spaces is close to dual spaces, which is used in the theory of function spaces. It is sometimes referred to as the Köthe dual. In the case of Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}^n)$ the definition is given as follows:

Definition 9.1. Let $p(\cdot) : \Omega \to [1, \infty]$ be a variable exponent. The associate space of $L^{p(\cdot)}(\Omega)$ and its norm are defined as follows:

$$\begin{aligned} L^{p(\cdot)}(\Omega)' &= \left\{ f \in L^0(\Omega) \, : \, \|f\|_{L^{p(\cdot)}(\Omega)'} < \infty \right\}, \\ \|f\|_{L^{p(\cdot)}(\Omega)'} &:= \sup \left\{ \int_\Omega |f(x)g(x)| \, dx \, : \, \|g\|_{L^{p(\cdot)}(\Omega)} \le 1 \right\}. \end{aligned}$$

Remark 9.2. The condition $\|g\|_{L^{p(\cdot)}(\Omega)} \le 1$ is equivalent to $\rho_p(g) \le 1$ by virtue of Lemma 8.4.

Theorem 9.2. *Let $p(\cdot) : \Omega \to [1, \infty]$ be a variable exponent. Then $L^{p(\cdot)}(\Omega)' = L^{p'(\cdot)}(\Omega)$ with norm equivalence*

$$\frac{1}{3}\|f\|_{L^{p'(\cdot)}(\Omega)} \le \|f\|_{L^{p(\cdot)}(\Omega)'} \le r_p \|f\|_{L^{p'(\cdot)}(\Omega)}, \tag{9.3}$$

where r_p is the constant defined in (9.2).

See [31] for the weighted case.

Remark 9.3. Let f be a measurable function. Define

$$\|f\|^{(0)}_{L^{p(\cdot)}(\Omega)'} := \sup \left\{ \int_\Omega |f(x)g(x)| \; dx \; : \; \|g\|^{(0)}_{L^{p(\cdot)}(\Omega)} \le 1 \right\}.$$

Then we have

$$\|f\|_{L^{p(\cdot)}(\Omega)'} \le \|f\|^{(0)}_{L^{p(\cdot)}(\Omega)'} \le 2\|f\|_{L^{p(\cdot)}(\Omega)'}, \tag{9.4}$$

from (8.3).

Lemma 9.3. *Let $p(\cdot) : \Omega \to [1, \infty]$ be a variable exponent. Then every simple function s is in $L^{p(\cdot)}(\Omega)$ and*

$$\rho_p(s/\|s\|_{L^{p(\cdot)}(\Omega)}) = 1. \tag{9.5}$$

It is worth noting that (9.5) holds even when $\tilde{p}_+ = \|\chi_{\Omega \setminus \Omega_\infty} p(\cdot)\|_{L^\infty}$ is not finite, which we assumed in Lemma 8.2(2).

Proof. Assume for the time being that s has an expression; $s := c\chi_E$, where $E \subset \Omega$, $0 < |E| < \infty$ and $c > 0$. and Then s is in $L^{p(\cdot)}(\Omega)$, since $\rho_p(s/c) \le |E| + 1 < \infty$. From the properties of modular it follows that $\lambda \mapsto \rho_p(\lambda s)$ is continuous and strictly increasing on some interval $[0, \lambda_0)$ with $\lim_{\lambda \to \lambda_0} \rho_p(\lambda s) = \infty$, where $\lambda_0 \in (0, \infty]$. Since $\lim_{\lambda \to 0} \rho_p(\lambda s) = 0$, we have (9.5). For general s, we have the same conclusion, since the finite sum of continuous and strictly increasing functions is also continuous and strictly increasing. $\square$

See [117, Lemma 3] for a similar technique.

Proof of Theorem 9.2. Let $f \in L^{p'(\cdot)}(\Omega)$. Then the second inequality in (9.3) holds by the definition of the norm $\|\cdot\|_{L^{p(\cdot)}(\Omega)'}$ and generalized Hölder's inequality Theorem 9.1. That is, $L^{p(\cdot)}(\Omega)' \supset L^{p'(\cdot)}(\Omega)$.

Conversely, let $f \in L^{p(\cdot)}(\Omega)'$. We may assume that $f \not\equiv 0$. Take a sequence $\{f_j\}_{j=1}^\infty$ of simple functions such that $f_j \not\equiv 0$ and that $0 \le f_1 \le f_2 \le \cdots$ and $f_j \to |f|$ a.e. as $j \to \infty$. Then each f_j is in $L^{p'(\cdot)}(\Omega)$. We normalize f; set $\tilde{f}_j \equiv f_j/\|f_j\|_{L^{p'(\cdot)}(\Omega)}$. We also abbreviate;

$$\begin{aligned} a_j &:= \|\tilde{f}_j\|_{L^1(\{x\in\Omega:p'(x)=1\})}, \\ b_j &:= \int_{\{x\in\Omega:1<p'(x)<\infty\}} |\tilde{f}_j(x)|^{p'(x)}\,dx, \\ c_j &:= \|\tilde{f}_j\|_{L^\infty(\{x\in\Omega:p'(x)=\infty\})}, \\ U_j &:= \{x\in\Omega : p'(x)=\infty,\ \tilde{f}_j(x)=c_j\}. \end{aligned}$$

Then $a_j + b_j + c_j = 1$ by Lemma 9.3.

Case 1: $a_j \ge 1/3$. Let $g_j := \chi_{\{x\in\Omega:p'(x)=1\}} = \chi_{\{x\in\Omega:p(x)=\infty\}}$. Then $\rho_p(g_j) = 1$, that is, $\|g_j\|_{L^{p(\cdot)}} = 1$, and

$$\frac{\|f_j\|_{L^{p(\cdot)}(\Omega)'}}{\|f_j\|_{L^{p'(\cdot)}(\Omega)}} = \|\tilde{f}_j\|_{L^{p(\cdot)}(\Omega)'} \ge \int_\Omega |\tilde{f}_j(x)g_j(x)|\,dx = a_j \ge \frac{1}{3}.$$

Case 2: $b_j \ge 1/3$. Let $g_j := |\tilde{f}_j(x)|^{p'(x)-1}\chi_{\{x\in\Omega:1<p'(x)<\infty\}}$. Then $\rho_p(g_j) = b_j \le 1$, that is, $\|g_j\|_{L^{p(\cdot)}} \le 1$, and

$$\frac{\|f_j\|_{L^{p(\cdot)}(\Omega)'}}{\|f_j\|_{L^{p'(\cdot)}(\Omega)}} = \|\tilde{f}_j\|_{L^{p(\cdot)}(\Omega)'} \ge \int_\Omega |\tilde{f}_j(x)g_j(x)|\,dx = b_j \ge \frac{1}{3}.$$

Case 3: $c_j \ge 1/3$. Let $g_j := |U_j|^{-1}\chi_{U_j}$. Then $\rho_p(g_j) = 1$, that is, $\|g_j\|_{L^{p(\cdot)}} = 1$, and

$$\frac{\|f_j\|_{L^{p(\cdot)}(\Omega)'}}{\|f_j\|_{L^{p'(\cdot)}(\Omega)}} = \|\tilde{f}_j\|_{L^{p(\cdot)}(\Omega)'} \ge \int_\Omega |\tilde{f}_j(x)g_j(x)|\,dx = c_j \ge \frac{1}{3}.$$

This shows

$$\frac{1}{3}\|f_j\|_{L^{p'(\cdot)}(\Omega)} \le \|f_j\|_{L^{p(\cdot)}(\Omega)'} \le \|f\|_{L^{p(\cdot)}(\Omega)'}.$$

As $j \to \infty$, we have the first inequality in (9.3) and $L^{p(\cdot)}(\Omega)' \subset L^{p'(\cdot)}(\Omega)$. □

10 Norm convergence, modular convergence and convergence in measure Here we investigate the relations between several types of convergences.

10.1 Elementary results The following theorem is recorded as [23, Theorem 1.3]. When $p_+ < \infty$, this goes back to [49, 101]. Let Ω be a measurable set in $\mathbb{R}^n$ again.

Theorem 10.1. *Let $p(\cdot) : \Omega \to [1,\infty]$ be a variable exponent and $f_j \in L^{p(\cdot)}(\Omega)$ $(j = 1,2,3,\ldots)$.*

(1) *If $\lim_{j\to\infty} \|f_j\|_{L^{p(\cdot)}(\Omega)} = 0$, then $\lim_{j\to\infty} \rho_p(f_j) = 0$.*

(2) *Assume that $|\Omega \setminus \Omega_\infty| > 0$. The following two conditions* (A) *and* (B) *are equivalent:*

(A) $\operatorname{ess\,sup}_{x\in\Omega\setminus\Omega_\infty} p(x) < \infty$.

(B) *If $\lim_{j\to\infty} \rho_p(f_j) = 0$, then $\lim_{j\to\infty} \|f_j\|_{L^{p(\cdot)}(\Omega)} = 0$.*

Proof. (1) Suppose $\lim_{j\to\infty}\|f_j\|_{L^{p(\cdot)}(\Omega)}=0$ and fix $0<\varepsilon<1$ arbitrarily. We can take $j\in\mathbb{N}$ so large that $\|f_j\|_{L^{p(\cdot)}(\Omega)}<\varepsilon$. By virtue of Lemma 8.4 (1) we obtain $\rho_p(f_j)\le\|f_j\|_{L^{p(\cdot)}(\Omega)}<\varepsilon$. This implies that $\lim_{j\to\infty}\rho_p(f_j)=0$ is true.

(2) Assume that (A) holds and that $\lim_{j\to\infty}\rho_p(f_j)=0$. Fix $0<\varepsilon<1$ arbitrarily. Below we write $\tilde{p}_+:=\operatorname{ess\,sup}_{x\in\Omega\setminus\Omega_\infty}p(x)$ and take $j\in\mathbb{N}$ so large that $\rho_p(f_j)<\varepsilon^{\tilde{p}_+}$. Since $\rho_p(f_j)<1$, we have

$$\begin{aligned}
\rho_p\left(\frac{f_j}{\rho_p(f_j)^{1/\tilde{p}_+}}\right) &= \int_{\Omega\setminus\Omega_\infty}\left|\frac{f_j(x)}{\rho_p(f_j)^{1/\tilde{p}_+}}\right|^{p(x)}dx+\left\|\frac{f_j}{\rho_p(f_j)^{1/\tilde{p}_+}}\right\|_{L^\infty(\Omega_\infty)}\\
&\le \rho_p(f_j)^{-1}\int_{\Omega\setminus\Omega_\infty}|f_j(x)|^{p(x)}\,dx+\rho_p(f_j)^{-1/\tilde{p}_+}\|f_j\|_{L^\infty(\Omega_\infty)}\\
&\le \rho_p(f_j)^{-1}\left(\int_{\Omega\setminus\Omega_\infty}|f_j(x)|^{p(x)}\,dx+\|f_j\|_{L^\infty(\Omega_\infty)}\right)\\
&= 1,
\end{aligned}$$

that is,

$$\|f_j\|_{L^{p(\cdot)}(\Omega)}\le\rho_p(f_j)^{1/\tilde{p}_+}<\varepsilon.$$

Therefore (B) is true.

Meanwhile, if $\operatorname{ess\,sup}_{x\in\Omega\setminus\Omega_\infty}p(x)=\infty$, then we can take a family of measurable sets $\{G_j\}_{j=1}^\infty$ such that the following are satisfied:

$$G_{j+1}\subset G_j\subset\Omega\setminus\Omega_\infty,\ |G_j|<\infty\text{ for all }j\in\mathbb{N},\tag{10.1}$$

$$\lim_{j\to\infty}|G_j|=0,\tag{10.2}$$

$$p(x)>j\text{ if }x\in G_j,\tag{10.3}$$

$$\sup\{j\in\mathbb{N}:|G_j\setminus G_{j+1}|>0\}=\infty.\tag{10.4}$$

Now we fix $0<\lambda<1$ and define

$$\begin{aligned}
\omega_j &:= |G_j\setminus G_{j+1}|,\\
a_j &:= \begin{cases}\lambda^j\omega_j^{-1} & (\omega_j>0),\\ 0 & (\omega_j=0),\end{cases}\\
f(x) &:= \left(\sum_{j=1}^\infty a_j\chi_{G_j\setminus G_{j+1}}(x)\right)^{1/p(x)}\qquad (x\in G).
\end{aligned}$$

We use

$$\rho_p(f)=\int_{\Omega\setminus\Omega_\infty}|f(x)|^{p(x)}\,dx.$$

By inserting the definition of f to the equality we have

$$\rho_p(f)=\int_{\Omega\setminus\Omega_\infty}\sum_{j=1}^\infty a_j\chi_{G_j\setminus G_{j+1}}(x)\,dx=\sum_{j=1}^\infty a_j\omega_j\le\sum_{j=1}^\infty\lambda^j<\infty.\tag{10.5}$$

Meanwhile, note that

$$\rho_p\left(\frac{f\chi_{G_j}}{\lambda}\right)\ge\int_{G_j}\left|\frac{f(x)}{\lambda}\right|^{p(x)}dx\tag{10.6}$$

for every $j \in \mathbb{N}$, and hence, by (10.1),

$$\rho_p\left(\frac{f\chi_{G_j}}{\lambda}\right) \geq \sum_{k=j}^{\infty} \int_{G_k \setminus G_{k+1}} \left|\frac{f(x)}{\lambda}\right|^{p(x)} dx = \sum_{k=j}^{\infty} \int_{G_k \setminus G_{k+1}} a_k \lambda^{-p(x)}\, dx.$$

Recall that $\{G_j\}_{j=1}^{\infty}$ satisfies (10.1), (10.2), (10.3) and (10.4). Thus, we obtain

$$\rho_p\left(\frac{f\chi_{G_j}}{\lambda}\right) \geq \sum_{k=j}^{\infty} \int_{G_k \setminus G_{k+1}} a_k \lambda^{-k}\, dx = \sum_{k=j}^{\infty} \omega_k a_k \lambda^{-k} = \sum_{k=j,\, \omega_k > 0}^{\infty} \omega_k a_k \lambda^{-k}.$$

The most right-hand side is not finite;

$$\rho_p\left(\frac{f\chi_{G_j}}{\lambda}\right) \geq \sum_{k=j,\, \omega_k > 0}^{\infty} \omega_k a_k \lambda^{-k} = \sharp\{k \in \mathbb{N} : k \geq j,\ \omega_k > 0\} = \infty,$$

that is,

$$\|f\chi_{G_j}\|_{L^{p(\cdot)}(\Omega)} \geq \lambda > 0. \tag{10.7}$$

Meanwhile, by virtue of (10.2), (10.5) and the Lebesgue dominated convergence theorem, we obtain

$$\lim_{j\to\infty} \rho_p(f\chi_{G_j}) = \lim_{j\to\infty} \int_{G_j} |f(x)|^{p(x)}\, dx = \int_{\Omega} \left(|f(x)|^{p(x)} \lim_{j\to\infty} \chi_{G_j}(x)\right) dx = 0. \tag{10.8}$$

(10.7) and (10.8) show that (B) is false. □

Remark 10.1. A similar construction is used to prove that

$$\{f \in L^0(\Omega) : \rho_p(f) < \infty\} = \{f \in L^0(\Omega) : \rho_p(f/\lambda) < \infty \text{ for some } \lambda > 0\},$$

if and only if $p_+ < \infty$, see [49, Theorem 1.1].

Theorem 10.2. *If a sequence $\{f_j\}_{j=1}^{\infty} \subset L^{p(\cdot)}(\Omega)$ converges to 0 in $L^{p(\cdot)}(\Omega)$, then it converges to 0 in the sense of the Lebesgue measure, namely,*

$$\lim_{j\to\infty} |\{x \in \Omega : |f_j(x)| > \varepsilon\}| = 0 \tag{10.9}$$

for all $\varepsilon > 0$.

Proof. We may assume that $p(x)$ is finite for all $x \in \mathbb{R}^n$. Indeed, on Ω_∞, $\{f_j\}_{j=1}^{\infty}$ is convergent to 0 in $L^\infty(\mathbb{R}^n)$, which is stronger than (10.9).

Assume that $p(x)$ is finite for all $x \in \mathbb{R}^n$. Then we have, if $\|f_j\|_{L^{p(\cdot)}(\Omega)} < \varepsilon$,

$$|\{x \in \Omega \setminus \Omega_\infty : |f_j(x)| > \varepsilon\}| \leq \int_{\Omega\setminus\Omega_\infty} \left|\frac{f_j(x)}{\varepsilon}\right|^{p(x)} dx \leq \rho_p\left(\frac{f_j}{\varepsilon}\right) \leq \left\|\frac{f_j}{\varepsilon}\right\|_{L^{p(\cdot)}} = \frac{\|f_j\|_{L^{p(\cdot)}}}{\varepsilon}.$$

Letting $j \to \infty$, we obtain

$$\lim_{j\to\infty} |\{x \in \Omega \setminus \Omega_\infty : |f_j(x)| > \varepsilon\}| = 0.$$

Hence it follows that f_j converges to 0 in the sense of the Lebesgue measure. □

See [23, Example 1.7] for an example showing that convergence in measure does not guarantee the convergence in modular.

As an example of $p(\cdot)$ satisfying the requirement of Theorem 10.2, we can list

$$p(x) = 2 + \infty \cdot \chi_{B(0,1)}(x) = \begin{cases} 2 & (x \notin B(0,1)), \\ \infty & (x \in B(0,1)). \end{cases}$$

Here we assumed $B(0,1) \subset \Omega$.

Remark that Sharapudinov considered the norm convergence in [208].

10.2 Nakano's results on convergence of functions The definition of Lebesgue spaces with variable exponent is clearly written in the book of Nakano [159, Section 89]. Nakano placed himself in the setting of the compact interval $[0,1]$ to define the function spaces. Let $p(\cdot) : [0,1] \to [1,\infty]$ be a measurable function. Unlike Definition 8.1, Nakano used the following modular;

$$\rho_p^{(N)}(f) = \int_0^1 \frac{1}{p(t)} |f(t)|^{p(t)}\, dt, \tag{10.10}$$

where it will be understood that

$$\frac{1}{\infty} r^\infty = \lim_{n\to\infty} \frac{1}{n} r^n = \begin{cases} 0, & 0 \le r \le 1, \\ \infty, & r > 1. \end{cases}$$

Let ρ_p and $\rho_p^{(0)}$ be as in Definition 8.1 and Remark 8.2 with $\Omega = [0,1]$, respectively. That is,

$$\rho_p(f) = \int_{[0,1]\setminus\{p(t)=\infty\}} |f(t)|^{p(t)}\, dt + \|f\|_{L^\infty(\{p(t)=\infty\})},$$

and

$$\rho_p^{(0)}(f) = \int_0^1 |f(t)|^{p(t)}\, dt,$$

where it is understood that

$$r^\infty = \begin{cases} 0, & 0 \le r \le 1, \\ \infty, & r > 1. \end{cases} \tag{10.11}$$

First we give an example to show the difference between ρ_p, $\rho_p^{(0)}$ and $\rho_p^{(N)}$. It can happen that $\rho_p^{(N)}(f) = \infty > \rho_p(f)$.

Example 10.1. For each $n \in \mathbb{N}$, we let a_n solve the equation

$$a_n \ge 0, \quad \frac{{a_n}^n}{n(n+1)} = \frac{1}{n\sqrt{n}}.$$

Then, if we let $p(t) := [t^{-1}], t \in [0,1]$, then

$$\rho_p\left(\sum_{n=1}^{\infty} a_n \chi_{((n+1)^{-1},n^{-1})}(t)\right) = \int_0^1 \left(\sum_{n=1}^{\infty} a_n \chi_{((n+1)^{-1},n^{-1})}(t)\right)^{p(t)} dt$$
$$= \sum_{n=1}^{\infty} \int_{((n+1)^{-1},n^{-1})} {a_n}^n \, dt < \infty$$
$$\rho_p^{(N)}\left(\sum_{n=1}^{\infty} a_n \chi_{((n+1)^{-1},n^{-1})}(t)\right) = \int_0^1 \frac{1}{p(t)} \left(\sum_{n=1}^{\infty} a_n \chi_{((n+1)^{-1},n^{-1})}(t)\right)^{p(t)} dt$$
$$= \sum_{n=1}^{\infty} \int_{((n+1)^{-1},n^{-1})} n{a_n}^n \, dt = \infty.$$

However, we can show that Nakano's $L^{p(\cdot)}([0,1])$ coincides with the one taken up in the present paper. Actually, we have the following:

Proposition 10.3. *Let $p(\cdot) : [0,1] \to [1,\infty]$ be a variable exponent. For a measurable function $f : [0,1] \to \mathbb{C}$, let*

$$\|f\|_{L^{p(\cdot)}}^{(0)} = \inf\left\{\lambda > 0 : \rho_p^{(0)}(f/\lambda) \le 1\right\} = \inf\left\{\lambda > 0 : \int_0^1 \left|\frac{f(t)}{\lambda}\right|^{p(t)} dt \le 1\right\},$$
$$\|f\|_{L^{p(\cdot)}}^{(N)} = \inf\left\{\lambda > 0 : \rho_p^{(N)}(f/\lambda) \le 1\right\} = \inf\left\{\lambda > 0 : \int_0^1 \frac{1}{p(t)}\left|\frac{f(t)}{\lambda}\right|^{p(t)} dt \le 1\right\}.$$

Then

$$\|f\|_{L^{p(\cdot)}}^{(N)} \le \|f\|_{L^{p(\cdot)}}^{(0)} \le 2\|f\|_{L^{p(\cdot)}}^{(N)}. \tag{10.12}$$

Proof. Using the inequality

$$1 \le p(t) < 2^{p(t)}, \tag{10.13}$$

we have

$$\frac{1}{p(t)}|f(t)|^{p(t)} \le |f(t)|^{p(t)} \le \frac{1}{p(t)}|2f(t)|^{p(t)},$$

that is,

$$\rho_p^{(N)}(f) \le \rho_p^{(0)}(f) \le \rho_p^{(N)}(2f). \tag{10.14}$$

This shows (10.12). □

From (8.3) and (10.12) we have the following:

Corollary 10.4. *Let $p(\cdot) : [0,1] \to [1,\infty]$ be a variable exponent. Then*

$$\|f\|_{L^{p(\cdot)}}^{(N)} \le \|f\|_{L^{p(\cdot)}}^{(0)} \le \|f\|_{L^{p(\cdot)}} \le 2\|f\|_{L^{p(\cdot)}}^{(0)} \le 4\|f\|_{L^{p(\cdot)}}^{(N)}$$

for $f \in L^0(\Omega)$.

Next we show the equivalence of the modular convergence ([159, Section 78]) with respect to ρ_p, $\rho_p^{(0)}$ and $\rho_p^{(N)}$.

Proposition 10.5. *Let $\{f_j\}_{j=1}^{\infty}$ be a sequence in $L^0([0,1])$, and let $p(\cdot):[0,1]\to[1,\infty]$ be a variable exponent. Then the following are equivalent:*

(i) $\lim_{j\to\infty}\rho_p(\xi f_j)=0$ *for every* $\xi>0$,

(ii) $\lim_{j\to\infty}\rho_p^{(0)}(\xi f_j)=0$ *for every* $\xi>0$,

(iii) $\lim_{j\to\infty}\rho_p^{(N)}(\xi f_j)=0$ *for every* $\xi>0$.

Proof. From the inequalities (10.14) we have the equivalence between (ii) and (iii). Assume that (i) holds, that is,

$$\lim_{j\to\infty}\rho_p(\xi f_j)=\lim_{j\to\infty}\int_{[0,1]\setminus\{p(t)=\infty\}}|\xi f_j(t)|^{p(t)}\,dt+\lim_{j\to\infty}\|\xi f_j\|_{L^\infty(\{p(t)=\infty\})}=0$$

for every $\xi>0$. Then, for every $\xi>0$, $\|\xi f_j\|_{L^\infty(\{p(t)=\infty\})}<1$ if j is large enough. In this case $\rho_p^{(0)}(\xi f_j)=\int_{[0,1]\setminus\{p(t)=\infty\}}|\xi f_j(t)|^{p(t)}\,dt\le\rho_p(\xi f_j)$. This shows that (ii) holds. Conversely, assume that (ii) holds. Then, for every $k\in\mathbb{N}$, there exists $j_0\in\mathbb{N}$ such that $\rho_p^{(0)}(kf_j)<\infty$ $(j\ge j_0)$. In this case $\|kf_j\|_{L^\infty(\{p(t)=\infty\})}\le 1$, that is, $\|f_j\|_{L^\infty(\{p(t)=\infty\})}\le 1/k$. This shows that, for every $\xi>0$,

$$\lim_{j\to\infty}\|\xi f_j\|_{L^\infty(\{p(t)=\infty\})}=0. \tag{10.15}$$

Moreover,

$$\lim_{j\to\infty}\int_{[0,1]\setminus\{p(t)=\infty\}}|\xi f_j(t)|^{p(t)}\,dt\le\lim_{j\to\infty}\rho_p^{(0)}(\xi f_j)=0. \tag{10.16}$$

(10.15) and (10.16) show that (i) holds. □

Let ρ be a modular on $L^0([0,1])$, and let

$$X:=\{f\in L^0([0,1]):\rho(\xi f)<\infty \text{ for some } \xi>0\}.$$

One says that the modulared space (X,ρ) is modular complete, if any sequence $\{f_j\}_{j=1}^{\infty}$ of X satisfying

$$\lim_{j,k\to\infty}\rho(\xi(f_j-f_k))=0 \text{ for every } \xi>0$$

has a unique element f satisfying

$$\lim_{j\to\infty}\rho(\xi(f_j-f))=0 \text{ for every } \xi>0.$$

See [159, p. 205].

Nakano proved in his book the modular completeness of $L^{p(\cdot)}([-1,1])$.

Theorem 10.6 ([159, Section 89, Theorem 1])**.** *Let $p(\cdot):[0,1]\to[1,\infty]$ be a measurable function. Then $L^{p(\cdot)}([-1,1])$ is modular complete.*

An element $f\in X$ is said to be finite if $\rho(\xi f)<\infty$ for all $\xi>0$, and X is said to be finite if every element of X is finite; see [159, Section 86]. From (10.14) it follows that f is finite with respect to $\rho_p^{(0)}$ if and only if f is finite with respect to $\rho_p^{(N)}$. Note that, if $p\equiv\infty$ and $f\equiv 1$, then f is finite with respect to ρ_p, but not finite with respect to $\rho_p^{(0)}$ or $\rho_p^{(N)}$. Nakano proved that Nakano's $L^{p(\cdot)}([-1,1])$ space is finite if and only if $p_+<\infty$.

Theorem 10.7 ([159, Section 89, Theorem 2]). *Nakano's $L^{p(\cdot)}([0,1])$ space equipped with the modular $\rho_p^{(N)}$ is finite if and only if $p_+ < \infty$.*

By assuming $p_+ = \infty$, Nakano constructed a function by a method akin to Theorem 10.1. More can be said for the case $p_+ < \infty$.

Theorem 10.8 ([159, Section 89, Theorem 3]). *Let $p_+ < \infty$. Nakano's $L^{p(\cdot)}([0,1])$ space equipped with the modular $\rho_p^{(N)}$ is uniformly finite and uniformly simple in the following senses, respectively;*

$$\sup\{\rho_p^{(N)}(\xi f) \,:\, \rho_p^{(N)}(f) \leq 1\} < \infty \quad \textit{and} \quad \inf\{\rho_p^{(N)}(\xi f) \,:\, \rho_p^{(N)}(f) \geq 1\} > 0,$$

for every $\xi > 0$.

The first inequality is the uniformly finiteness [159, p. 224] and the second inequality is the uniformly simpleness [159, p. 221]. The above two theorems are valid for $\rho_p^{(0)}$ by (10.14). In [159, Section 89, Theorems 4 and 5], the converse is proved; if Nakano's $L^{p(\cdot)}([0,1])$ space is uniformly finite or uniformly simple, then $p_+ < \infty$. Next, Nakano defined the finite subspace of all finite elements in Nakano's $L^{p(\cdot)}([0,1])$ space and Nakano showed the finiteness and the modular completeness in [159, Section 89, Theorem 6].

The Lebesgue convergence theorem can be carried over to Nakano's $L^{p([0,1])}([0,1])$ space.

Theorem 10.9. [159, Section 89, Theorem 7] *Let $\{f_j\}_{j=1}^\infty$ belong to Nakano's $L^{p(\cdot)}([0,1])$ space equipped with the modular $\rho_p^{(N)}$. Assume that $\{f_j\}_{j=1}^\infty$ converges a.e. to 0 and that there exists a finite element f_0 in Nakano's $L^{p(\cdot)}([0,1])$ space such that $|f_j| \leq f_0$. Then $\|f_j\|_{L^{p(\cdot)}}^{(N)} \to 0$ as $j \to \infty$.*

Nakano investigated duality (see Section 12.2 below) in [159, Section 89]. In his book Nakano's $L^{p(\cdot)}([0,1])$ space shows up as another context; he investigated the product space [159, Section 93, Theorem 4].

11 Completeness We go back to the initial setting, where we are given a measurable set Ω with $|\Omega| > 0$. Next, we show that $L^{p(\cdot)}(\Omega)$ is a complete space.

Theorem 11.1. *Let $p(\cdot) : \Omega \to [1,\infty]$ be a measurable function. Then the norm $\|\cdot\|_{L^{p(\cdot)}(\Omega)}$ is complete, that is, $L^{p(\cdot)}(\Omega)$ is a Banach space.*

Proof. Take a Cauchy sequence $\{f_j\}_{j=1}^\infty$ in $L^{p(\cdot)}(\Omega)$ arbitrarily. We prove that $\{f_j\}_{j=1}^\infty$ converges to a function in $L^{p(\cdot)}(\Omega)$. We can take a subsequence $\{f_{j_k}\}_{k=1}^\infty \subset \{f_j\}_{j=1}^\infty$ so that

$$\|f_{j_{k+1}} - f_{j_k}\|_{L^{p(\cdot)}(\Omega)} < 2^{-k}$$

holds for every $k \in \mathbb{N}$. Thus Lemma 8.4 implies that

$$\rho_p(f_{j_{k+1}} - f_{j_k}) < 2^{-k}.$$

Now we define

$$\begin{aligned} g_N(x) &:= \sum_{k=1}^N |f_{j_{k+1}}(x) - f_{j_k}(x)| \quad (N \in \mathbb{N}, x \in \Omega), \\ g(x) &:= \sum_{k=1}^\infty |f_{j_{k+1}}(x) - f_{j_k}(x)| \quad (x \in \Omega). \end{aligned}$$

Note that

$$\|g_N\|_{L^{p(\cdot)}(\Omega)} \le \sum_{k=1}^{N} \|f_{j_{k+1}} - f_{j_k}\|_{L^{p(\cdot)}(\Omega)} \le \sum_{k=1}^{N} 2^{-k} < 1,$$

in particular, $\rho_p(g_N) < 1$ holds. By virtue of the Lebesgue monotone convergence theorem we get

$$\int_{\Omega\setminus\Omega_\infty} g(x)^{p(x)}\, dx = \lim_{N\to\infty} \int_{\Omega\setminus\Omega_\infty} g_N(x)^{p(x)}\, dx \le 1.$$

Hence we see that $g^{p(\cdot)} \in L^1(\Omega\setminus\Omega_\infty)$ and that $g < \infty$ a.e. $\Omega\setminus\Omega_\infty$. If $x \in \Omega_\infty$, then we get

$$g(x) \le \sum_{k=1}^{\infty} \|f_{j_{k+1}} - f_{j_k}\|_{L^\infty(\Omega_\infty)} \le \sum_{k=1}^{\infty} 2^{-k} = 1. \tag{11.1}$$

Namely we have $g \in L^\infty(\Omega_\infty)$, that is, $g \in L^{p(\cdot)}(\Omega)$. We also see that the series

$$\sum_{k=1}^{\infty} (f_{j_{k+1}} - f_{j_k})$$

converges absolutely a.e. Ω. Now we additionally define

$$\begin{aligned} f(x) &:= f_{j_1}(x) + \sum_{k=1}^{\infty} (f_{j_{k+1}}(x) - f_{j_k}(x)) = \lim_{k\to\infty} f_{j_k}(x), \\ F(x) &:= |f_{j_1}(x)| + g(x) \end{aligned}$$

for $x \in \Omega$. We see that f, $f_{j_l} \in L^{p(\cdot)}(\Omega)$ and that $F \in L^{p(\cdot)}(\Omega)$ for all $l \in \mathbb{N}$, since $|f|$, $|f_{j_l}| \le F$ a.e. Ω. For $m > l$,

$$\|f_{j_m} - f_{j_l}\|_{L^{p(\cdot)}(\Omega)} \le \sum_{k=l}^{m-1} \|f_{j_{k+1}} - f_{j_k}\|_{L^{p(\cdot)}(\Omega)} < 2^{-l+1}.$$

Hence, it follows that

$$\int_{\Omega\setminus\Omega_\infty} \left| \frac{f_{j_m}(x) - f_{j_l}(x)}{2^{-l+1}} \right|^{p(x)} dx \le 1.$$

By the Fatou lemma, we deduce

$$\begin{aligned} \int_{\Omega\setminus\Omega_\infty} \left| \frac{f(x) - f_{j_l}(x)}{2^{-l+1}} \right|^{p(x)} dx &= \int_{\Omega\setminus\Omega_\infty} \liminf_{m\to\infty} \left| \frac{f_{j_m}(x) - f_{j_l}(x)}{2^{-l+1}} \right|^{p(x)} dx \\ &\le \liminf_{m\to\infty} \int_{\Omega\setminus\Omega_\infty} \left| \frac{f_{j_m}(x) - f_{j_l}(x)}{2^{-l+1}} \right|^{p(x)} dx \\ &\le 1. \end{aligned}$$

As a result, it follows that

$$\|(f - f_{j_l})\chi_{\Omega\setminus\Omega_\infty}\|_{L^{p(\cdot)}(\Omega)} \le 2^{-l+1}. \tag{11.2}$$

Hence, we conclude from (11.1) and (11.2) that the Cauchy sequence $\{f_j\}_{j=1}^\infty$ converges to f in $L^{p(\cdot)}(\Omega)$. $\square$

12 Duality (The generalized F. Riesz representation theorem) Here we show that a counterpart to the $L^p(\Omega)$-$L^{p'}(\Omega)$ duality is available in the variable setting.

12.1 A fundamental result Let $p(\cdot) : \Omega \to [1, \infty]$ be a variable exponent. The dual space of $L^{p(\cdot)}(\Omega)$ and its norm are given by

$$\begin{aligned} L^{p(\cdot)}(\Omega)^* &:= \left\{ T : L^{p(\cdot)}(\Omega) \to \mathbb{C} \,:\, T \text{ is linear and bounded} \right\}, \\ \|T\|_{L^{p(\cdot)}(\Omega)^*} &:= \sup \left\{ |T(u)| \,:\, \|u\|_{L^{p(\cdot)}(\Omega)} \leq 1 \right\}. \end{aligned}$$

It is natural to ask ourselves whether $L^{p'(\cdot)}(\Omega)$ is naturally identified with the dual of $L^{p(\cdot)}(\Omega)$ when $p_+ < \infty$. Part of the answer is given by the next theorem.

Theorem 12.1. *Let $p(\cdot) : \Omega \to [1, \infty]$ be a variable exponent. Given a measurable function $f \in L^{p'(\cdot)}(\Omega)$, define the functional T_f by*

$$T_f(u) := \int_\Omega f(x)u(x)\, dx \quad (u \in L^{p(\cdot)}(\Omega)).$$

Then, the integral defining $T_f u$ converges absolutely. Also, the functional T_f belongs to $L^{p(\cdot)}(\Omega)^*$ and the estimate below holds;

$$\frac{1}{3}\|f\|_{L^{p'(\cdot)}(\Omega)} \leq \|T_f\|_{L^{p(\cdot)}(\Omega)^*} \leq \left(1 + \frac{1}{p_-} - \frac{1}{p_+}\right) \|f\|_{L^{p'(\cdot)}(\Omega)}. \tag{12.1}$$

In particular $L^{p'(\cdot)}(\Omega) \subset L^{p(\cdot)}(\Omega)^*$ is true.

Proof. The generalized Hölder inequality gives us

$$|T_f(u)| \leq \left(1 + \frac{1}{p_-} - \frac{1}{p_+}\right) \|f\|_{L^{p'(\cdot)}(\Omega)} \|u\|_{L^{p(\cdot)}(\Omega)}$$

for all $u \in L^{p(\cdot)}(\Omega)$, namely, the right inequality (12.1) holds. Meanwhile, using the associate norm (Theorem 9.2), we obtain

$$\frac{1}{3}\|f\|_{L^{p'(\cdot)}(\Omega)} \leq \|f\|_{L^{p(\cdot)}(\Omega)'} = \sup \left\{ \left| \int_\Omega f(x)g(x)\, dx \right| \,:\, \|g\|_{L^{p(\cdot)}(\Omega)} \leq 1 \right\} = \|T_f\|_{L^{p(\cdot)}(\Omega)^*},$$

which prove the left inequality of (12.1). □

When $p_+ < \infty$, then we can give a positive answer to the above question. Remark that Theorems 12.1 and 12.2 can be found in [49, 101].

Theorem 12.2. *Let $p(\cdot) : \Omega \to [1, \infty)$ be a variable exponent such that*

$$p_+ < \infty.$$

Then, for all linear functionals $F \in L^{p(\cdot)}(\Omega)^*$, there exists a unique function $f \in L^{p'(\cdot)}(\Omega)$ such that

$$F(u) = \int_\Omega f(x)u(x)\, dx \quad (u \in L^{p(\cdot)}(\Omega)).$$

Moreover, we have the norm estimate;

$$\frac{1}{3}\|f\|_{L^{p'(\cdot)}(\Omega)} \leq \|F\|_{L^{p(\cdot)}(\Omega)^*} \leq \left(1 + \frac{1}{p_-} - \frac{1}{p_+}\right) \|f\|_{L^{p'(\cdot)}(\Omega)}. \tag{12.2}$$

In particular $L^{p(\cdot)}(\Omega)^* \subset L^{p'(\cdot)}(\Omega)$ is true.

Proof. We will prove this theorem by five steps.

Step 1. We first show the uniqueness of the function f. If there exist two functions $f,\ f_1 \in L^{p'(\cdot)}(\Omega)$ such that

$$F(u) = \int_\Omega f(x)u(x)\,dx = \int_\Omega f_1(x)u(x)\,dx \quad (u \in L^{p(\cdot)}(\Omega)),$$

then by virtue of Theorem 12.1 we get

$$\frac{1}{3}\|f - f_1\|_{L^{p'(\cdot)}(\Omega)} \le \|T_{f-f_1}\|_{L^{p(\cdot)}(\Omega)^*} = \|T_f - T_{f_1}\|_{L^{p(\cdot)}(\Omega)^*} = 0,$$

that is, $f = f_1$.

Step 2. Next construct a function $f \in L^{p'(\cdot)}(\Omega)$ such that

$$\frac{1}{3}\|f\|_{L^{p'(\cdot)}(\Omega)} \le \|F\|_{L^{p(\cdot)}(\Omega)^*}, \quad F(u) = \int_\Omega f(x)u(x)\,dx \quad (u \in L^{p(\cdot)}(\Omega)), \tag{12.3}$$

provided $|\Omega| < \infty$.

To begin with, we define $\nu(E) := F(\chi_E)$ for a measurable set $E \subset \Omega$ and then we shall prove that ν is a finite complex measure. By virtue of $|E| \le |\Omega| < \infty$ we see that

$$|\nu(E)| \le \|F\|_{L^{p(\cdot)}(\Omega)^*}\|\chi_E\|_{L^{p(\cdot)}(\Omega)} < \infty.$$

If we choose a sequence of disjoint measurable sets $\{E_j\}_{j=1}^\infty$, then we have for each $k \in \mathbb{N}$,

$$\sum_{j=1}^k \nu(E_j) = \sum_{j=1}^k F(\chi_{E_j}) = F(\chi_{\bigcup_{j=1}^k E_j}).$$

Hence we obtain

$$\begin{aligned}
\left|\nu\left(\bigcup_{j=1}^\infty E_j\right) - \sum_{j=1}^k \nu(E_j)\right| &= \left|F(\chi_{\bigcup_{j=1}^\infty E_j}) - F(\chi_{\bigcup_{j=1}^k E_j})\right| \\
&= \left|F(\chi_{\bigcup_{j=k+1}^\infty E_j})\right| \\
&\le \|F\|_{L^{p(\cdot)}(\Omega)^*}\|\chi_{\bigcup_{j=k+1}^\infty E_j}\|_{L^{p(\cdot)}(\Omega)}.
\end{aligned}$$

By virtue of the fact that

$$\rho_{p'}(\chi_{\bigcup_{j=k+1}^\infty E_j}) = \sum_{j=k+1}^\infty \rho_{p'}(\chi_{E_j}) \to 0 \quad (k \to \infty),$$

we get $\lim_{k\to\infty}\|\chi_{\bigcup_{j=k+1}^\infty E_j}\|_{L^{p(\cdot)}(\Omega)} = 0$ by Theorem 10.1 (2) (B). Thus, we have

$$\nu\left(\bigcup_{j=1}^\infty E_j\right) = \sum_{j=1}^\infty \nu(E_j),$$

that is, ν is a finite complex measure. Meanwhile, if a measurable set E satisfies $|E| = 0$, then $\chi_E = 0$ a.e. holds. Hence, we have

$$|\nu(E)| \le \|F\|_{L^{p(\cdot)}(\Omega)^*}\|\chi_E\|_{L^{p(\cdot)}(\Omega)} = 0.$$

Namely, ν is absolutely continuous with respect to the Lebesgue measure. Therefore, we can apply the Radon-Nykodym theorem to get a function $f \in L^1(\Omega)$ such that

$$\nu(E) = F(\chi_E) = \int_\Omega f(x)\chi_E(x)\,dx$$

holds for all measurable sets E. In particular, given a simple function

$$s(x) = \sum_j a_j \chi_{E_j}(x) \quad (x \in \mathbb{R}^n)$$

we obtain

$$F(s) = \sum_j a_j F(\chi_{E_j}) = \sum_j a_j \int_\Omega f(x)\chi_{E_j}(x)\,dx = \int_\Omega f(x)s(x)\,dx. \tag{12.4}$$

We shall prove that (12.4) is still true replacing s by any $u \in L^{p(\cdot)}(\Omega)$. Take $u \in L^{p(\cdot)}(\Omega)$ and $k \in \mathbb{N}$ arbitrarily, and define $u_k(x) := u(x)\chi_{\{|u|\le k\}}(x)$. Then we have $|u_k(x)| \le k$ and the next lemma.

Lemma 12.3. *There exists a sequence of simple functions $\{s_j^k\}_j$ such that $|s_j^k| \le |u_k|$ and that $\lim_{j\to\infty} s_j^k = u_k$ hold for a.e. Ω.*

We postpone the proof of Lemma 12.3 till Step 5. The Lebesgue dominated convergence theorem implies that $\lim_{j\to\infty} \rho_p(u_k - s_j^k) = 0$ and that $\lim_{j\to\infty} \|u_k - s_j^k\|_{L^{p(\cdot)}(\Omega)} = 0$. Hence we have

$$F(u_k) = \lim_{j\to\infty} F(s_j^k) = \lim_{j\to\infty} \int_\Omega f(x)s_j^k(x)\,dx.$$

Meanwhile, $|fs_j^k| \le k|f| \in L^1(\Omega)$ holds. Thus using the Lebesgue dominated convergence theorem again we conclude $\lim_{j\to\infty} \int_\Omega f(x)s_j^k(x)\,dx = \int_\Omega f(x)u_k(x)\,dx$ and that

$$F(u_k) = \int_\Omega f(x)u_k(x)\,dx. \tag{12.5}$$

Now we consider (12.5) replacing u by $|u| \cdot \frac{\bar f}{|f|} \in L^{p(\cdot)}(\Omega)$. Since

$$u_k(x) = \chi_{\{|u_k|\le k\}}(x) \cdot |u(x)| \cdot \frac{\overline{f(x)}}{|f(x)|} \quad (x \in \mathbb{R}^n, f(x) \neq 0),$$

we have $F(u_k) = \int_{\{|u_k|\le k\}} |f(x)| \cdot |u(x)|\,dx$. Because $F(u_k) \le \|F\|_{L^{p(\cdot)}(\Omega)^*} \|u\|_{L^{p(\cdot)}(\Omega)}$ holds, by $k \to \infty$ we obtain

$$\int_\Omega |f(x)| \cdot |u(x)|\,dx \le \|F\|_{L^{p(\cdot)}(\Omega)^*} \|u\|_{L^{p(\cdot)}(\Omega)}. \tag{12.6}$$

Taking the supremum of (12.6) over u such that $\|u\|_{L^{p(\cdot)}(\Omega)} \le 1$, we conclude that $\|T_{|f|}\|_{L^{p(\cdot)}(\Omega)^*} \le \|F\|_{L^{p(\cdot)}(\Omega)^*}$. Applying Theorem 12.1, we get

$$\frac{1}{3}\|f\|_{L^{p'(\cdot)}(\Omega)} \le \|F\|_{L^{p(\cdot)}(\Omega)^*},$$

in particular $f \in L^{p'(\cdot)}(\Omega)$. Moreover we obtain

$$\begin{aligned}
&\left| F(u) - \int_\Omega f(x)u(x)\,dx \right| \\
&\le |F(u) - F(u_k)| + \left| F(u_k) - \int_\Omega f(x)u(x)\,dx \right| \\
&= |F(u - u_k)| + \left| \int_\Omega f(x)(u_k(x) - u(x))\,dx \right| \\
&\le \left\{ \|F\|_{L^{p(\cdot)}(\Omega)^*} + \left(1 + \frac{1}{p_-} - \frac{1}{p_+}\right) \|f\|_{L^{p'(\cdot)}(\Omega)} \right\} \|u - u_k\|_{L^{p(\cdot)}(\Omega)}.
\end{aligned}$$

By virtue of $|u_k| \le |u|$ and the Lebesgue dominated convergence theorem once again, we conclude $\lim_{k\to\infty} \rho_p(u - u_k) = 0$ and that $\lim_{k\to\infty} \|u - u_k\|_{L^{p(\cdot)}(\Omega)} = 0$. Therefore,

$$F(u) = \int_\Omega f(x)u(x)\,dx$$

is true.

Step 3. We prove (12.3) in the case of $|\Omega| = \infty$. Take a sequence of measurable sets $\{\Omega_m\}_{m=1}^\infty$ so that

(A) $\Omega_m \subset \Omega_{m+1}$ for all $m \in \mathbb{N}$,

(B) $|\Omega_m| < \infty$ for all $m \in \mathbb{N}$,

(C) $\Omega = \bigcup_{m=1}^\infty \Omega_m$.

By virtue of *Step 2.*, for all $m \in \mathbb{N}$, there exists a unique function $f_m \in L^{p'(\cdot)}(\Omega_m)$ such that

$$F(u\chi_{\Omega_m}) = \int_{\Omega_m} f_m(x)u(x)\,dx = \int_\Omega f_m(x)\chi_{\Omega_m}(x)u(x)\,dx \quad (u \in L^{p(\cdot)}(\Omega)). \tag{12.7}$$

Because f_m is unique, $f_j = f_m$ a.e. Ω_j if $j \le m$. Now if we define

$$f(x) := f_m(x) \quad (x \in \Omega_m),$$

then the function f is well-defined for a.e. Ω. By (12.7) we see that

$$F(u\chi_{\Omega_m}) = \int_{\Omega_m} f(x)u(x)\,dx.$$

Replacing u by $\frac{|u|\bar{f}}{|f|}$ and using $\int_{\Omega_m} |f(x)| \cdot |u(x)|\,dx = \int_{\Omega_m} f(x) \cdot \frac{|u(x)|\overline{f(x)}}{|f(x)|}\,dx$, we have

$$\int_{\Omega_m} |f(x)| \cdot |u(x)|\,dx = F\left(\frac{|u|\bar{f}}{|f|} \cdot \chi_{\Omega_m}\right) \le \|F\|_{L^{p(\cdot)}(\Omega)^*} \|u\|_{L^{p(\cdot)}(\Omega)}.$$

Thus, we obtain as $m \to \infty$,

$$\int_\Omega |f(x)| \cdot |u(x)|\,dx \le \|F\|_{L^{p(\cdot)}(\Omega)^*} \|u\|_{L^{p(\cdot)}(\Omega)}.$$

By the same argument as *Step 2.* with taking the supremum over $\|u\|_{L^{p(\cdot)}(\Omega)} \le 1$, we get

$$\frac{1}{3}\|f\|_{L^{p'(\cdot)}(\Omega)} \le \|F\|_{L^{p(\cdot)}(\Omega)^*}, \quad f \in L^{p'(\cdot)}(\Omega).$$

Meanwhile, we see that

$$\begin{aligned}
&\left|F(u) - \int_\Omega f(x)u(x)\,dx\right| \\
&\quad\le |F(u) - F(u\chi_{\Omega_m})| + \left|F(u\chi_{\Omega_m}) - \int_\Omega f(x)u(x)\,dx\right| \\
&\quad= |F(u - u\chi_{\Omega_m})| + \left|\int_\Omega f(x)(u(x)\chi_{\Omega_m}(x) - u(x))\,dx\right| \\
&\quad\le \left\{\|F\|_{L^{p(\cdot)}(\Omega)^*} + \left(1 + \frac{1}{p_-} - \frac{1}{p_+}\right)\|f\|_{L^{p'(\cdot)}(\Omega)}\right\}\|u - u\chi_{\Omega_m}\|_{L^{p(\cdot)}(\Omega)}.
\end{aligned}$$

Since $\|u - u\chi_{\Omega_m}\|_{L^{p(\cdot)}(\Omega)} \to 0$ as $m \to \infty$, we have

$$F(u) = \int_\Omega f(x)u(x)\,dx.$$

Step 4. We shall complete the proof of the norm estimate (12.2). By the generalized Hölder inequality we get

$$|F(u)| = \left|\int_\Omega f(x)u(x)\,dx\right| \le \left(1 + \frac{1}{p_-} - \frac{1}{p_+}\right)\|f\|_{L^{p'(\cdot)}(\Omega)}$$

for all $u \in L^{p(\cdot)}(\Omega)$. Taking the supremum over $\|u\|_{L^{p(\cdot)}(\Omega)} \le 1$, we obtain

$$\|F\|_{L^{p(\cdot)}(\Omega)^*} \le \left(1 + \frac{1}{p_-} - \frac{1}{p_+}\right)\|f\|_{L^{p'(\cdot)}(\Omega)}.$$

Step 5. We prove Lemma 12.3.

(i) In the case of $u_k \ge 0$, we define for each $j \in \mathbb{N}$

$$s_j^k(x) := \begin{cases} 2^{-j}(l-1) & \text{if } |x| \le j \text{ and } 2^{-j}(l-1) \le u_k(x) < 2^{-j}l, \\ & \qquad l = 1, 2, \ldots, j \cdot 2^j, \\ j & \text{if } |x| \le j \text{ and } u_k(x) \ge j, \\ 0 & \text{if } |x| > j. \end{cases}$$

Then s_j^k satisfies $0 \le |s_j^k| \le |u_k|$ and $\lim_{j\to\infty} s_j^k = u_k$ a.e. Ω.

(ii) In the case of $u_k \in \mathbb{R}$, we have

$$u_k = \max\{u_k,\ 0\} - \max\{-u_k,\ 0\},\ \max\{u_k,\ 0\} \ge 0,\ \max\{-u_k,\ 0\} \ge 0.$$

Thus it suffices to consider (i).

(iii) In the case of $u_k \in \mathbb{C}$, we have

$$u_k = \Re(u_k) + \sqrt{-1}\Im(u_k),\ \Re(u_k) \in \mathbb{R},\ \Im(u_k) \in \mathbb{R}.$$

Hence we have only to prove the case (ii). □

12.2 Nakano's contribution of the dual spaces Let $\rho_p^{(N)}$ be the modular given by (10.10). In [159, Section 89], Nakano considered the dual space of $L^{p(\cdot)}([0,1])$ assuming that $1 \le p(t) < \infty$ for all $t \in [0,1]$.

Nakano ([159, Sections 78-80, 84]) called a linear space $\mathcal{R}$ a modulared space associated with a modular $\rho : \mathcal{R} \to [0,\infty]$, if ρ satisfies (a)–(e) in Definition 8.2 for $\mathcal{R}$ instead of $L^0(\Omega)$, and if,

(h) for any $f \in \mathcal{R}$ there exists $\lambda > 0$ such that $\rho(\lambda f) < \infty$.

Moreover, if ρ satisfies (f) in Definition 8.2, then Nakano said that $\mathcal{R}$ is simple, or that the modular ρ of $\mathcal{R}$ is simple. A linear functional φ on $\mathcal{R}$ is said to be modular bounded if

$$\sup\{|\varphi(f)| \,:\, \rho(f) \le 1\} < \infty.$$

Let $\overline{\mathcal{R}}$ be the set of all modular bounded linear functionals on $\mathcal{R}$, and define

$$\overline{\rho}(\varphi) := \sup_f \{\varphi(f) - \rho(f)\}, \quad \varphi \in \overline{\mathcal{R}}.$$

Then $\overline{\mathcal{R}}$ is a modulared space associated with the adjoint modular $\overline{\rho}$ of ρ. The space $\overline{\mathcal{R}}$ is called the modular adjoint space of $\mathcal{R}$. We can consider further the modular adjoint space $\overline{\overline{\mathcal{R}}}$ of $\overline{\mathcal{R}}$. Then $\mathcal{R}$ may be considered as a subspace of $\overline{\overline{\mathcal{R}}}$ by the relation

$$\rho(\overline{\rho}) = \overline{\rho}(\rho) \quad \text{for } \rho \in \mathcal{R} \text{ and } \overline{\rho} \in \overline{\mathcal{R}}.$$

If $\mathcal{R}$ coincides with the whole $\overline{\overline{\mathcal{R}}}$, then Nakano said that $\mathcal{R}$ is regular, or that the modular ρ of $\mathcal{R}$ is regular.

Denote by $p'(t)$ the harmonic conjugate of $p(t)$ as before. For $g \in L^{p'(\cdot)}([0,1])$, putting

$$\varphi_g(f) := \int_0^1 f(t)g(t)\,dt, \quad f \in L^{p(\cdot)}([0,1]),$$

we see that $L^{p'(\cdot)}([0,1])$ is contained in the modular adjoint space of $L^{p(\cdot)}([0,1])$. Let

$$L_F^{p(\cdot)}([0,1]) = \{f \in L^{p(\cdot)}([0,1]) : \rho_p^{(N)}(\xi f) < \infty \text{ for all } \xi > 0\}.$$

Nakano proved the following three theorems:

Theorem 12.4 ([159, Section 89, Theorem 8]). *If $L^{p(\cdot)}([0,1])$ is simple, that is, if $1 \le p(t) < \infty$, then the modular adjoint space of $L_F^{p(\cdot)}([0,1])$ coincides with $L^{p'(\cdot)}([0,1])$.*

Theorem 12.5 ([159, Section 89, Theorem 9]). *The modular adjoint space of $L^{p(\cdot)}([0,1])$ coincides with $L^{p'(\cdot)}([0,1])$ if and only if $p_+ < \infty$.*

Theorem 12.6 ([159, Section 89, Theorem 10]). *The space $L^{p(\cdot)}([0,1])$ is regular if and only if $1 < p_- \le p_+ < \infty$.*

Also, Nakano proved

$$\lim_{\xi \downarrow 0} \frac{1}{\xi} \sup\{\rho(\xi x) \,:\, \rho(x) \le 1\} = 0, \quad \lim_{\xi \uparrow \infty} \frac{1}{\xi} \inf\{\rho(\xi x) \,:\, \rho(x) \ge 1\} = \infty$$

when $p_- > 1$ [159, Section 89, Theorem 11]. These two properties are referred to as uniformly monotone [159, Section 85] and uniformly increasing [159, Section 86], respectively.

Finally, when $1 < p_- \le p_+ < \infty$, Nakano proved the uniform convexity and the uniform smoothness [159, Section 89, Theorem 12]. Remark that in Nakano's book [159], the terminology of uniform evenness is used instead of uniform smoothness. The uniformly convexity of $L^{p(\cdot)}([0,1])$ with $1 < p_- \le p_+ < \infty$ is extended to the space $L^{p(\cdot)}(\mathbb{R}^n)$ in [49, Theorem 1.10]. Since Nakano's $L^{p(\cdot)}([0,1])$ space is a normed space equipped with $\|\cdot\|^{(N)}_{L^{p(\cdot)}}$, we can also use terminology of normed spaces. Recall that a normed space $(X, \|\cdot\|)$ is uniformly convex, if for any $\varepsilon > 0$ there exists $\delta > 0$ such that $\|x\| \le 1$, $\|y\| \le 1$, $\|x-y\| \ge \varepsilon$ implies $\left\|\dfrac{x+y}{2}\right\| \le 1-\delta$. Recall also that a normed space $(X, \|\cdot\|)$ is uniformly smooth [112], if for any $\varepsilon > 0$, there exists $\delta > 0$ such that $\|x\| = 1$, $\|y\| \le \delta$ implies $\|x+y\|+\|x-y\| \le 2+\varepsilon\|y\|$. The triple $(\varepsilon, f, g) := (1, 2\chi_{[0,1/2)}, 2\delta\chi_{[1/2,1]})$ with $\delta > 0$ disprove that $L^1([0,1])$ is a uniformly convex Banach space and the triple $(\varepsilon, f, g) := (1, \chi_{[0,1]}, \delta(\chi_{[0,1/2)} - \chi_{[1/2,1]}))$ with $\delta > 0$ disproves $L^\infty([0,1])$ is a uniformly smooth Banach space.

13 Density We shall state and prove basic properties about density. The results in this section are in [101, (2.47), Theorem 2.11, Corollary 2.12]. See also [49, Theorems 1.5 and 1.6]. As an application of what we have obtained, we consider a density condition. We are interested in the condition that $C^\infty_{\rm comp}(\Omega)$ is dense in $L^{p(\cdot)}(\Omega)$.

Theorem 13.1. *If a variable exponent $p(\cdot) : \Omega \to [1,\infty]$ satisfies*

$$\operatorname*{ess\,sup}_{x\in\Omega\setminus\Omega_\infty} p(x) < \infty,$$

then the set

$$\mathcal{G} := \{g \in L^{p(\cdot)}(\Omega) \,:\, g \text{ is essentially bounded}\} = L^{p(\cdot)}(\Omega) \cap L^\infty(\Omega)$$

is dense in $L^{p(\cdot)}(\Omega)$.

Proof. Take $f \in L^{p(\cdot)}(\Omega)$ arbitrarily and for each $j \in \mathbb{N}$ define

$$\begin{aligned} G_j &:= \{x \in \Omega \setminus \Omega_\infty \,:\, |x| < j\}, \\ f_j(x) &:= \begin{cases} f(x) & (x \in G_j \cup \Omega_\infty,\ |f(x)| \le j), \\ jf(x)|f(x)|^{-1} & (x \in G_j \cup \Omega_\infty,\ |f(x)| > j), \\ 0 & (x \notin G_j \cup \Omega_\infty). \end{cases} \end{aligned}$$

Then we see $f_j \in \mathcal{G}$ and that $|f_j| \le \min\{j,\ |f|\}$. Thus, we are in the position of using the Lebesgue dominated convergence theorem and we obtain

$$\lim_{j\to\infty} \rho_p(f_j - f) = 0, \tag{13.1}$$

that is, $\lim_{j\to\infty} \|f_j - f\|_{L^{p(\cdot)}(\Omega)} = 0$ by virtue of Theorem 10.1 (2) (A). $\square$

If $\Omega \subset \mathbb{R}^n$ is an open set, we define

$$C^\infty_{\rm comp}(\Omega) := \{f \in C^\infty(\Omega) \,:\, \text{supp}(f) \text{ is compact}\},$$

where $\text{supp}(f) := \overline{\{x \in \Omega \,:\, f(x) \ne 0\}}$.

Theorem 13.2. *If a variable exponent $p(\cdot) : \Omega \to [1,\infty)$ satisfies $p_+ < \infty$, then the following hold:*

(1) *The set* $C(\Omega) \cap L^{p(\cdot)}(\Omega)$ *is dense in* $L^{p(\cdot)}(\Omega)$.

(2) *If* Ω *is an open set, then* $C^\infty_{\mathrm{comp}}(\Omega)$ *is dense in* $L^{p(\cdot)}(\Omega)$.

Proof. Take $f \in L^{p(\cdot)}(\Omega)$ and $\varepsilon > 0$ arbitrarily.

We first prove (1). By virtue of Theorem 13.1, we can take a bounded function $g \in L^{p(\cdot)}(\Omega)$ so that $\|f - g\|_{L^{p(\cdot)}(\Omega)} < \varepsilon$. Now we use the Luzin theorem (cf. [69, 71, 215]) to obtain a function $h \in C(\Omega)$ and an open set U such that

$$|U| < \min\left\{1,\ \left(\frac{\varepsilon}{2\|g\|_{L^\infty(\Omega)}}\right)^{p_+}\right\}, \tag{13.2}$$

that

$$\sup_{x\in\Omega}|h(x)| = \sup_{x\in\Omega\setminus U}|g(x)| \le \|g\|_{L^\infty(\Omega)}, \tag{13.3}$$

and that

$$g(x) = h(x) \text{ for all } x \in \Omega \setminus U. \tag{13.4}$$

By the triangle inequality, we have

$$\|g - h\|_{L^\infty(\Omega)} \le \|g\|_{L^\infty(\Omega)} + \|h\|_{L^\infty(\Omega)} \le 2\|g\|_{L^\infty(\Omega)}.$$

We write $\rho_p\left(\frac{g-h}{\varepsilon}\right)$ out in full;

$$\rho_p\left(\frac{g-h}{\varepsilon}\right) = \int_\Omega \left|\frac{g(x)-h(x)}{\varepsilon}\right|^{p(x)} dx.$$

Since $t^{p(x)} \le \max(1, t^{p_+})$ holds for $t > 0$, we obtain

$$\rho_p\left(\frac{g-h}{\varepsilon}\right) \le |U| \max\left\{1,\ \left(\frac{2\|g\|_{L^\infty(\Omega)}}{\varepsilon}\right)^{p_+}\right\} \le 1, \tag{13.5}$$

namely, $\|g - h\|_{L^{p(\cdot)}(\Omega)} \le \varepsilon$. Therefore we have

$$\|f - h\|_{L^{p(\cdot)}(\Omega)} \le \|f - g\|_{L^{p(\cdot)}(\Omega)} + \|g - h\|_{L^{p(\cdot)}(\Omega)} < 2\varepsilon.$$

Next we assume that Ω is open and prove (2). Again we fix $\varepsilon > 0$. For $f \in L^{p(\cdot)}(\Omega)$, take $h \in C(\Omega)$ such that $\|f - h\|_{L^{p(\cdot)}(\Omega)} < 2\varepsilon$. Since $p_+ < \infty$, we have $C^\infty_{\mathrm{comp}}(\Omega) \subset L^{p(\cdot)}(\Omega)$ and

$$\rho_p\left(\frac{h}{\varepsilon}\right) \le \max\{\varepsilon^{-p_+},\ \varepsilon^{-p_-}\}\rho_p(h) < \infty.$$

Thus if we take a bounded open set $G \subset \Omega$ so that

$$\rho_p\left(\frac{h\chi_{\Omega\setminus G}}{\varepsilon}\right) \le 1,$$

then we get

$$\|h - h\chi_G\|_{L^{p(\cdot)}(\Omega)} \le \varepsilon. \tag{13.6}$$

Observe that $\overline{G}$ is compact since G is bounded. Now we take a polynomial $Q(x)$ so that

$$\sup_{x \in G} |h(x) - Q(x)| < \varepsilon \min\{1, \ |G|^{-1}\}$$

by using the Weierstrass theorem. Then, since $\min\{1, |G|^{-1}\}^{p(x)} \leq \min\{1, |G|^{-1}\}$ for all $x \in G$, we have

$$\rho_p\left(\frac{h\chi_G - Q\chi_G}{\varepsilon}\right) \leq |G| \min\{1, \ |G|^{-1}\} \leq 1,$$

that is,

$$\|h\chi_G - Q\chi_G\|_{L^{p(\cdot)}(\Omega)} \leq \varepsilon. \tag{13.7}$$

By virtue of $\rho_p\left(\frac{Q\chi_G}{\varepsilon}\right) < \infty$, we can take a small constant $a > 0$ so that

$$\rho_p\left(\frac{Q\chi_{G \setminus K_a}}{\varepsilon}\right) \leq 1,$$

where K_a is a compact set defined by

$$K_a := \{x \in G \, : \, \mathrm{dist}(x, \partial G) \geq a\}.$$

Thus we obtain

$$\|Q\chi_G - Q\chi_{K_a}\|_{L^{p(\cdot)}(\Omega)} \leq \varepsilon. \tag{13.8}$$

Now we fix a function $\varphi \in C^\infty_{\mathrm{comp}}(\Omega)$ such that

$$\mathrm{supp}(\varphi) \subset G, \quad 0 \leq \varphi \leq 1 \text{ on } G, \quad \varphi \equiv 1 \text{ on } K_a$$

to have;

$$\|Q\chi_G - Q\varphi\|_{L^{p(\cdot)}(\Omega)} = \||Q| \cdot |\chi_G - \varphi|\|_{L^{p(\cdot)}(\Omega)} \leq \||Q| \cdot |\chi_G - \chi_{K_a}|\|_{L^{p(\cdot)}(\Omega)} \leq \varepsilon, \tag{13.9}$$

where the last inequality follows from (13.8). Combining (13.5), (13.6), (13.7) and (13.9), we have $Q\varphi \in C^\infty_{\mathrm{comp}}(\Omega)$ and

$$\begin{aligned}
\|f - Q\varphi\|_{L^{p(\cdot)}(\Omega)} &\leq \|f - h\|_{L^{p(\cdot)}(\Omega)} + \|h - h\chi_G\|_{L^{p(\cdot)}(\Omega)} \\
&\quad + \|h\chi_G - Q\chi_G\|_{L^{p(\cdot)}(\Omega)} + \|Q\chi_G - Q\varphi\|_{L^{p(\cdot)}(\Omega)} \\
&< 2\varepsilon + \varepsilon + \varepsilon + \varepsilon \\
&= 5\varepsilon.
\end{aligned}$$

Thus, the proof is therefore complete. □

Corollary 13.3. *If a variable exponent $p(\cdot) : \Omega \to [1, \infty)$ satisfies $p_+ < \infty$, then $L^{p(\cdot)}(\Omega)$ is separable.*

We also remark that the property of mollifier is investigated in [114, Theorems 1.1 and 1.2] together with some examples in [114, Remarks 3.5 and 3.6], where the authors extended the result to $L^{p(\cdot)}(\log L)^{q(\cdot)}(\mathbb{R}^n)$, where the norm is given by

$$\|f\|_{L^{p(\cdot)}(\log L)^{q(\cdot)}} = \inf\left\{\lambda > 0 \, : \, \int_{\mathbb{R}^n} \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} \left\{\log\left(3 + \frac{|f(x)|}{\lambda}\right)\right\}^{q(x)} dx \leq 1\right\}.$$

Part III
Hardy-Littlewood maximal operator on $L^{p(\cdot)}(\mathbb{R}^n)$

On the generalized Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$ with variable exponents, the boundedness of the Hardy-Littlewood maximal operator was proved by Diening [36] and Cruz-Uribe, Fiorenza and Neugebauer [26, 27] at the beginning of this century. In this chapter we rearrange their proof. Our proof may be simpler than the original. Moreover, we state some basic results around 2005.

14 Variable exponent and norm Recall that $L^0(\mathbb{R}^n;T)$ is the set of all measurable functions from $\mathbb{R}^n$ to T, where $T \subset \mathbb{C}$ or $T \subset [0,\infty]$. If $T = \mathbb{C}$, then we denote $L^0(\mathbb{R}^n;\mathbb{C})$ by $L^0(\mathbb{R}^n)$ simply.

In this section, by a variable exponent we mean any measurable function from $\mathbb{R}^n$ to a subset of $(-\infty,\infty]$.

14.1 Log-Hölder condition For a variable exponent $p(\cdot)$, let

$$p_- = \operatorname*{ess\,inf}_{x\in\mathbb{R}^n} p(x), \quad p_+ = \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} p(x).$$

We consider the local log-Hölder continuity condition;

$$|p(x)-p(y)| \le \frac{c_*}{\log(1/|x-y|)} \quad \text{for} \quad |x-y| \le \frac{1}{2},\ x,y\in\mathbb{R}^n, \tag{14.1}$$

and a log-Hölder type decay condition at infinity;

$$|p(x)-p_\infty| \le \frac{c^*}{\log(e+|x|)} \quad \text{for} \quad x\in\mathbb{R}^n, \tag{14.2}$$

where c_*, c^* and p_∞ are positive constants independent of x and y. Let

$$\begin{aligned} LH_0 &:= \{p(\cdot)\in L^0(\mathbb{R}^n;\mathbb{R}) : p(\cdot) \text{ satisfies (14.1)}\},\\ LH_\infty &:= \{p(\cdot)\in L^0(\mathbb{R}^n;\mathbb{R}) : p(\cdot) \text{ satisfies (14.2)}\},\\ LH &:= LH_0 \cap LH_\infty. \end{aligned}$$

From $p_+ < \infty$ and (14.1) it follows that

$$|p(x)-p(y)| \le \frac{C}{\log(e+1/|x-y|)} \quad \text{for all } x,y\in\mathbb{R}^n. \tag{14.3}$$

From (14.2) it follows that

$$|p(x)-p(y)| \le \frac{2c^*}{\log(e+|x|)} \quad \text{for all } x,y\in\mathbb{R}^n \text{ with } |y|\ge|x|. \tag{14.4}$$

The condition (14.2) is equivalent to

$$|(p(x)-p_\infty)\log(e+|x|)| \le c^* \quad \text{for all } x\in\mathbb{R}^n,$$

that is,

$$\frac{1}{e^{c^*}} \le \frac{(e+|x|)^{p(x)}}{(e+|x|)^{p_\infty}} \le e^{c^*} \quad \text{for all } x \in \mathbb{R}^n. \tag{14.5}$$

Recall that, for a variable exponent $p(\cdot) \in L^0(\mathbb{R}^n; [1,\infty])$, its conjugate exponent $p'(\cdot) \in L^0(\mathbb{R}^n; [1,\infty])$ is defined as

$$1 = \frac{1}{p(x)} + \frac{1}{p'(x)},$$

where $1/\infty = 0$. If $p(\cdot) \in LH_0$ and $p_- > 1$, then $p'(\cdot) \in LH_0$. If $p(\cdot) \in LH_\infty$ and $p_- > 1$, then $p'(\cdot) \in LH_\infty$.

14.2 Norm of $L^{p(\cdot)}(\mathbb{R}^n)$ Unlike the usual Lebesgue spaces L^p, we have to be careful for the proof of the boundedness of operators. To this end, we reconsider Definition 8.1. For a variable exponent $p(\cdot) \in L^0(\mathbb{R}^n; (0,\infty])$, here we let $L^{p(\cdot)}(\mathbb{R}^n)$ be the set of all measurable functions f on $\mathbb{R}^n$ such that

$$\|f\|_{L^{p(\cdot)}} := \inf\left\{\lambda > 0 : \int_{\mathbb{R}^n} \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} dx \le 1\right\} < \infty.$$

In the above we regard

$$r^\infty = \begin{cases} 0, & 0 \le r \le 1, \\ \infty, & r > 1. \end{cases}$$

See (10.11). In this subsection we supplement some properties of the above norm.

Remark 14.1. If $p_+ = \infty$, then this definition is not exactly the same as in Definition 8.1. However, both definitions give the same space up to equivalence of (quasi) norms if $p_- > 0$, see Remark 8.2.

If $1 \le p_- \le p_+ \le \infty$, then $\|f\|_{L^{p(\cdot)}}$ is a norm and thereby $L^{p(\cdot)}(\mathbb{R}^n)$ is a Banach space (see Part II).

From the definition, for a positive constant C, if

$$\int_{\mathbb{R}^n} \left(\frac{|f(x)|}{C}\right)^{p(x)} dx \le 1,$$

then $\|f\|_{L^{p(\cdot)}} \le C$. Conversely, from

$$\int_{\mathbb{R}^n} \left(\frac{|f(x)|}{\|f\|_{L^{p(\cdot)}} + \varepsilon}\right)^{p(x)} dx \le 1, \quad \varepsilon > 0,$$

it follows that

$$\int_{\mathbb{R}^n} \left(\frac{|f(x)|}{\|f\|_{L^{p(\cdot)}}}\right)^{p(x)} dx \le 1,$$

by $\varepsilon \to +0$. Therefore, we have the following conclusions:

Lemma 14.1. *For $f \in L^0(\mathbb{R}^n)$,*

$$\int_{\mathbb{R}^n} |f(x)|^{p(x)}\, dx \le 1 \iff \|f\|_{L^{p(\cdot)}} \le 1.$$

It is not so hard to prove;

Lemma 14.2. *If $0 < p_- \le p_+ < \infty$, then we have*

$$\int_{\mathbb{R}^n} \left(\frac{|f(x)|}{\|f\|_{L^{p(\cdot)}}} \right)^{p(x)} dx = 1$$

for all $f \in L^{p(\cdot)}(\mathbb{R}^n) \setminus \{0\}$ and

$$L^{p(\cdot)}(\mathbb{R}^n) = \left\{ f \in L^0(\mathbb{R}^n) : \int_{\mathbb{R}^n} |f(x)|^{p(x)}\, dx < \infty \right\}.$$

15 Boundedness of the Hardy-Littlewood maximal operator Recall that the uncentered Hardy-Littlewood maximal operator M is given by

$$Mf(x) := \sup_{B \ni x} \frac{1}{|B|} \int_B |f(y)|\, dy,$$

where the supremum is taken over all open balls B containing x. We can replace the open balls $\{B\}$ by the open cubes $\{Q\}$. As we have shown in Theorem 4.2, the operator M is bounded on $L^p(\mathbb{R}^n)$ if $1 < p \le \infty$. That is, $M \in B(L^p(\mathbb{R}^n))$ if $1 < p \le \infty$.

Let

$$\mathcal{B}(\mathbb{R}^n) := \{p(\cdot) \in L^0(\mathbb{R}^n; [1, \infty]) : M \in B(L^{p(\cdot)}(\mathbb{R}^n))\}. \tag{15.1}$$

If p is a constant in $(1, \infty]$, then $p \in \mathcal{B}(\mathbb{R}^n)$.

Remark 15.1. Let $q(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ and $1 < r < \infty$. Then $rq(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. Actually, by Hölder's inequality, we have

$$\|Mf\|_{L^{rq(\cdot)}} \le \|(M|f|^r)^{1/r}\|_{L^{rq(\cdot)}} = \|M|f|^r\|_{L^{q(\cdot)}}^{1/r} \lesssim \|\,|f|^r\,\|_{L^{q(\cdot)}}^{1/r} = \|f\|_{L^{rq(\cdot)}}.$$

15.1 Log-Hölder condition as a sufficient condition: Diening's result Here we consider the following theorem proved in Diening [36] and Cruz-Uribe, Fiorenza and Neugebauer [26, 27].

Theorem 15.1. *If $p(\cdot) \in LH$ and $1 < p_- \le p_+ < \infty$, then $M \in B(L^{p(\cdot)}(\mathbb{R}^n))$.*

This boundedness relies upon the next pointwise estimate and the boundedness of M on $L^{p_-}(\mathbb{R}^n)$ for $p_- > 1$.

Theorem 15.2. *If $p(\cdot) \in LH$ and $1 \le p_- \le p_+ < \infty$, then there exists a positive constant C, dependent only on n and $p(\cdot)$, such that, for all measurable functions f with $\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le 1$,*

$$Mf(x)^{p(x)} \le C(M(|f|^{p(\cdot)/p_-})(x)^{p_-} + (e + |x|)^{-np_-}) \quad \text{for all } x \in \mathbb{R}^n.$$

We prove Theorem 15.2 in Section 16.1. We remark that similar technique is used in [131, 132]. In [131, Lemma 3.5] and [53], an estimate was obtained with the help of the Hardy operator. A similar technique to Theorem 15.2 is used to prove the boundedness of one-sided maximal operator, see [32].

Proof of Theorem 15.1. It is enough to prove that, there exists a positive constant C such that $\|Mf\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C$ for all $f \in L^{p(\cdot)}(\mathbb{R}^n)$ with $\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le 1$. Note that $\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le 1$ is equivalent to

$$\int_{\mathbb{R}^n} |f(x)|^{p(x)}\, dx \le 1.$$

In this case, letting $g(x) = |f(x)|^{p(x)/p_-}$, we have $\|g\|_{L^{p_-}(\mathbb{R}^n)} \le 1$. By Theorem 15.2 and the boundedness of M on $L^{p_-}(\mathbb{R}^n)$ for $p_- > 1$, we have

$$\int_{\mathbb{R}^n} Mf(x)^{p(x)}\,dx \lesssim \int_{\mathbb{R}^n} Mg(x)^{p_-}\,dx + \int_{\mathbb{R}^n} (e+|x|)^{-np_-}\,dx \lesssim \int_{\mathbb{R}^n} g(x)^{p_-}\,dx + 1 \lesssim 1.$$

This shows that, for some $C > 0$,

$$\int_{\mathbb{R}^n} \left(\frac{Mf(x)}{C}\right)^{p(x)} dx \le 1,$$

since $1 < p_- \le p_+ < \infty$. That is, $\|Mf\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C$. □

We give a proof of Theorem 15.2 in the next section.

Here, we present examples of $p(\cdot) \notin \mathcal{P}(\mathbb{R}^n) \setminus \mathcal{B}(\mathbb{R}^n)$, whose details are investigated in Section 17.

Example 15.1. Let $n = 1$. Then the operator M is not bounded on $L^{p(\cdot)}(\mathbb{R})$ in the following cases:

(i) $p(\cdot) = 4\chi_{(-\infty,0)} + 2\chi_{[0,\infty)}$, see Proposition 17.1.

(ii) $p(\cdot) = 2\chi_{(-\infty,-2]} + 4\chi_{(-2,0)} + 2\chi_{[0,\infty)}$, see Corollary 17.2.

(iii) $p(\cdot)$ is continuous, $p(x) = 2$ on $(-\infty,-1]$ and $p(x) = 4$ on $[1,\infty)$, see Proposition 17.3.

(iv) $p(\cdot) = 3 + \cos(2\pi\cdot)$, see Proposition 17.6.

Moreover, there is a Lipschitz continuous function $p(\cdot) \notin \mathcal{B}(\mathbb{R}^n)$ such that

$$p(x) = p_\infty > 1 \text{ for } x \le 0, \quad \lim_{x\to\infty} |p(x) - p_\infty| = 0, \quad \lim_{x\to\infty} |p(x) - p_\infty| \log x = \infty;$$

see Proposition 17.5.

15.2 Other sufficient conditions While examples in Example 15.1 are not in the class LH, LH is not always necessary. In [131], the following function was considered;

$$p(x) = p_\infty + \frac{a\log(e + \log(e+|x|))}{\log(e+|x|)} + \frac{b}{\log(e+|x|)}$$

when $a = 0$ this function is in LH. If $a \ne 0$, then $p(\cdot)$ does not belong to LH_∞. Mizuta and Shimomura [131] showed that the maximal operator is bounded in $L^{p(\cdot)}(\mathbb{R}^n)$ with $a \ne 0$.

Let IC_∞ be the set of all variable exponents satisfying the following condition: There exist constants $p_\infty \in (1,\infty)$ and $c \in (0,\infty)$ such that

$$\int_{\mathbb{R}^n} |p(x) - p_\infty| c^{1/|p(x)-p_\infty|}\,dx < \infty. \tag{15.2}$$

We shall recall the proof of the following theorem later.

Theorem 15.3 (Nekvinda [161] (2004))**.** *Let $p(\cdot) \in LH_0 \cap IC_\infty$ and $1 < p_- \le p_+ < \infty$. Then $M \in B(L^{p(\cdot)}(\mathbb{R}^n))$.*

A simple calculation shows that

$$LH_\infty \subset IC_\infty. \tag{15.3}$$

The following theorem shows that LH_0, LH_∞ and (15.2) are not always necessary for the boundedness of M:

Theorem 15.4 (Lerner [102] (2005))**.** *Let $p(\cdot) \in L^0(\mathbb{R}^n;\mathbb{R})$. If $p(\cdot)$ is a pointwise multiplier on* $\mathrm{BMO}(\mathbb{R}^n)$*, then $\alpha + p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ for some nonnegative constant α.*

Let g_1 and g_2 be the functions in (5.3) and (5.4), respectively. Then

$$g_1 \in \mathrm{PWM}(\mathrm{BMO}(\mathbb{R}^n)) \setminus LH_0 \quad (g_1 \text{ is not continuous at the origin}),$$

and

$$g_2 \in \mathrm{PWM}(\mathrm{BMO}(\mathbb{R}^n)) \setminus IC_\infty \subset \mathrm{PWM}(\mathrm{BMO}(\mathbb{R}^n)) \setminus LH_\infty.$$

The following inclusion relation is a special case of [157, Proposition 5.1] (1985):

$$LH = LH_0 \cap LH_\infty \subset \mathrm{PWM}(\mathrm{BMO}(\mathbb{R}^n)).$$

Let

$$g_3(x) := p_\infty + \sum_{k=1}^{\infty} (1/k - |x - e^{k^2}|)\chi_{[e^{k^2}-1/k,\, e^{k^2}+1/k]}(x) \quad (x \in \mathbb{R}).$$

Then $g_3 \in LH_0$ and

$$g_3 \in LH_0 \cap IC_\infty \setminus (LH_\infty \cup \mathrm{PWM}(\mathrm{BMO}(\mathbb{R}^n))),$$

see [16, 102]. Note that Lerner's idea is valid for the martingale setting, see [153].

Meanwhile, Diening gave an equivalent condition to the boundedness of M:

Theorem 15.5 (Diening [38] (2005))**.** *Let $p(\cdot)$ be a positive variable exponent and $1 < p_- \le p_+ < \infty$. Then $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ if and only if there exists a positive constant c such that for any family of pairwise disjoint cubes π,*

$$\left\| \sum_{Q\in\pi} (|f|_Q)\chi_Q \right\|_{L^{p(\cdot)}} \le c\|f\|_{L^{p(\cdot)}}.$$

Necessity is clear; $\displaystyle\sum_{Q\in\pi} (|f|_Q)\chi_Q \le Mf$. As other equivalent conditions, we can list the following ones.

Remark 15.2. Let $p(\cdot)$ be a positive variable exponent with $1 < p_- \le p_+ < \infty$. Then $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ if and only if either one of the following conditions holds;

(i) $p'(\cdot) \in \mathcal{B}(\mathbb{R}^n)$,

(ii) for some $a > (1/p_-)$, $ap(\cdot) \in \mathcal{B}(\mathbb{R}^n)$.

In 2009 the following was proved so as to cover the case when $p_+ = \infty$:

Theorem 15.6 (Diening, Harjulehto, Hästö, Mizuta and Shimomura [39])**.** *Let $p(\cdot)$ be in $L^0(\mathbb{R}^n;(1,\infty])$. Assume that $1/p(\cdot) \in LH$ and $p_- > 1$. Then $M \in B(L^{p(\cdot)}(\mathbb{R}^n))$.*

Note that although $1/p(\cdot)$ is bounded, the variable exponent $p(\cdot)$ itself can be unbounded. For the proof, see [40] also.

Again, for any positive constant α,

$$g_1 \in \mathrm{PWM}(\mathrm{BMO}(\mathbb{R}^n)), \quad 1/(\alpha + g_1) \notin LH_0,$$

and

$$g_2 \in \mathrm{PWM}(\mathrm{BMO}(\mathbb{R}^n)), \quad 1/(\alpha + g_2) \notin IC_\infty.$$

Observe $IC_\infty \supset LH_\infty$ again, see (15.3).

Later, we point out that $p_- > 1$ is a necessary condition. See Theorem 21.2 below.

16 Proofs of Theorems 15.2, 15.3 and 15.4

16.1 Proof of the pointwise estimate (Theorem 15.2) In this section we prove the pointwise estimate (Theorem 15.2).The method is due to Mizuta and Shimomura (see [129, 125]).

For a nonnegative function f and a ball $B(x,r)$, let

$$I = I(x,r) = \fint_{B(x,r)} f(y)\,dy, \quad J = J(x,r) = \fint_{B(x,r)} f(y)^{p(y)}\,dy. \tag{16.1}$$

Then

$$Mf(x) \sim \sup_{r>0} I \quad \text{and} \quad M(|f(\cdot)|^{p(\cdot)})(x) \sim \sup_{r>0} J.$$

Let

$$\mathcal{F}_{p(\cdot)} := \left\{ f \in L^1_{\mathrm{loc}}(\mathbb{R}^n) \cap L^0(\mathbb{R}^n; \{0\} \cup [1,\infty)) : \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le 1 \right\},$$
$$\mathcal{G} := L^0(\mathbb{R}^n; [0,1))$$

To prove Theorem 15.2, we state and prove two basic lemmas.

Lemma 16.1. *Let $p(\cdot) \in LH_0$ and $1 \le p_- \le p_+ < \infty$. Then there exists a positive constant C, dependent only on n and $p(\cdot)$, such that, for all functions $f \in \mathcal{F}_{p(\cdot)}$ and for all balls $B(x,r)$,*

$$I \le C J^{1/p(x)}.$$

Lemma 16.2. *Let $p(\cdot) \in LH_\infty$ and $1 \le p_- \le p_+ < \infty$. Then there exists a positive constant C, dependent only on n and $p(\cdot)$, such that, for all functions $f \in \mathcal{G}$ and for all balls $B(x,r)$,*

$$I \le C(J^{1/p(x)} + (e+|x|)^{-n}).$$

Proof of Lemma 16.1. Let $B = B(x,r)$ and let $f \in \mathcal{F}_{p(\cdot)}$.

Case 1: $J > 1$. In this case $1 < J \le 1/|B| = 1/(v_n r^n)$, since $\int f(y)^{p(y)}\,dy \le 1$, where v_n is the volume of the unit ball in $\mathbb{R}^n$. Take an integer m such that $1/v_n \le e^m$. Then $1 < J \le 1/(v_n r^n) \le e^m/r^n \le (e+1/r)^{m+n}$. Let $K := J^{1/p(x)}$. Then, for $y \in B(x,r)$, using (14.3), we have

$$\begin{aligned}
|(p(x)-p(y))\log K| &= \frac{|p(x)-p(y)|}{p(x)} \log J \\
&\le \frac{|p(x)-p(y)|}{p_-}(m+n)\log(e+1/r) \\
&\le \frac{m+n}{p_-} \frac{C}{\log(e+1/|x-y|)} \log(e+1/r) \le C,
\end{aligned}$$

that is, $K^{p(x)} \sim K^{p(y)}$. Hence

$$\begin{aligned}
I &= \fint_B f(y)\chi_{\{x\in B\,:\,f(x)\le K\}}(y)\,dy + \fint_B f(y)\chi_{\{x\in B\,:\,f(x)>K\}}(y)\,dy \\
&\le \fint_B K\,dy + \fint_B f(y)\left(\frac{f(y)}{K}\right)^{p(y)-1} dy \\
&= K + \fint_B \frac{K}{K^{p(y)}} f(y)^{p(y)}\,dy \\
&\lesssim K + \frac{K}{K^{p(x)}} J = 2K = 2J^{1/p(x)}.
\end{aligned}$$

Case 2: $J \le 1$. In this case, using $f(y) \le f(y)^{p(y)}$, we have $I \le J \le J^{1/p(x)}$. Therefore we have the conclusion. $\square$

Proof of Lemma 16.2. Let $B = B(x,r)$ and $f \in \mathcal{G}$. Let

$$\begin{aligned}
E_1 &:= \{y \in B : |y| < |x|,\ (e+|y|)^{-n-1} \le f(y) < 1\},\\
E_2 &:= \{y \in B : |y| < |x|,\ 0 \le f(y) < (e+|y|)^{-n-1}\}.\\
E_3 &:= \{y \in B : |y| \ge |x|,\ (e+|x|)^{-n-1} \le f(y) < 1\},\\
E_4 &:= \{y \in B : |y| \ge |x|,\ 0 \le f(y) < (e+|x|)^{-n-1}\}.
\end{aligned}$$

Case 1: Integration over E_1. Let $y \in E_1$. By (14.2)

$$\begin{aligned}
|(p(x)-p(y))\log f(y)| &= |p(x)-p(y)|\log(1/f(y))\\
&\le \frac{C}{\log(e+|y|)}\log((e+|y|)^{n+1}) = C,
\end{aligned}$$

that is, $f(y)^{p(x)} \sim f(y)^{p(y)}$. Let $K := J^{1/p(x)}$ again. Then

$$\begin{aligned}
\frac{1}{|B|}\int_{E_1} f(y)\,dy &\le \frac{1}{|B|}\int_{E_1} K\,dy + \frac{1}{|B|}\int_{E_1} f(y)\left(\frac{f(y)}{K}\right)^{p(x)-1} dy\\
&\lesssim K + \frac{K}{K^{p(x)}}\frac{1}{|B|}\int_{E_1} f(y)^{p(y)}\,dy\\
&\le K + \frac{K}{K^{p(x)}}J = 2K = 2J^{1/p(x)}.
\end{aligned}$$

Case 2: Integration over E_2. Let $y \in E_2$. If $r \le |x|/2$, then $|x| \sim |y|$. Hence

$$\frac{1}{|B|}\int_{E_2} f(y)\,dy \le \frac{1}{|B|}\int_{E_2}(e+|y|)^{-n-1}\,dy \le C(e+|x|)^{-n-1}.$$

If $r > |x|/2$ and $|x| > 1$, then

$$\frac{1}{|B|}\int_{E_2} f(y)\,dy \le \frac{1}{|B|}\int_{E_2}(e+|y|)^{-n-1}\,dy \le \frac{1}{|B|}\int_{\mathbb{R}^n}(e+|y|)^{-n-1}\,dy \lesssim r^{-n} \lesssim (e+|x|)^{-n}.$$

If $|x| \le 1$, then

$$\frac{1}{|B|}\int_{E_2} f(y)\,dy \le 1 \le C(e+|x|)^{-n}.$$

Case 3: Integration over E_3. Let $y \in E_3$. By (14.2)

$$\begin{aligned}
|(p(x)-p(y))\log f(y)| &= |p(x)-p(y)|\log(1/f(y))\\
&\le \frac{C}{\log(e+|x|)}\log((e+|x|)^{n+1}) \le C,
\end{aligned}$$

that is, $f(y)^{p(x)} \sim f(y)^{p(y)}$. Then, by the same calculation as Case 1, we have

$$\frac{1}{|B|}\int_{E_3} f(y)\,dy \le 2J^{1/p(x)}.$$

Case 4: Integration over E_4. A crude estimate $f(y) \le (e+|x|)^{-n-1}$ for $y \in E_4$ suffices;

$$\frac{1}{|B|}\int_{E_4} f(y)\,dy \le \frac{1}{|B|}\int_{E_4}(e+|x|)^{-n-1}\,dy \le (e+|x|)^{-n-1}.$$

Therefore we have the conclusion. $\square$

Proof of Theorem 15.2. Let $\|f\|_{L^{p(\cdot)}} \le 1$. We may assume that f is nonnegative. Decompose $f = f_1 + f_2$, where

$$f_1 := f\chi_{\{x\in\mathbb{R}^n\,:\,f(x)\ge 1\}}, \quad f_2 := f\chi_{\{x\in\mathbb{R}^n\,:\,0<f(x)<1\}}.$$

Let $\overline{p}(x) = p(x)/p_-$. Then $\overline{p}(\cdot)$ satisfies (14.1), (14.2) and $1 \le \overline{p}_- \le \overline{p}_+ < \infty$. In this case $\|f_1\|_{L^{\overline{p(\cdot)}}} \le 1$, since $f_1(y)^{\overline{p}(y)} \le f_1(y)^{p(y)} \le f(y)^{p(y)}$, that is, $f_1 \in \mathcal{F}_{\overline{p}(\cdot)}$ and $f_2 \in \mathcal{G}$. Let

$$I = I(x,r) = \fint_{B(x,r)} f(y)\,dy, \quad \bar{J} = \bar{J}(x,r) = \fint_{B(x,r)} f(y)^{\overline{p}(y)}\,dy,$$
$$I_i = I_i(x,r) = \fint_{B(x,r)} f_i(y)\,dy, \quad \bar{J}_i = \bar{J}_i(x,r) = \fint_{B(x,r)} f_i(y)^{\overline{p}(y)}\,dy, \quad i = 1,2.$$

By Lemmas 16.1 and 16.2 we have

$$I = I_1 + I_2 \le C\bar{J}_1{}^{1/\overline{p}(x)} + C(\bar{J}_2{}^{1/\overline{p}(x)} + (e+|x|)^{-n}) \le C(\bar{J}^{1/\overline{p}(x)} + (e+|x|)^{-n}).$$

Then

$$I^{p(x)} \le C(\bar{J}^{p_-} + (e+|x|)^{-np_-}),$$

that is,

$$\left(\fint_{B(x,r)} f(y)\,dy\right)^{p(x)} \le C\left(\left(\fint_{B(x,r)} f(y)^{p(y)/p_-}\,dy\right)^{p_-} + (e+|x|)^{-np_-}\right),$$

for all balls $B(x,r)$. Then we have the conclusion. □

16.2 Proof of Nekvinda's theorem (Theorem 15.3) The following lemma is a fundamental one in that this lemma can be transformed for other operators when we consider the boundedness:

Lemma 16.3. *Let $p(\cdot) \in L^0(\mathbb{R}^n;[1,\infty))$ and $p_+ < \infty$. Then the following are equivalent:*

(i) *There exists a positive constant C such that $\|Mf\|_{L^{p(\cdot)}} \le C\|f\|_{L^{p(\cdot)}}$ for all $f \in L^{p(\cdot)}(\mathbb{R}^n)$.*

(ii) *$\int_{\mathbb{R}^n} Mf(x)^{p(x)}\,dx < \infty$ provided $\int_{\mathbb{R}^n} |f(x)|^{p(x)}\,dx \le 1$.*

Proof. (i) ⇒ (ii): If $\int_{\mathbb{R}^n} |f(x)|^{p(x)}\,dx \le 1$, then $\|f\|_{L^{p(\cdot)}} \le 1$. By (i) we have

$$\frac{1}{(1+C)^{p_+}}\int_{\mathbb{R}^n} Mf(x)^{p(x)}\,dx \le \int_{\mathbb{R}^n}\left(\frac{Mf(x)}{1+C}\right)^{p(x)} dx \le \int_{\mathbb{R}^n}\left(\frac{Mf(x)}{C}\right)^{p(x)} dx < \infty.$$

This shows (ii).

(ii) ⇒ (i): Assume the contrary. Then there exists a sequence of functions $f_m \ge 0$ with $\|f_m\|_{L^{p(\cdot)}} \le 1$ and $\|Mf_m\|_{L^{p(\cdot)}} \ge 4^m$. Set $f := \sum_{m=1}^\infty 2^{-m} f_m$. Then $\|f\|_{L^{p(\cdot)}} \le 1$ by virtue of the triangle inequality for $L^{p(\cdot)}(\mathbb{R}^n)$. This implies $\int_{\mathbb{R}^n} |f(x)|^{p(x)}\,dx \le 1$ in view of the definition of the norm. On the other hand, we obtain $\|Mf\|_{L^{p(\cdot)}} \ge \|2^{-m}Mf_m\|_{L^{p(\cdot)}} \ge 2^m$ for each m, that is, $\|Mf\|_{L^{p(\cdot)}} = \infty$. Since $p_+ < \infty$, this means that $\int_{\mathbb{R}^n} Mf(x)^{p(x)}\,dx = \infty$. □

The key observation Nekvinda made is that his assumption (15.2) enables us to freeze the variable exponent $p(\cdot)$, namely, his assumption can be used to replace $p(\cdot)$ with p_∞ when $f \in L^\infty(\mathbb{R}^n)$. Nekvinda generalized his idea in the following form:

Lemma 16.4. *Let* $p(\cdot), q(\cdot) \in L^0(\mathbb{R}^n; [1,\infty))$ *satisfy*

$$\int_{\mathbb{R}^n} |p(x) - q(x)| c^{1/|p(x)-q(x)|}\, dx < \infty \tag{16.2}$$

for some positive constant c*. Assume that* $0 \le f(x) \le 1$ *a.e. Then*

$$\int_{\mathbb{R}^n} f(x)^{p(x)}\, dx < \infty \quad \text{if and only if} \quad \int_{\mathbb{R}^n} f(x)^{q(x)}\, dx < \infty.$$

Proof. Without loss of generality we can assume $c \le 1$ by replacing c with $\min(c,1)$ if necessary. Symmetry reduces the matter to proving

$$\int_{\mathbb{R}^n} f(x)^{q(x)}\, dx < \infty,$$

under the condition that

$$\int_{\mathbb{R}^n} f(x)^{p(x)}\, dx < \infty.$$

Let $G_1 := \{x \in \mathbb{R}^n : p(x) > q(x)\}$ and $G_2 := \mathbb{R}^n \setminus G_1$. Then, since $0 \le f(x) \le 1$ for a.e. $x \in \mathbb{R}^n$, we have

$$\int_{G_2} f(x)^{q(x)}\, dx \le \int_{G_2} f(x)^{p(x)}\, dx < \infty.$$

Let $g(x) := f(x)^{p(x)}$ and $\varepsilon(x) := (p(x)-q(x))/p(x)$. Then $\int_{\mathbb{R}^n} g(x)\, dx < \infty$ and $0 < \varepsilon(x) < 1$ for $x \in G_1$. We will show that

$$\int_{G_1} f(x)^{q(x)}\, dx \left(= \int_{G_1} g(x)^{1-\varepsilon(x)}\, dx \right) < \infty. \tag{16.3}$$

Then inserting the definition of $\varepsilon(x)$, we obtain

$$\int_{G_1} \varepsilon(x) c^{1/\varepsilon(x)}\, dx \le \int_{G_1} (p(x) - q(x)) c^{1/(p(x)-q(x))}\, dx < \infty, \tag{16.4}$$

since $p(x) \ge 1$ and $c \le 1$. Let $G_3 = \{x \in G_1 : g(x) > \varepsilon(x) c^{1/\varepsilon(x)}\}$ and $G_4 = G_1 \setminus G_3$. Observe that $\varepsilon(x)^{-\varepsilon(x)} \le e^{1/e}$. If $x \in G_3$, then $g(x)^{-\varepsilon(x)} \le (\varepsilon(x) c^{1/\varepsilon(x)})^{-\varepsilon(x)} \le c^{-1} e^{1/e}$ and

$$\int_{G_3} g(x)^{1-\varepsilon(x)}\, dx \le c^{-1} e^{1/e} \int_{G_3} g(x)\, dx < \infty. \tag{16.5}$$

If $x \in G_4$, then $g(x)^{1-\varepsilon(x)} \le (\varepsilon(x) c^{1/\varepsilon(x)})^{1-\varepsilon(x)} \le c^{-1} e^{1/e} (\varepsilon(x) c^{1/\varepsilon(x)})$ and

$$\int_{G_4} g(x)^{1-\varepsilon(x)}\, dx \le c^{-1} e^{1/e} \int_{G_4} \varepsilon(x) c^{1/\varepsilon(x)}\, dx < \infty \tag{16.6}$$

by virtue of (16.4). Therefore, (16.3) follows from (16.5) and (16.6). $\square$

Proof of Theorem 15.3. Let $\int_{\mathbb{R}^n} |f(x)|^{p(x)}\, dx \le 1$ as we considered in Lemma 16.3. We may assume that f is nonnegative. Let $f_1 := f \chi_{\{x \in \mathbb{R}^n : f(x) \ge 1\}}$ and $f_2 := f - f_1$. We show that

$$\int_{\mathbb{R}^n} Mf(x)^{p(x)}\, dx \lesssim \int_{\mathbb{R}^n} Mf_1(x)^{p(x)}\, dx + \int_{\mathbb{R}^n} Mf_2(x)^{p(x)}\, dx < \infty.$$

Then we have the conclusion by Lemma 16.3.

For f_1, by using the same argument as in the proof of Theorem 15.1, in particular by using Lemma 16.1, we have

$$\int_{\mathbb{R}^n} Mf_1(x)^{p(x)}\,dx \le C\int_{\mathbb{R}^n} M[|f_1|^{p(\cdot)/p_-}](x)^{p_-}\,dx < \infty.$$

Next, we remark that $0 \le f_2(x) < 1$ and that

$$\int_{\mathbb{R}^n} f_2(x)^{p(x)}\,dx \le \int_{\mathbb{R}^n} f(x)^{p(x)}\,dx \le 1.$$

Then, by Lemma 16.4, we have

$$\int_{\mathbb{R}^n} f_2(x)^{p_\infty}\,dx < \infty.$$

By the boundedness of M on $L^{p_\infty}(\mathbb{R}^n)$, we have

$$\int_{\mathbb{R}^n} Mf_2(x)^{p_\infty}\,dx < \infty.$$

From the fact that $Mf_2(x) \le \|f_2\|_{L^\infty} \le 1$ for almost all $x \in \mathbb{R}^n$, we can use again Lemma 16.4 to obtain

$$\int_{\mathbb{R}^n} Mf_2(x)^{p(x)}\,dx < \infty.$$

Therefore, we have the conclusion. □

16.3 Proof of Lerner's theorem (Theorem 15.4) We will use Theorem 5.3 in the following form:

Proposition 16.5. *There exists a constant c_n, depending only on n, such that for any $\varphi \in \mathrm{BMO}(\mathbb{R}^n)$ with $\|\varphi\|_{\mathrm{BMO}} \le c_n$ one has $e^\varphi \in A_2$ with $\|e^\varphi\|_{A_2} \le 4$.*

Proof. Let b be the constant in Theorem 5.2 and let $c_n := b/3$. If $\|\varphi\|_{\mathrm{BMO}} \le c_n$, then

$$\int_Q e^{|\varphi(x)-\varphi_Q|}\,dx = |Q| + \int_1^\infty |\{x \in Q : e^{|\varphi(x)-\varphi_Q|} > \lambda\}|\,d\lambda \le |Q| + 2|Q|\int_1^\infty \lambda^{-3}\,d\lambda = 2|Q|.$$

Thus, it follows that

$$\begin{aligned}\|e^\varphi\|_{A_2} &= \sup_{Q\in\mathcal{Q}}\left(⨍_Q e^{\varphi(x)}\,dx\right)\left(⨍_Q e^{-\varphi(x)}\,dx\right)\\ &= \sup_{Q\in\mathcal{Q}}\left(⨍_Q e^{\varphi(x)-\varphi_Q}\,dx\right)\left(⨍_Q e^{-\varphi(x)+\varphi_Q}\,dx\right) \le 4.\end{aligned}$$

□

We consider another BMO estimate and we recall that we adopted the notation $Q(0,1) = (-1/2,1/2)^n$.

Lemma 16.6. *Let $p(\cdot) \in L^0(\mathbb{R}^n,[1,\infty))$ with $p_+ < \infty$. For any nonnegative function $f \in L^{p(\cdot)}(\mathbb{R}^n)$ with $\|f\|_{L^{p(\cdot)}} \le 1$, write $\tilde{f} := f + \chi_{Q(0,1)}$ and set $\varphi := \log(M\tilde{f})$. Then*

$$\|\varphi\|_{\mathrm{BMO}} + |\varphi_{Q(0,1)}| \le \tilde{\gamma}_n, \tag{16.7}$$

where $\tilde{\gamma}_n > 1$ depends only on n.

Proof. Let $f_1 = f\chi_{\{x\in\mathbb{R}^n\,:\,f(x)>1\}}$. Then $f_1 \in L^1(\mathbb{R}^n)$, since $0 \le f_1 \le f^{p(\cdot)}$ and $f^{p(\cdot)} \in L^1(\mathbb{R}^n)$. Hence

$$M\tilde{f}(x) \le M\chi_{Q(0,1)}(x) + Mf(x) \le 2 + Mf_1(x) < \infty, \quad \text{a.e. } x.$$

Let $\varphi = \log(M\tilde{f})$. By Theorem 5.4 we have

$$\|\varphi\|_{\mathrm{BMO}} \le \gamma_n. \tag{16.8}$$

Next, using the relation

$$(\log(1 + Mf))^* = \log(1 + (Mf)^*),$$

and (4.7), we have

$$0 \le \int_{Q(0,1)} \varphi(x)\,dx \le \int_{Q(0,1)} \log(1 + Mf(x))\,dx = \int_0^1 \log(1 + (Mf)^*(t))\,dt. \tag{16.9}$$

Let $0 < t < 1$. By (4.6) and (4.7) we have

$$(Mf)^*(t) \le \nu_n f^{**}(t) = \frac{\nu_n}{t} \sup_{|E|=t} \int_E f(x)\,dx. \tag{16.10}$$

Let E be any measurable set with $|E| = t$. Then $\|\chi_E\|_{L^{p'(\cdot)}} \le 1$, since

$$\int_{\mathbb{R}^n} \chi_E(x)^{p'(x)}\,dx = \int_E dx = t < 1.$$

Hence

$$\int_E f(x)\,dx \le 2\|f\|_{L^{p(\cdot)}}\|\chi_E\|_{L^{p'(\cdot)}} \le 2. \tag{16.11}$$

Thus, it follows from (16.9)–(16.11) that

$$\int_{Q(0,1)} \varphi(x)\,dx \le \int_0^1 \log(1 + 2\nu_n/t)\,dt := \gamma_n' < \infty.$$

That is,

$$|\varphi_{Q(0,1)}| \le \gamma_n'. \tag{16.12}$$

If we combine (16.8) and (16.12), then we obtain (16.7). □

Proof of Theorem 15.4. Let $p(\cdot) \in L^0(\mathbb{R}^n;\mathbb{R}) \cap \mathrm{PWM}(\mathrm{BMO}(\mathbb{R}^n))$. First we assume that $p(\cdot)$ is nonnegative and that $\|p\|_{L^\infty}$ and that the operator norm $\|p\|_{Op}$ are small enough as to have;

$$\|p\|_{Op}\tilde{\gamma}_n \le c_n \quad 0 \le p(x) \le 1/2,$$

where c_n is from Proposition 16.5. We use the notation in Lemma 16.6. For any nonnegative function $f \in L^{2-p(\cdot)}(\mathbb{R}^n)$ with $\|f\|_{L^{2-p(\cdot)}} \le 1$, by Theorem 5.7 and Lemma 16.6, we have

$$\|(-p)\log(M\tilde{f})\|_{\mathrm{BMO}} \le \|p\|_{Op}(|(\log(M\tilde{f}))_{Q(0,1)}| + \|\log(M\tilde{f})\|_{\mathrm{BMO}}) \le \|p\|_{Op}\tilde{\gamma}_n \le c_n.$$

Hence, by Proposition 16.5, $(M\tilde{f}(x))^{-p(x)}$ is an A_2-weight and its A_2 constant is less than or equal to 4. Then, using $M\tilde{f} \geq \tilde{f} \geq 0$ also, we have

$$\begin{aligned}\int_{\mathbb{R}^n} (Mf(x))^{2-p(x)}\,dx &\leq \int_{\mathbb{R}^n} (M\tilde{f}(x))^{2-p(x)}\,dx \\ &= \int_{\mathbb{R}^n} (M\tilde{f}(x))^2 (M\tilde{f}(x))^{-p(x)}\,dx \\ &\lesssim \int_{\mathbb{R}^n} (\tilde{f}(x))^2 (M\tilde{f}(x))^{-p(x)}\,dx \\ &\leq \int_{\mathbb{R}^n} (\tilde{f}(x))^{2-p(x)}\,dx \lesssim 1.\end{aligned}$$

This shows that M is bounded on $L^{2-p(\cdot)}(\mathbb{R}^n)$.

For general $p(\cdot) \in L^0(\mathbb{R}^n;\mathbb{R}) \cap \mathrm{PWM}(\mathrm{BMO}(\mathbb{R}^n))$, let $\tilde{p}(x) = (p_+ - p(x))/r$, for large $r > 1$. Then $\tilde{p}(\cdot)$ is nonnegative and $\|\tilde{p}\|_{Op}$ is small. Hence M is bounded on $L^{2-\tilde{p}(\cdot)}(\mathbb{R}^n)$. In this case M is bounded on $L^{r(2-\tilde{p}(\cdot))}(\mathbb{R}^n) = L^{2r+p(\cdot)-p_+}(\mathbb{R}^n)$ by Remark 15.1. The proof is complete. □

17 Counterexamples In this section, to guarantee the boundedness of M, we need to postulate some regularity assumption on $p(\,\cdot\,)$, we give several examples of $p(\,\cdot\,)$ for which the Hardy-Littlewood maximal operator M is not bounded on $L^{p(\,\cdot\,)}(\mathbb{R}^n)$ with $n = 1$.

We will use the following fundamental facts in Propositions 17.1 and 17.3, respectively: For $a > 0$,

$$M[|\cdot|^{-\theta}\chi_{(0,a]}](x) \geq C|x|^{-\theta}\chi_{[-a,0)\cup(0,a]}(x), \quad \text{if} \quad 0 < \theta < 1,$$

and

$$M[|\cdot|^{-\theta}\chi_{[a,\infty)}](x) \geq C|x|^{-\theta}\chi_{(-\infty,-a]\cup[a,\infty)}(x), \quad \text{if} \quad \theta > 0.$$

The authors learned these propositions below from Diening's talk.

The variable exponent $p(\,\cdot\,)$ in the following proposition doesn't satisfy the local log-Hölder continuity condition (14.1):

Proposition 17.1. *Let $n = 1$ and $p(\cdot) := 4\chi_{(-\infty,0)} + 2\chi_{[0,\infty)}$. Then the operator M is not bounded on $L^{p(\cdot)}(\mathbb{R})$.*

Proof. Let $f(x) := |x|^{-1/3}\chi_{(0,1)}(x)$. Then

$$\int_{-\infty}^{\infty} \left|\frac{f(x)}{\sqrt{3}}\right|^{p(x)} dx = \int_0^1 \left|\frac{x^{-1/3}}{\sqrt{3}}\right|^2 dx = \int_0^1 \frac{x^{-2/3}}{3}\,dx = 1.$$

Hence $\|f\|_{L^{p(\cdot)}} = \sqrt{3}$. On the other hand, for $x \in (-1, 0)$,

$$Mf(x) \geq \frac{1}{2|x|}\int_x^{-x} f(y)\,dy = \frac{1}{2|x|}\int_0^{-x} |x|^{-1/3}\,dy \geq \frac{|x|^{-1/3}}{2}.$$

Then, for any $\lambda > 1$,

$$\int_{-1}^0 \left|\frac{Mf(x)}{\lambda}\right|^4 dx \geq \frac{1}{(2\lambda)^4}\int_{-1}^0 |x|^{-4/3}\,dx = \infty.$$

That is, $\|Mf\|_{L^{p(\cdot)}} = \infty$. □

By the same argument as Proposition 17.1, we can prove the following:

Corollary 17.2. *Let $n = 1$ and $p(\cdot) := 2\chi_{(-\infty,-2]} + 4\chi_{(-2,0)} + 2\chi_{[0,\infty)}$. Then the operator M is not bounded on $L^{p(\cdot)}(\mathbb{R})$.*

The variable exponent $p(\,\cdot\,)$ in the following proposition doesn't satisfy the log-Hölder type decay condition (14.2):

Proposition 17.3. *Let $n = 1$ and $p(\cdot) : \mathbb{R} \to (0, \infty)$. If $p(x) \leq 2$ on $(-\infty, -k)$ and $p(x) \geq 4$ on $[k, \infty)$ for some $k \geq 0$, then the operator M is not bounded on $L^{p(\cdot)}(\mathbb{R})$.*

Proof. Let $f(x) := |x|^{-1/3}\chi_{[\max(1,k),\infty)}(x)$. Then

$$\int_{-\infty}^{\infty} \left|\frac{f(x)}{\sqrt[4]{3}}\right|^{p(x)} dx = \int_{\max(1,k)}^{\infty} \left|\frac{|x|^{-1/3}}{\sqrt[4]{3}}\right|^{4} dx \leq \int_{1}^{\infty} \frac{x^{-4/3}}{3}\, dx = 1.$$

Hence $\|f\|_{L^{p(\cdot)}} \leq \sqrt[4]{3}$. On the other hand, for $x < -2\max(1,k)$,

$$(1 \geq) Mf(x) \geq \frac{1}{2|x|}\int_{x}^{-x} f(y)\, dy \geq \frac{1}{2|x|}\int_{\max(1,k)}^{-x} |x|^{-1/3}\, dy \geq \frac{|x|^{-1/3}}{4}.$$

Then, for any $\lambda > 1$,

$$\int_{-\infty}^{\infty} \left|\frac{Mf(x)}{\lambda}\right|^{p(x)} dx \geq \int_{-\infty}^{-2\max(1,k)} \left|\frac{Mf(x)}{\lambda}\right|^{2} dx \geq \frac{1}{(4\lambda)^2}\int_{-\infty}^{-2\max(1,k)} |x|^{-2/3}\, dx = \infty.$$

That is, $\|Mf\|_{L^{p(\cdot)}} = \infty$. □

The next corollary follows immediately from the above proposition.

Corollary 17.4. *Let $n = 1$ and $p(\cdot) : \mathbb{R} \to (0, \infty)$ be a variable exponent. If*

$$\limsup_{x\to-\infty} p(x) < 2 \quad \liminf_{x\to\infty} p(x) > 4$$

then the operator M is not bounded on $L^{p(\cdot)}(\mathbb{R})$.

The next example shows that the log-Hölder type decay condition (14.2) is necessary in a sense.

Proposition 17.5 ([26])**.** *Fix $p_\infty \in (1, \infty)$. Let $\phi : [0, \infty) \to [0, p_\infty - 1)$ be such that*

$$\phi(0) = \lim_{x\to\infty} \phi(x) = 0, \quad \lim_{x\to\infty} \phi(x) \log x = \infty.$$

Assume in addition that ϕ is decreasing on $[1, \infty)$. Define

$$p(x) := p_\infty - \phi(\max(x, 0)) \quad (x \in \mathbb{R}).$$

Then M is not bounded on $L^{p(\cdot)}(\mathbb{R})$.

A key idea is that $M\chi_{(R,2R)} \geq \dfrac{1}{4}\chi_{(-2R,2R)}$.

Proof. Since $1 \leq p_\infty - \phi(2x, 0) \leq p_\infty$, we have

$$\lim_{x\to\infty}\left(1 - \frac{p_\infty}{p(2x)}\right)\log x = -\lim_{x\to\infty} \frac{\phi(\max(2x,0))\log x}{p_\infty - \phi(\max(2x,0))} = -\infty,$$

or equivalently,

$$\lim_{x\to\infty} x^{1-p_\infty/p(2x)} = 0.$$

Thus, we can find a negative sequence $\{c_n\}_{n=1}^{\infty}$ such that

$$c_{n+1} < 2c_n < -4, \quad |c_n|^{1-p_\infty/p(2|c_n|)} \le 2^{-n} \quad (\text{for all} n \in \mathbb{N}).$$

Define

$$f(x) := \sum_{n=1}^{\infty} |c_n|^{-1/p(2|c_n|)} \chi_{(2c_n,c_n)}(x).$$

Since

$$\int_{\mathbb{R}} |f(x)|^{p(x)}\, dx = \sum_{n=1}^{\infty} |c_n|^{-p_\infty/p(2|c_n|)} |c_n| \le 1,$$

we have $f \in L^{p(\cdot)}(\mathbb{R})$. Meanwhile, if $x \in (-c_n, -2c_n)$, then

$$Mf(x) \ge \frac{-1}{4c_n} \int_{2c_n}^{-2c_n} f(y)\, dy \ge \frac{-1}{4c_n} \int_{2c_n}^{c_n} f(y)\, dy = \frac{1}{4} |c_n|^{-1/p(-2c_n)}.$$

Hence

$$\begin{aligned}
\int_{\mathbb{R}} \{Mf(x)\}^{p(x)}\, dx &\ge \frac{1}{4} \sum_{n=1}^{\infty} \int_{-c_n}^{-2c_n} |c_n|^{-p(x)/p(-2c_n)}\, dx \\
&\ge \frac{1}{4} \sum_{n=1}^{\infty} \int_{-c_n}^{-2c_n} |c_n|^{-p(-2c_n)/p(-2c_n)}\, dx \\
&= \frac{1}{4} \sum_{n=1}^{\infty} 1 = \infty.
\end{aligned}$$

This shows that $Mf \notin L^{p(\cdot)}(\mathbb{R})$. □

Remark 17.1. Keep to the same setting as Proposition 17.5. The above proof shows that the Hardy operator

$$Hf(x) = \frac{1}{|x|} \int_{-|x|}^{|x|} f(t)\, dt \quad (x \in \mathbb{R})$$

is not bounded on $L^{p(\cdot)}(\mathbb{R})$.

The next example is from Cruz-Uribe's web page. This example shows that it does not suffice to assume the continuity solely.

Proposition 17.6. *For $x \in \mathbb{R}$, let $p(x) := 3 + \cos(2\pi x)$. Then, M is not bounded on $L^{p(\cdot)}(\mathbb{R})$.*

The point is again that M recovers the missing part of f defined by (17.1) below: $Mf(x) \ge C_0|x|^{-1/3}$ for all $x > 0$. See (17.3).

Proof. Note that $p(x) \ge 3 + \cos(\pi/4)$ for $x \in [j, j + 1/8]$, $j = 1, 2, \ldots$.

Let

$$f(x) := |x|^{-1/3} \sum_{j=1}^{\infty} \chi_{[0,1/8]}(x - j) \quad (x \in \mathbb{R}). \tag{17.1}$$

Then

$$\begin{aligned}(17.2)\qquad \int_{\mathbb{R}} |f(x)|^{p(x)}\,dx &= \sum_{j=1}^{\infty}\int_j^{j+1/8} |x|^{-p(x)/3}\,dx \\ &\le \sum_{j=1}^{\infty}\int_j^{j+1/8} |x|^{-(3+\cos(\pi/4))/3}\,dx \\ &\le \frac{1}{8}\sum_{j=1}^{\infty} j^{-(3+\cos(\pi/4))/3} < \infty.\end{aligned}$$

On the other hand, for $x \in (j, j+1)$, $j = 1, 2, \ldots$,

$$(17.3)\qquad Mf(x) \ge \int_j^{j+1} f(y)\,dy = \int_j^{j+1/8} |y|^{-1/3}\,dy \ge \frac{(j+1/8)^{-1/3}}{8} > \frac{(j+1)^{-1/3}}{8}.$$

Since $p(x) \le 3$ for $x \in [j+1/4, j+3/4]$, $j = 1, 2, \ldots$,

$$\begin{aligned}(17.4)\qquad \int_{\mathbb{R}} Mf(x)^{p(x)}\,dx &\ge \sum_{j=1}^{\infty}\int_j^{j+1} \left(\frac{(j+1)^{-1/3}}{8}\right)^{p(x)} dx \\ &\ge \sum_{j=1}^{\infty}\int_{j+1/4}^{j+3/4} \left(\frac{(j+1)^{-1/3}}{8}\right)^{3} dx \\ &= \frac{1}{2\times 8^3}\sum_{j=1}^{\infty}(j+1)^{-1} = \infty.\end{aligned}$$

(17.2) and (17.4) disprove that M is bounded on $L^{p(\cdot)}(\mathbb{R})$. □

Part IV
Related topics

In this part, we give results related to the boundedness of the Hardy-Littlewood maximal operators on $L^{p(\cdot)}(\mathbb{R}^n)$. For results in this part, refer also to surveys; Harjulehto and Hästö [63], Harjulehto, Hästö, Lê and Nuortio [66], Mizuta [121], S. Samko [189], and, a book; Diening, Harjulehto, Hästö and Růžička [40].

18 Modular inequalities In this section, we will make a supplemental but important remark about the proof of the boundedness of the Hardy-Littlewood maximal operator M. As mentioned in Subsection 14.2, we have to be careful when we prove the boundedness of M; it seems natural to try to prove

$$\int_{\mathbb{R}^n} Mf(x)^{p(x)}\,dx \le C\int_{\mathbb{R}^n} |f(x)|^{p(x)}\,dx.$$

However, this idea does not work. An interesting result is proved by Lerner [103], in which he used the A_∞-weights. In this section, we give an alternative proof. Our proof can be extended to the setting of the non-doubling measures readily.

Theorem 18.1 (Lerner [103, Theorem 1.1] (2005))**.** *Let $p(\cdot) \in L^0(\mathbb{R}^n; (1, \infty))$ be such that $1 < p_- \le p_+ < \infty$. Then the following two conditions are equivalent:*

(a) *There exists a constant $C > 0$ such that for all $f \in L^{p(\cdot)}(\mathbb{R}^n)$*

$$\int_{\mathbb{R}^n} Mf(x)^{p(x)}\,dx \le C \int_{\mathbb{R}^n} |f(x)|^{p(x)}\,dx. \tag{18.1}$$

(b) *The variable exponent $p(\cdot)$ equals to a constant.*

The implication (b) $\Rightarrow$ (a) is well known, see Section 4 for its proof. It counts that (a) $\Rightarrow$ (b) is true. This implies a difference between the norm inequality and the modular inequality (18.1). In particular we see that the inequality (18.1) shows a stronger condition than the norm one. Izuki [72] has considered the similar problems for some operators arising from multiresolution analysis and wavelets.

Here we shall supply a new proof without using the notion of A_∞-weights, which was obtained by carefully reexamining the original proof of Lerner [103].

Proof of Theorem 18.1. As is remarked above, the heart of the matters is to prove that (a) implies (b). The indicator function testing (18.1) essentially suffices. Assume that (a) holds and that $p(\cdot)$ is not a.e. equal to a constant function on a ball B. Let

$$p_-(B) := \operatorname*{ess\,inf}_{x \in B} p(x), \quad p_+(B) := \operatorname*{ess\,sup}_{x \in B} p(x).$$

For $\varepsilon > 0$, we write

$$E_\varepsilon := \{x \in B \,:\, p(x) > p_+(B) - \varepsilon\}.$$

Since $p_-(B) < p_+(B)$, there exists $\varepsilon > 0$ such that $p_+(B) - 2\varepsilon > p_-(B) + \varepsilon$. In this case we have $0 < |B \setminus E_{2\varepsilon}| < |B|$ and $|E_\varepsilon| > 0$ in view of the definition of $p_\pm(B)$.

Let $t > 1$. Then, from (18.1) by letting $f := t\chi_{B \setminus E_{2\varepsilon}}$, we obtain

$$\begin{aligned}
\int_B M[t\chi_{B\setminus E_{2\varepsilon}}](x)^{p(x)}\,dx &\le \int_{\mathbb{R}^n} M[t\chi_{B\setminus E_{2\varepsilon}}](x)^{p(x)}\,dx \\
&\le C \int_{\mathbb{R}^n} (t\chi_{B\setminus E_{2\varepsilon}}(x))^{p(x)}\,dx \\
&= C \int_{B\setminus E_{2\varepsilon}} t^{p(x)}\,dx \\
&\le C t^{p_+(B)-2\varepsilon} |B \setminus E_{2\varepsilon}|.
\end{aligned}$$

Since $M[t\chi_{B\setminus E_{2\varepsilon}}](x) \ge \frac{|B\setminus E_{2\varepsilon}|}{|B|}\chi_{E_\varepsilon}(x)t$, it follows that

$$\begin{aligned}
\int_B M[t\chi_{B\setminus E_{2\varepsilon}}](x)^{p(x)}\,dx &\ge \int_B \left(\frac{|B\setminus E_{2\varepsilon}|}{|B|}\right)^{p(x)} \chi_{E_\varepsilon}(x) t^{p(x)}\,dx \\
&\ge \left(\frac{|B\setminus E_{2\varepsilon}|}{|B|}\right)^{p_+(B)} |E_\varepsilon| t^{p_+(B)-\varepsilon}.
\end{aligned}$$

From both inequalities we have

$$t^\varepsilon \le C \left(\frac{|B|}{|B\setminus E_{2\varepsilon}|}\right)^{p_+(B)} \frac{|B\setminus E_{2\varepsilon}|}{|E_\varepsilon|},$$

for any $t > 1$. This is a contradiction. $\square$

The proof carries over the setting of the (non-doubling) metric measure spaces, where the notion of A_∞-weights is immature. Recall that in the metric measure space $(\mathcal{X}, d, \mu)$, the uncentered maximal operator

$$M'_k f(x) := \sup \left\{ \frac{1}{\mu(B(y,kr))} \int_{B(y,r)} |f(z)|\, d\mu(z) \,:\, B(y,r) \ni x \right\}$$

and the centered maximal operator

$$M_k f(x) := \sup \left\{ \frac{1}{\mu(B(x,kr))} \int_{B(x,r)} |f(y)|\, d\mu(y) \,:\, r > 0 \right\}$$

satisfy

$$\|M'_3 f\|_{L^p(\mu)} \le \frac{p\,2^p}{p-1} \|f\|_{L^p(\mu)}, \quad \|M_2 f\|_{L^p(\mu)} \le \frac{p\,2^p}{p-1} \|f\|_{L^p(\mu)}, \tag{18.2}$$

respectively. Here

$$\|f\|_{L^p(\mu)} := \left(\int_{\mathcal{X}} |f(x)|^p \, d\mu(x) \right)^{1/p}.$$

For estimates (18.2) for M'_3 and M_2 we refer to [160] and [201, 217] respectively.

Mimicking the above proof, we can prove the following for a measurable function $p(\cdot)$:

Theorem 18.2. *Let $p(\cdot) : \mathcal{X} \to [1, \infty)$ be a μ-measurable function.*

(i) *Let $k \ge 3$. If there exists a constant $C > 0$ such that*

$$\int_{\mathcal{X}} M'_k f(x)^{p(x)} \, d\mu(x) \le C \int_{\mathcal{X}} |f(x)|^{p(x)} \, d\mu(x),$$

if and only if $p(\cdot)$ is equal to a μ-a.e. constant function.

(ii) *Let $k \ge 2$. If there exists a constant $C > 0$ such that*

$$\int_{\mathcal{X}} M_k f(x)^{p(x)} \, d\mu(x) \le C \int_{\mathcal{X}} |f(x)|^{p(x)} \, d\mu(x),$$

if and only if $p(\cdot)$ is equal to a μ-a.e. constant function.

19 The norm of the characteristic function of a cube The following is a crucial inequality and it is used many times in Part V.

Lemma 19.1 ([154, Lemma 2.2])**.** *Suppose that $p(\cdot)$ is a function satisfying* (14.1), (14.2) *and $0 < p_- \le p_+ < \infty$.*

(i) *For all cubes $Q = Q(z,r)$ with $z \in \mathbb{R}^n$ and $r \le 1$, we have $|Q|^{1/p_-(Q)} \lesssim |Q|^{1/p_+(Q)}$. In particular, we have*

$$|Q|^{1/p_-(Q)} \sim |Q|^{1/p_+(Q)} \sim |Q|^{1/p(z)} \sim \|\chi_Q\|_{L^{p(\cdot)}}. \tag{19.1}$$

(ii) *For all cubes $Q = Q(z,r)$ with $z \in \mathbb{R}^n$ and $r \ge 1$, we have*

$$\|\chi_Q\|_{L^{p(\cdot)}} \sim |Q|^{1/p_\infty}.$$

Here the implicit constants in $\sim$ do not depend on z and $r > 0$.

Proof. If $r \leq 1$, then (14.3) yields $|Q|^{1/p(x)} \sim |Q|^{1/p(z)} \sim |Q|^{1/p_-(Q)} \sim |Q|^{1/p_+(Q)}$ for all $x \in Q$. Hence, it follows, for example, that

$$\int_{\mathbb{R}^n} \left(\frac{\chi_Q(x)}{|Q|^{1/p(z)}} \right)^{p(x)} dx \sim \int_{\mathbb{R}^n} \frac{\chi_Q(x)}{|Q|} \, dx = 1.$$

Consequently we obtain $\|\chi_Q\|_{L^{p(\cdot)}} \sim |Q|^{1/p(z)} \sim |Q|^{1/p_-(Q)} \sim |Q|^{1/p_+(Q)}$. Let $\{z_j\}_{j=1}^\infty$ be a rearrangement of $\mathbb{Z}^n$ and $\{Q_j\}_{j=1}^\infty = \{Q(z_j, 1)\}_{j=1}^\infty$ be cubes. Then, invoking the localization principle [67, Theorem 2.4], we have that, for $r > 1$,

$$\|\chi_Q\|_{L^{p(\cdot)}} \sim \left\| \|\chi_Q\|_{L^{p(\cdot)}(Q_j)} \right\|_{\ell^{p_\infty}} \sim |Q|^{1/p_\infty}.$$

Thus, the proof of the lemma is now complete. □

Remark 19.1. The equivalence (19.1) can be implicitly found in [39, Lemma 2.5].

20 Weight class A_p Recently it turns out that the theory of maximal operators on variable Lebesgue spaces has a lot to do with the theory of weights.

Recall that, by "a weight" w, we mean that it is a non-negative a.e. $\mathbb{R}^n$ and locally integrable function. Below we write

$$w(S) := \int_S w(x) \, dx$$

for a weight w and a measurable set S. Recall also that a weight w is said to satisfy the Muckenhoupt A_p condition, $1 \leq p < \infty$, if

$$[w]_{A_p} := [w]_{A_p(\mathbb{R}^n)} = \sup_{Q \in \mathcal{Q}} w_Q \left([w^{-1/(p-1)}]_Q \right)^{p-1} < \infty, \quad 1 < p < \infty, \tag{20.1}$$

and

$$[w]_{A_1} := [w]_{A_1(\mathbb{R}^n)} = \sup_{Q \in \mathcal{Q}} w_Q \left(\operatorname*{ess\,sup}_{x \in Q} \frac{1}{w(x)} \right) < \infty, \qquad p = 1. \tag{20.2}$$

Let A_p be the set of all weights satisfying the Muckenhoupt A_p condition.

Theorem 20.1 (Muckenhoupt [137]). *Let $w > 0$ a.e. $\mathbb{R}^n$ be a weight.*

(1) *If $1 < p < \infty$, then the following three conditions are equivalent:*

  (a) *$w \in A_p$.*

  (b) *The Hardy-Littlewood maximal operator M is bounded on $L^p_w(\mathbb{R}^n)$.*

  (c) *M is of weak type (p,p) on $L^p_w(\mathbb{R}^n)$, namely, for all $f \in L^p_w(\mathbb{R}^n)$ and all $\lambda > 0$,*

$$w\left(\{x \in \mathbb{R}^n \, : \, Mf(x) > \lambda\}\right)^{1/p} \leq C \, \lambda^{-1} \, \|f\|_{L^p_w}.$$

(2) *The following two conditions are equivalent:*

  (a) *$w \in A_1$.*

(b) *M is of weak type $(1,1)$ on $L^1_w(\mathbb{R}^n)$, namely, for all $f \in L^1_w(\mathbb{R}^n)$ and all $\lambda > 0$,*

$$w\left(\{x \in \mathbb{R}^n : Mf(x) > \lambda\}\right) \le C\,\lambda^{-1}\,\|f\|_{L^1_w}.$$

Example 20.1. Let $a \in \mathbb{R}$. We consider the power weight $|x|^a$ defined on $\mathbb{R}^n$.

(1) Let $1 < p < \infty$. Then the weight $|x|^a$ is the Muckenhoupt A_p weight if and only if $-n < a < n(p-1)$.

(2) The weight $|x|^a$ is the Muckenhoupt A_1 weight if and only if $-n < a \le 0$.

The theory carries over to the spaces on open sets. Let Ω be an open set in $\mathbb{R}^n$ and, for measurable functions f on Ω, define

$$Mf(x) := \sup_B \fint_{B\cap\Omega} |f(y)|\,dy, \quad \fint_{B\cap\Omega} |f(y)|\,dy := \frac{1}{|B|}\int_{B\cap\Omega} |f(y)|\,dy. \tag{20.3}$$

where the supremum is taken over all balls B containing x.

In analogy with (20.1) and (20.2) for an open set Ω, we write

$$[w]_{A_p(\Omega)} := \sup_Q w_{Q\cap\Omega}\left([w^{-1/(p-1)}]_{Q\cap\Omega}\right)^{p-1}, \quad 1 < p < \infty,$$

where Q runs over all cubes and

$$[w]_{A_1(\Omega)} := \sup_Q w_{Q\cap\Omega}\left(\operatorname*{ess\,sup}_{x\in Q\cap\Omega} \frac{1}{w(x)}\right).$$

For $1 \le p < \infty$, let $A_p(\Omega)$ be the set of all weights w such that $[w]_{A_p(\Omega)} < \infty$. We also define

$$[w]^{\mathrm{balls}}_{A_1(\Omega)} := \sup_B w_{B\cap\Omega}\left(\operatorname*{ess\,sup}_{x\in B\cap\Omega} \frac{1}{w(x)}\right), \tag{20.4}$$

where the supremum is taken over all balls B. Then $[w]^{\mathrm{balls}}_{A_1(\Omega)} \sim [w]_{A_1(\Omega)}$. Note that

$$[w]^{\mathrm{balls}}_{A_1(\Omega)} = \operatorname*{ess\,sup}_{x\in\Omega} \frac{Mw(x)}{w(x)},$$

or equivalently

$$Mw(x) \le [w]^{\mathrm{balls}}_{A_1(\Omega)}\, w(x) \quad \text{a.e. } x \in \Omega,$$

where M is the operator defined by (20.3).

The next theorem is an analogy of the result due to Lerner, Ombrosi and Pérez [106]. Let Q be a cube and $x \in \mathbb{R}^n$. Define $\mathcal{D}(Q)$ the set of all dyadic cubes with respect to Q. More precisely, let $Q = Q(x,r)$. Then a dyadic cube with respect to Q is a cube that can be expressed as

$$Q \cap (x + (r/2^{\nu+1})m + [0, r/2^{\nu+1}]^n), \quad m \in \mathbb{Z}^n,\ \nu = 0,1,2,\ldots.$$

Denote by $\mathcal{D}(Q)_x$ the subset of all cubes in $\mathcal{D}(Q)$ that contain x.

Theorem 20.2 (reverse Hölder inequality)**.** *Let Q be a cube. Let $w \in A_1(\Omega)$. Define*

$$M_{Q,\mathrm{dyadic};\Omega}w(x) := \sup_{R \in \mathcal{D}(Q)_x} \frac{1}{|R|} \int_{R \cap \Omega} w(y)\,dy \quad (x \in \mathbb{R}^n).$$

If we set $\delta := \frac{1}{2^{n+1}[w]^{\mathrm{balls}}_{A_1(\Omega)}}$, then we have

$$\left(\fint_{Q \cap \Omega} M_{Q,\mathrm{dyadic};\Omega}w(x)^{\delta} w(x)\,dx \right)^{\frac{1}{1+\delta}} \le 2 \fint_{Q \cap \Omega} w(x)\,dx$$

for all cubes Q.

Observe that $M_{Q,\mathrm{dyadic};\Omega}$ is controlled by M; $M_{Q,\mathrm{dyadic};\Omega}w \le CMw$.

Proof. First we note that, for any positive constant r, we have

$$[\min(w,r)]^{\mathrm{balls}}_{A_1(\Omega)} \le [w]^{\mathrm{balls}}_{A_1(\Omega)},$$

from the definition (20.4). Then, by replacing w with $\min(w,r)$ with $r > 0$, we can and do assume that $w \in L^{\infty}(\mathbb{R}^n)$. Abbreviate $\fint_{Q \cap \Omega} w(x)\,dx$ to μ. Then we have

$$\begin{aligned}
&\fint_{Q \cap \Omega} M_{Q,\mathrm{dyadic};\Omega}w(x)^{\delta} w(x)\,dx \\
&= \frac{1}{|Q|} \int_0^{\infty} \delta\lambda^{\delta-1} w\left\{x \in Q \cap \Omega \,:\, M_{Q,\mathrm{dyadic};\Omega}w(x) > \lambda\right\} d\lambda \\
&= \frac{1}{|Q|} \left(\int_0^{\mu} + \int_{\mu}^{\infty} \right) \delta\lambda^{\delta-1} w\left\{x \in Q \cap \Omega \,:\, M_{Q,\mathrm{dyadic};\Omega}w(x) > \lambda\right\} d\lambda \\
&\le \mu^{\delta+1} + \frac{1}{|Q|} \int_{\mu}^{\infty} \delta\lambda^{\delta-1} w\left\{x \in Q \cap \Omega \,:\, M_{Q,\mathrm{dyadic};\Omega}w(x) > \lambda\right\} d\lambda.
\end{aligned}$$

Let $\lambda > \mu$. Then we can decompose

$$\{x \in Q \cap \Omega \,:\, M_{Q,\mathrm{dyadic};\Omega}w(x) > \lambda\} = \bigcup_j Q_j \cap \Omega \tag{20.5}$$

into a union of dyadic cubes $\{Q_j\}_j$ with respect to Q such that

$$\frac{1}{|Q_j|} \int_{Q_j \cap \Omega} w(x)\,dx > \lambda \ge \frac{1}{2^n|Q_j|} \int_{Q_j \cap \Omega} w(x)\,dx = \frac{1}{2^n|Q_j|} w(Q_j \cap \Omega) \tag{20.6}$$

and that

$$|Q_j \cap Q_{j'}| = 0 \quad (j \ne j'). \tag{20.7}$$

Hence from (20.5)–(20.7) we have

$$\begin{aligned}
w\{x \in Q \cap \Omega \,:\, M_{Q,\mathrm{dyadic};\Omega}w(x) > \lambda\} &= \sum_j w(Q_j \cap \Omega) \\
&\le 2^n \sum_j |Q_j|\lambda \\
&= 2^n\lambda\,|\{x \in Q \cap \Omega \,:\, M_{Q,\mathrm{dyadic};\Omega}w(x) > \lambda\}|.
\end{aligned}$$

Inserting this estimate, we obtain

$$\begin{aligned}
&\frac{1}{|Q|}\int_\mu^\infty \delta\lambda^{\delta-1} w\left\{x \in Q\cap\Omega \,:\, M_{Q,\mathrm{dyadic};\Omega}w(x)>\lambda\right\} d\lambda \\
&\le \frac{2^n}{|Q|}\int_\mu^\infty \delta\lambda^{\delta} \left|\left\{x \in Q\cap\Omega \,:\, M_{Q,\mathrm{dyadic};\Omega}w(x)>\lambda\right\}\right| d\lambda \\
&\le \frac{2^n}{|Q|}\int_0^\infty \delta\lambda^{\delta} \left|\left\{x \in Q\cap\Omega \,:\, M_{Q,\mathrm{dyadic};\Omega}w(x)>\lambda\right\}\right| d\lambda \\
&= \frac{2^n\delta}{1+\delta} \fint_{Q\cap\Omega} M_{Q,\mathrm{dyadic};\Omega}w(x)^{1+\delta}\, dx.
\end{aligned}$$

Therefore, it follows that

$$\begin{aligned}
\fint_{Q\cap\Omega} M_{Q,\mathrm{dyadic};\Omega}w(x)^{\delta} w(x)\, dx &\le \mu^{\delta+1} + \frac{2^n\delta}{\delta+1}\fint_{Q\cap\Omega} M_{Q,\mathrm{dyadic};\Omega}w(x)^{1+\delta}\, dx \\
&\le \mu^{\delta+1} + \frac{2^n\delta}{\delta+1}\fint_{Q\cap\Omega} M_{Q,\mathrm{dyadic};\Omega}w(x)^{\delta} Mw(x)\, dx \\
&\le \mu^{\delta+1} + \frac{2^n\delta[w]_{A_1(\Omega)}^{\mathrm{balls}}}{\delta+1}\fint_{Q\cap\Omega} M_{Q,\mathrm{dyadic};\Omega}w(x)\, dx \\
&\le \mu^{\delta+1} + \frac{1}{2}\fint_{Q\cap\Omega} M_{Q,\mathrm{dyadic};\Omega}w(x)^{\delta} w(x)\, dx.
\end{aligned}$$

Now that we are assuming that $w \in L^\infty(\mathbb{R}^n)$, it follows from the absorbing argument that

$$\fint_{Q\cap\Omega} M_{Q,\mathrm{dyadic};\Omega}w(x)^{\delta} w(x)\, dx \le 2\mu^{\delta+1} = 2\left(\fint_{Q\cap\Omega} w(x)\, dx\right)^{1+\delta}.$$

The proof is therefore complete. $\square$

Remark 20.1. Since

$$w^{1+\delta}(x) \le M_{Q,\mathrm{dyadic};\Omega}w(x)^{\delta} w(x) \quad (a.e. x \in \mathbb{R}^n),$$

using Theorem 20.2 we have the reverse Hölder inequality for $w \in A_1(\Omega)$, that is,

$$\left(\fint_{Q\cap\Omega} w^{1+\delta}(x)\, dx\right)^{1/(1+\delta)} \le 2\fint_{Q\cap\Omega} w(x)\, dx.$$

21 Boundedness of the Hardy-Littlewood maximal operator on domains In this section we recall some known results. To formulate results let us use the following notations, which are standard in the setting of variable exponents:

Recall the definition of the Hardy-Littlewood maximal operator M on the domain $\Omega \subset \mathbb{R}^n$;

$$Mf(x) = \sup_{B \ni x}\fint_B |f(y)|\, dy, \quad \fint_B |f(y)|\, dy := \frac{1}{|B|}\int_{B\cap\Omega} |f(y)|\, dy.$$

We write

$$p_- := \operatorname*{ess\,inf}_{x\in\Omega} p(x), \quad p_+ := \operatorname*{ess\,sup}_{x\in\Omega} p(x).$$

Definition 21.1.

(1) The set $\mathcal{P}(\Omega)$ consists of all variable exponents $p(\cdot) : \Omega \to [1, \infty]$ such that $1 < p_- \leq p_+ < \infty$.

(2) The set $\mathcal{B}(\Omega)$ consists of all variable exponents $p(\cdot) \in \mathcal{P}(\Omega)$ such that the Hardy-Littlewood maximal operator M is bounded on $L^{p(\cdot)}(\Omega)$.

(3) A measurable function $r(\cdot) : \Omega \to (0, \infty)$ is said to be locally log-Hölder continuous if there exists a positive constant C such that

$$|r(x) - r(y)| \leq \frac{C}{-\log(|x-y|)} \quad (|x-y| \leq 1/2)$$

is satisfied. The set $LH_0(\Omega)$ consists of all locally log-Hölder continuous functions.

(4) A measurable function $r(\cdot) : \Omega \to (0, \infty)$ is said to be log-Hölder type decay condition at ∞ if there exist positive constants C and r_∞ such that

$$|r(x) - r_\infty| \leq \frac{C}{\log(e + |x|)} \quad (x \in \Omega).$$

The set $LH_\infty(\Omega)$ consists of all measurable functions satisfying the log-Hölder decay condition at ∞.

(5) The set $LH(\Omega)$ consists of all measurable functions satisfying the two log-Hölder continuous properties above, namely, $LH(\Omega) := LH_0(\Omega) \cap LH_\infty(\Omega)$.

Before we proceed further, a helpful remark may be in order.

Remark 21.1. We can easily check the following facts:

(1) Given a measurable function $r(\cdot) : \Omega \to (0, \infty)$, we see that the following two conditions are equivalent:

 (a) $r(\cdot) \in LH_\infty(\Omega)$.

 (b) There exists a positive constant C such that

$$|r(x) - r(y)| \leq \frac{C}{\log(e + |x|)} \quad (|y| \geq |x|)$$

(2) Let a variable exponent $p(\cdot) : \Omega \to [1, \infty)$ satisfy $p_+ < \infty$. Then $p(\cdot) \in LH(\Omega)$ if and only if $1/p(\cdot) \in LH(\Omega)$.

(3) Let $p(\cdot) \in \mathcal{P}(\Omega)$. Then $p(\cdot) \in LH(\Omega)$ holds if and only if $1/p(\cdot) \in LH(\Omega)$ holds.

There are some famous results on sufficient conditions of variable exponents for the boundedness of the Hardy-Littlewood maximal operator. If a variable exponent $p(\cdot) : \Omega \to [1, \infty]$ satisfies $1 < p_- \leq p_+ \leq \infty$, we define

$$\|f\|_{L^{p(\cdot)}(\Omega)} := \|\chi_{\{x \in \Omega \,:\, p(x) < \infty\}} f\|_{L^{p(\cdot)}(\Omega)} + \|\chi_{\{x \in \Omega \,:\, p(x) = \infty\}} f\|_{L^\infty(\Omega)}.$$

Proposition 21.1.

(1) [36] (2004): *If Ω is bounded, then $\mathcal{P}(\Omega) \cap LH_0(\Omega) \subset \mathcal{B}(\Omega)$.*

(2) [26] (2004): *Let Ω be an open set of $\mathbb{R}^n$. Then $\mathcal{P}(\Omega) \cap LH(\Omega) \subset \mathcal{B}(\Omega)$.*

(3) [20, 39] (2009): *If a variable exponent* $p(\cdot) : \mathbb{R}^n \to [1,\infty]$ *satisfies* $1 < p_- \le p_+ \le \infty$ *and* $1/p(\cdot) \in LH(\mathbb{R}^n)$*, then the Hardy-Littlewood maximal operator* M *is bounded on* $L^{p(\cdot)}(\mathbb{R}^n)$*.*

Next we state a necessary condition for the boundedness of the Hardy-Littlewood maximal operator.

Proposition 21.2 ([39])**.** *Let* Ω *be a subset in* $\mathbb{R}^n$ *with positive measure. Let* $p(\cdot) : \Omega \to [1,\infty]$ *be a variable exponent. If* M *is bounded on* $L^{p(\cdot)}(\Omega)$*, then* $p_- > 1$ *holds.*

The proof is originally by Diening, Harjulehto, Hästö, Mizuta and Shimomura [39]. However, Lerner extended this result to Banach function spaces when $\Omega = \mathbb{R}^n$ (see [105, Theorem 1.2] and [107, Corollary 1.3]). Here we transform Lerner's proof to our setting. Denote by M^j the j-fold composition of M.

Proof of Proposition 21.2. First we show that, if M is bounded on $L^{p(\cdot)}(\Omega)$, then M is also bounded on $L^{p(\cdot)/(1+\delta)}(\Omega)$ for some $\delta > 0$: Since M is assumed bounded on $L^{p(\cdot)}(\Omega)$, there exists a constant $C_0 > 0$ such that

$$\|Mf\|_{L^{p(\cdot)}(\Omega)} \le C_0 \|f\|_{L^{p(\cdot)}(\Omega)}.$$

Define

$$g(x) := \sum_{j=0}^{\infty} \frac{1}{(2C_0)^j} M^j f(x),$$

where it will be understood that $M^0 f(x) = |f(x)|$. Observe also that $\|g\|_{L^{p(\cdot)}} \sim \|f\|_{L^{p(\cdot)}}$. Since M is sublinear, we have

$$\begin{aligned} Mg(x) &= M\left[\sum_{j=0}^{\infty} \frac{1}{(2C_0)^j} M^j f\right](x) \\ &= \lim_{J\to\infty} M\left[\sum_{j=0}^{J} \frac{1}{(2C_0)^j} M^j f\right](x) \\ &\le \sum_{j=0}^{\infty} \frac{1}{(2C_0)^j} M^{j+1} f(x) \\ &\le 2C_0 g(x). \end{aligned}$$

This means that g is an A_1-weight and that the A_1-norm is less than $2C_0$. Thus, we are in the position of using the reverse Hölder inequality (Theorem 20.2 and Remark 20.1) and we obtain

$$M[g^{1+\delta}](x) \le C|g(x)|^{1+\delta} \quad (x \in \Omega).$$

Here the constants C and δ depend only upon n and C_0. Thus, we obtain

$$\|M[|f|^{1+\delta}]\|_{L^{p(\cdot)/(1+\delta)}(\Omega)} \le \|M[g^{1+\delta}]\|_{L^{p(\cdot)/(1+\delta)}(\Omega)} \le C(\|g\|_{L^{p(\cdot)}(\Omega)})^{1+\delta} \le C(\|f\|_{L^{p(\cdot)}(\Omega)})^{1+\delta}.$$

The function $f \in L^{p(\cdot)}(\Omega)$ being arbitrary, it follows that the operator M is bounded on $L^{p(\cdot)/(1+\delta)}(\Omega)$.

Next, with this in mind, assume that M is bounded on $L^{p(\cdot)}(\Omega)$ with $p_- = 1$. Then M is also bounded on $L^{p(\cdot)/(1+\delta)}(\Omega)$ for some $\delta > 0$. In this case, the set

$$U := \left\{ x \in \Omega \cap B(0,R) : \frac{p(x)}{1+\delta} \le \frac{1}{1+\delta/2} \right\}$$

has positive measure for large $R > 0$. Hence there exists $f \in L^{p(\cdot)/(1+\delta)}(\Omega)$ such that $\int_{\Omega \cap B(0,R)} |f(x)|\,dx = \infty$. For example, we partition U into a collection $\{U_j\}_{j=1}^{\infty}$ of measurable sets such that

$$|U_j| = 2^{-j}|U|,\ j = 1, 2, \dots,$$

and we let

$$f := \sum_{j=1}^{\infty} |U_j|^{-1} \chi_{U_j}.$$

Then $f \in L^{p(\cdot)/(1+\delta)}(\Omega)$ and $Mf \equiv \infty$ on Ω. Actually, by the generalized Hölder inequality (Theorem 9.1) we have

$$\|f\|_{L^{p(\cdot)/(1+\delta)}(\Omega)} = \|f\chi_U\|_{L^{p(\cdot)/(1+\delta)}(\Omega)} \le C_R \|f\|_{L^{1/(1+\delta/2)}(\Omega)} = C_R \left(\sum_{j=1}^{\infty} |U_j|^{\delta/2} \right)^{1+\delta/2} < \infty,$$

and

$$Mf(x) \ge \frac{1}{|B(x, |x| + 2R)|} \int_{\Omega \cap B(x,|x|+2R)} f(y)\,dy = \infty \quad (x \in \Omega).$$

Hence the inequality $\|Mf\|_{L^{p(\cdot)/(1+\delta)}(\Omega)} \le C\|f\|_{L^{p(\cdot)/(1+\delta)}(\Omega)}$ fails. This is a contradiction. Therefore, we have the conclusion. □

22 Weighted Lebesgue spaces with variable exponents In this section we state known results on weighted Lebesgue spaces $L^{p(\cdot)}_w(\mathbb{R}^n)$ with variable exponents without proof. First we define the space $L^{p(\cdot)}_w(\mathbb{R}^n)$ as the following:

Definition 22.1. Let $p(\cdot) \in L^0(\mathbb{R}^n; [1, \infty))$. Suppose that a measurable function w satisfies that $0 < w(x) < \infty$ a.e. $x \in \mathbb{R}^n$ and $w^{1/p(\cdot)} \in L^{p(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n)$. Then $L^{p(\cdot)}_w(\mathbb{R}^n)$ is the set of all $f \in L^0(\mathbb{R}^n)$ such that

$$\|f\|_{L^{p(\cdot)}_w} := \|f w^{1/p(\cdot)}\|_{L^{p(\cdot)}} < \infty.$$

22.1 Muckenhoupt weights with variable exponents The classical Muckenhoupt A_p class has been generalized to the setting $A_{p(\cdot)}$ of variable exponents by [21, 29, 41] and some equivalent conditions to the boundedness of M on $L^{p(\cdot)}_w(\mathbb{R}^n)$ has been given (see also [24, 86]).

Definition 22.2. For a variable exponent $p(\cdot) \in L^0(\mathbb{R}^n; [1, \infty))$, a measurable function w is said to be an $A_{p(\cdot)}$ weight if $0 < w(x) < \infty$ a.e. $x \in \mathbb{R}^n$ and

$$\sup_Q \frac{1}{|Q|} \|w^{1/p(\cdot)}\chi_Q\|_{L^{p(\cdot)}} \|w^{-1/p(\cdot)}\chi_Q\|_{L^{p'(\cdot)}} < \infty \tag{22.1}$$

holds, where the supremum is taken over all open cubes $Q \subset \mathbb{R}^n$ whose sides are parallel to the coordinate axes and $p'(\cdot)$ is the conjugate exponent of $p(\cdot)$, that is, $1/p(x) + 1/p'(x) = 1$. Note that $p'(\cdot) : \mathbb{R}^n \to (1, \infty]$ when $p(\cdot) : \mathbb{R}^n \to [1, \infty)$. The set $A_{p(\cdot)}$ consists of all $A_{p(\cdot)}$ weights.

If $p(\cdot)$ is a constant p, then $A_{p(\cdot)}$ is the classical A_p class.

The following is an extension of Theorem 20.1.

Theorem 22.1 ([21, 29, 41])**.** *Suppose that $p(\cdot) \in LH(\mathbb{R}^n)$ and $p_+ < \infty$. If $p_- > 1$, then the following three conditions are equivalent:*

(C1) $w \in A_{p(\cdot)}$.

(C2) *The Hardy-Littlewood maximal operator* M *is bounded on* $L_w^{p(\cdot)}(\mathbb{R}^n)$.

(C3) M *is of weak type* $(p(\cdot), p(\cdot))$ *on* $L_w^{p(\cdot)}(\mathbb{R}^n)$, *namely*

$$\|\chi_{\{Mf>\lambda\}} w^{1/p(\cdot)}\|_{L^{p(\cdot)}} \le C\lambda^{-1} \|f w^{1/p(\cdot)}\|_{L^{p(\cdot)}}$$

for all $\lambda > 0$ *and all* $f \in L_w^{p(\cdot)}(\mathbb{R}^n)$.

If $p_- \ge 1$, *then two conditions* (C1) *and* (C3) *are equivalent.*

22.2 Remarks on weighted norms It seems that there are two notations in weighted Lebesgue spaces with variable exponents. Let $w \in L^0(\mathbb{R}^n; [0,\infty))$. Two different expressions are in order;

$$\|f w^{1/p(\cdot)}\|_{L^{p(\cdot)}} \quad \text{and} \quad \|f w\|_{L^{p(\cdot)}}.$$

For example, the former is used in [200], and, the latter is used in [90, 91, 98]. See [89, 90, 91, 93, 94, 95, 96, 97] for related results.

23 Density in Sobolev spaces with variable exponents In this section we give alternative proofs for two theorems on density.

Recall that the Schwartz class is defined by

$$\mathcal{S}(\mathbb{R}^n) := \{u \in C^\infty(\mathbb{R}^n) \,:\, \sup_{x\in\mathbb{R}^n} |x^\alpha D^\beta u(x)| < \infty \text{ for all } \alpha,\, \beta \in {\mathbb{N}_0}^n\}.$$

The Schwartz space $\mathcal{S}(\mathbb{R}^n)$ is topologized by the family $\{p_N\}_{N\in\mathbb{N}}$, where

$$p_N(\varphi) = \sum_{|\alpha|\le N} \sup_{x\in\mathbb{R}^n} (1+|x|)^N |D^\alpha \varphi(x)|.$$

As the topological dual, $\mathcal{S}'(\mathbb{R}^n)$ is defined and usually it is equipped with the weak-* topology. We aim here to deal with Sobolev spaces associated to variable Lebesgue spaces. This is initially considered by [101] and independently investigated to [49]. Fan and Zhao [49, p. 444–445] gave an important remark on the variational problem.

23.1 Sobolev spaces based on Banach function spaces Given a function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $\alpha \in {\mathbb{N}_0}^n$, we define the derivative $D^\alpha f$ in the weak sense by

$$\int_{\mathbb{R}^n} D^\alpha f(x) u(x)\, dx = (-1)^{|\alpha|} \int_{\mathbb{R}^n} f(x) D^\alpha u(x)\, dx \quad (u \in \mathcal{S}(\mathbb{R}^n)).$$

Definition 23.1. Let $s \in \mathbb{N}$ and $X(\mathbb{R}^n) \subset L^1_{\mathrm{loc}}(\mathbb{R}^n)$ be a subspace equipped with a norm $\|\cdot\|_X$. Suppose that for every $f \in X(\mathbb{R}^n)$ there exists $N \in \mathbb{N}$ such that

$$\int_{\mathbb{R}^n} |f(x)\varphi(x)|\, dx \le N \times p_N(\varphi) \quad (\varphi \in \mathcal{S}(\mathbb{R}^n)).$$

The Sobolev space $X_s(\mathbb{R}^n)$ and its norm are defined respectively by

$$\begin{aligned} X_s(\mathbb{R}^n) &:= \left\{f \in X(\mathbb{R}^n) \,:\, D^\alpha f \in X(\mathbb{R}^n) \text{ for all } \alpha \in {\mathbb{N}_0}^n,\, |\alpha| \le s\right\}, \\ \|f\|_{X_s} &:= \sum_{|\alpha|\le s} \|D^\alpha f\|_X. \end{aligned}$$

The above is a very general framework. Here we survey a recent result in connection with $L^{p(\cdot)}(\mathbb{R}^n) =: X$. Assume that $p(\cdot)$ satisfies (14.1) and (14.2) as well as $1 < p_- \le p_+ < \infty$. Then in [154], we proved that

$$\|f\|_{L^{p(\cdot)}} \sim \left\| \left(\sum_{j=-\infty}^{\infty} |\mathcal{F}^{-1}[\varphi(2^{-j}\cdot)\mathcal{F}f]|^2 \right)^{1/2} \right\|_{L^{p(\cdot)}}, \tag{23.1}$$

where $\varphi \in \mathcal{S}(\mathbb{R}^n)$ satisfies

$$\mathrm{supp}(\varphi) \subset B(8) \setminus B(1) \text{ and } \sum_{j=-\infty}^{\infty} \varphi(2^{-j}\xi) \equiv \chi_{\mathbb{R}^n\setminus\{0\}}(\xi).$$

Thus, by using the vector-valued boundedness of the Hardy-Littlewood maximal operator, we have

$$\|f\|_{L^{p(\cdot)}_s} \sim \left\| \left(\sum_{j=-\infty}^{\infty} (1+2^{2js})|\mathcal{F}^{-1}[\varphi(2^{-j}\cdot)\mathcal{F}f]|^2 \right)^{1/2} \right\|_{L^{p(\cdot)}}.$$

Let $\varphi_j(D)f$ denote the function given by (1.3) with φ replaced by φ_j. Note that

$$f = \sum_{j=-\infty}^{\infty} \varphi_j(D)f$$

takes place in $\mathcal{S}'(\mathbb{R}^n)$. See [99], where the case of rectangle Littlewood-Paley patch is investigated. Indeed, we can characterize $L^{p(\cdot)}(\mathbb{R}^n)$ by means of the rectangle Littlewood-Paley patch if and only if $p(\cdot)$ is constant. See [76] for a similar approach, where Izuki used wavelet.

We remark that (23.1) above is a consequence of the extrapolation result in [25]. We refer to [73, 100] for related results.

Remark that Almeida and S. Samko characterized $X_s(\mathbb{R}^n)$ by using the Fourier multiplier, see [4].

When we study the differential equations

$$\mathrm{div}(|\nabla u(x)|^{p(x)-2}\nabla u(x)) = |u(x)|^{\sigma(x)-1}u(x) + f(x),$$

we need to deal with the Dirichlet integral of the form

$$\int_\Omega \left(|\nabla f(x)|^{p(x)} + |u(x)|^{\sigma(x)} \right) dx.$$

Therefore, $W^{m,p(\cdot)}(\Omega)$ is a natural function space. See [189, p. 461].

23.2 Fundamental results Now we state and reprove two theorems on density. Recall that $\mathcal{B}(\mathbb{R}^n)$ is the set of all measurable functions $p(\cdot) : \mathbb{R}^n \to [1, \infty]$ such that M is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$ (see (15.1)), namely, there exists a constant $C > 0$ such that

$$\|Mf\|_{L^{p(\cdot)}} \le C\|f\|_{L^{p(\cdot)}}$$

for all $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. The following result is proved by Diening:

Theorem 23.1 (Diening [37])**.** *If $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, then $C^\infty_{\mathrm{comp}}(\mathbb{R}^n)$ is dense in $L^{p(\cdot)}_s(\mathbb{R}^n)$.*

Recall that the set $LH_0(\mathbb{R}^n)$ consists of all locally log-Hölder continuous functions.

Theorem 23.2 (Cruz-Uribe and Fiorenza [22]). *If $p(\cdot) \in LH_0(\mathbb{R}^n)$ and $1 \le p_- \le p_+ < \infty$, then $C^\infty_{\mathrm{comp}}(\mathbb{R}^n)$ is dense in $L^{p(\cdot)}_s(\mathbb{R}^n)$.*

We will give alternative proofs of Theorems 23.1 and 23.2 above. In order to prove Theorem 23.1 we invoke the next theorem due to Nakai, Tomita and Yabuta [156].

Theorem 23.3 (Nakai, Tomita and Yabuta [156]). *Let $X(\mathbb{R}^n)$ be a subspace of $L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Assume the following four conditions:*

(1) *$\chi_B \in X(\mathbb{R}^n)$ for all open balls $B \subset \mathbb{R}^n$.*

(2) *If $g \in X(\mathbb{R}^n)$ and f is a measurable function such that $|f| \le |g|$ a.e. on $\mathbb{R}^n$, then $f \in X(\mathbb{R}^n)$.*

(3) *If $g \in X(\mathbb{R}^n)$, and each f_j $(j = 1, 2, \dots)$ is a measurable function such that $|f_j| \le |g|$ a.e. on $\mathbb{R}^n$ and that $\lim_{j\to\infty} f_j = 0$ a.e. on $\mathbb{R}^n$, then $\lim_{j\to\infty} \|f_j\|_X = 0$.*

(4) *The Hardy-Littlewood maximal operator M is bounded on $X(\mathbb{R}^n)$.*

Then $C^\infty_{\mathrm{comp}}(\mathbb{R}^n)$ is dense in $X_s(\mathbb{R}^n)$.

We give a proof of Theorem 23.3 later for convenience. Theorem 23.1 is a direct consequence of Theorem 23.3.

Proof of Theorem 23.1. We suppose $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ and we shall apply Theorem 23.3 with $X = L^{p(\cdot)}(\mathbb{R}^n)$. Theorem 23.3 (1), (2) and (4) are obviously true. We shall check (3). If $g \in L^{p(\cdot)}(\mathbb{R}^n)$, $|f_j| \le |g|$ $(j = 1, 2, \dots)$ a.e. $\mathbb{R}^n$ and $\lim_{j\to\infty} f_j = 0$ a.e. $\mathbb{R}^n$, then we have

$$\rho_p(f_j) = \int_{\mathbb{R}^n} |f_j(x)|^{p(x)}\, dx \le \int_{\mathbb{R}^n} |g(x)|^{p(x)}\, dx, \quad |g|^{p(\cdot)} \in L^1(\mathbb{R}^n).$$

Thus by the Lebesgue dominated convergence theorem we obtain

$$\lim_{j\to\infty} \rho_p(f_j) = \int_{\mathbb{R}^n} \lim_{j\to\infty} |f_j(x)|^{p(x)}\, dx = 0.$$

Therefore we get $\lim_{j\to\infty} \|f_j\|_{L^{p(\cdot)}} = 0$ by Theorem 10.1. $\square$

Note that we can prove the following by the same way as Theorem 23.1.

Theorem 23.4. *Let $p(\cdot) \in LH$, $1 < p_- \le p_+ < \infty$ and $w \in A_{p(\cdot)}$. Then $C^\infty_{\mathrm{comp}}(\mathbb{R}^n)$ is dense in $(L^{p(\cdot)}_w)_s(\mathbb{R}^n)$.*

Now we prove Theorem 23.3. Note that the assumptions (1) and (2) imply that $C^\infty_{\mathrm{comp}}(\mathbb{R}^n) \subset X_s(\mathbb{R}^n)$. We will use the following lemma:

Lemma 23.5. *Define*

$$X_{s,\mathrm{comp}}(\mathbb{R}^n) := \{f \in X_s(\mathbb{R}^n) : \mathrm{supp}(f) \text{ is compact}\}$$

and assume the condition (3) of Theorem 23.3. Then, $X_{s,\mathrm{comp}}(\mathbb{R}^n)$ is dense in $X_s(\mathbb{R}^n)$.

Proof. Take a cut-off function $\zeta \in C^\infty_{\mathrm{comp}}(\mathbb{R}^n)$ so that

$$0 \le \zeta \le 1, \quad \zeta(x) = \begin{cases} 1 & (|x| \le 1), \\ 0 & (|x| > 2). \end{cases}$$

Given a function $f \in X_s(\mathbb{R}^n)$, we define

$$f_j(x) := f(x)\zeta(x/j) \quad (j \in \mathbb{N}).$$

Then we have $f_j \in X_{s,\mathrm{comp}}(\mathbb{R}^n)$ and by condition (3),

$$\lim_{j\to\infty} \|f - f_j\|_{X_s} = 0.$$

Thus, the proof is complete. □

Proof of Theorem 23.3. First note that (1) and (2) imply that $C^\infty_{\mathrm{comp}}(\mathbb{R}^n)$ is a subset of $X(\mathbb{R}^n)$. Fix a non-negative and radial decreasing function $\psi \in C^\infty_{\mathrm{comp}}(\mathbb{R}^n)$ such that $\|\psi\|_{L^1} = 1$ and define ψ_t by (4.4) as before. By virtue of Lemma 23.5, we shall prove

$$\lim_{t\to 0} \|f - \psi_t * f\|_{X_s} = 0 \quad \text{for all } f \in X_{s,\mathrm{comp}}(\mathbb{R}^n). \tag{23.2}$$

Remark that

$$D^\alpha(\psi_t * f)(x) = \int_{\mathbb{R}^n} (D^\alpha f)(x-y)\psi_t(y)\,dy$$

for every $\alpha \in {\mathbb{N}_0}^n$ with $|\alpha| \le s$. Thus if we prove

$$\lim_{t\to 0} \|f - \psi_t * f\|_X = 0 \quad \text{for all } f \in X(\mathbb{R}^n) \text{ with compact support,} \tag{23.3}$$

then (23.2) is obtained. Take $f \in X(\mathbb{R}^n)$ with compact support. Then Lemma 4.5 gives us the estimate

$$|\psi_t * f(x)| \le Mf(x)$$

and due to condition (4) we see that $Mf \in X(\mathbb{R}^n)$. On the other hand, we have that $\lim_{t\to 0}(f - \psi_t * f) = 0$ a.e. $\mathbb{R}^n$. Therefore, by virtue of condition (3), we conclude that $\lim_{t\to 0} \|f - \psi_t * f\|_X = 0$. □

Next we give a proof of Theorem 23.2. In order to prove the theorem, we will use the following lemmas:

Lemma 23.6. *If a variable exponent $p(\cdot) : \mathbb{R}^n \to [1,\infty)$ satisfies $p_+ < \infty$, then the set*

$$L^\infty_{\mathrm{comp}}(\mathbb{R}^n) := \{f \in L^\infty(\mathbb{R}^n) : \mathrm{supp}(f) \text{ is compact}\}$$

is dense in $L^{p(\cdot)}(\mathbb{R}^n)$.

Proof. Take $f \in L^{p(\cdot)}(\mathbb{R}^n)$ and $\varepsilon > 0$ arbitrarily. By Theorem 13.1 we can take a bounded function $g \in L^{p(\cdot)}(\mathbb{R}^n)$ so that $\|f - g\|_{L^{p(\cdot)}} < \varepsilon$. Now we define $g_j := g\chi_{B(0,j)} \in L^\infty_{\mathrm{comp}}(\mathbb{R}^n)$ $(j \in \mathbb{N})$. Then, since $p_+ < \infty$, the Lebesgue dominated convergence theorem implies that

$$\lim_{j\to\infty} \rho_p(g - g_j) = 0. \tag{23.4}$$

Thus there exists $J \in \mathbb{N}$ such that $\|g - g_j\|_{L^{p(\cdot)}} < \varepsilon$ for all $j \ge J$. Namely we get

$$\|f - g_j\|_{L^{p(\cdot)}} \le \|f - g\|_{L^{p(\cdot)}} + \|g - g_j\|_{L^{p(\cdot)}} < 2\varepsilon.$$

Thus, the proof is complete. □

Defining $\psi_t, t > 0$ by (4.4) as before, we have the following local estimate in the variable setting.

Lemma 23.7. *Let $\psi \in C^\infty_{\rm comp}(\mathbb{R}^n)$. If $p(\cdot) \in LH_0(\mathbb{R}^n)$ and $1 \le p_- \le p_+ < \infty$, then, for all $N \in \mathbb{N}$, for all $f \in L^{p(\cdot)}(\mathbb{R}^n)$ supported on $B(0,N)$ and for all $t \in (0,1]$,*

$$\|\psi_t * f\|_{L^{p(\cdot)}} \le C_N \|f\|_{L^{p(\cdot)}},$$

in particular, $\psi_t * f \in L^{p(\cdot)}(\mathbb{R}^n)$.

The proof of Lemma 23.7 is based on the next lemma.

Lemma 23.8. *Let $p(\cdot) \in LH_0(\mathbb{R}^n)$ and $1 \le p_- \le p_+ < \infty$. Then there exists a constant $C > 0$ such that*

$$\left(⨍_{B(x,t)} |f(y)|\, dy \right)^{p(x)} \le C \left(⨍_{B(x,t)} |f(y)|^{p(y)}\, dy + 1 \right) \tag{23.5}$$

for all $t > 0$, all $x \in \mathbb{R}^n$ and all $f \in L^{p(\cdot)}(\mathbb{R}^n)$ with $\|f\|_{L^{p(\cdot)}} \le 1$.

Proof. In (16.1), we defined

$$I := I(x,t) = ⨍_{B(x,t)} |f(y)|\, dy, \quad J := J(x,t) = ⨍_{B(x,t)} |f(y)|^{p(y)}\, dy.$$

By Lemma 16.1 and the decomposition $f = f\chi_{\{x\in\mathbb{R}^n\,:\,|f(x)|>1\}} + f\chi_{\{x\in\mathbb{R}^n\,:\,|f(x)|\le 1\}}$, then we have

$$I \le CJ^{1/p(x)} + 1.$$

If we insert the definition of I and J, then we have the desired result. $\square$

Proof of Lemma 23.7. Assume $\|f\|_{L^{p(\cdot)}} = 1$ and the support of f is included in $B(0,N)$. Let $t \in (0,1]$. Then the support of $\psi_t * f$ is included in $B(0,N+2)$. We write $\rho_p(\psi_t * f)$ out in full:

$$\rho_p(\psi_t * f) = \int_{B(0,N+2)} \left| \int_{\mathbb{R}^n} t^{-n}\psi((x-y)/t) f(y)\, dy \right|^{p(x)} dx.$$

Applying (23.5) we obtain

$$\begin{aligned}
\rho_p(\psi_t * f) &\le C \int_{B(0,N+2)} \left(⨍_{B(x,t)} |f(y)|\, dy \right)^{p(x)} dx \\
&\le C \int_{B(0,N+2)} \left(1 + ⨍_{B(x,t)} |f(y)|^{p(y)}\, dy \right) dx \\
&= C|B(0,N+2)| \\
&\quad + \frac{C}{|B(0,t)|} \int_{B(0,N+2)} \left(\int_{\mathbb{R}^n} \chi_{\{(x,y)\in\mathbb{R}^{2n}\,:\,|x-y|<t\}}(x,y) |f(y)|^{p(y)}\, dy \right) dx \\
&\le C(|B(0,N+2)| + 1).
\end{aligned}$$

Therefore by Lemma 8.4 we get $\|\psi_t * f\|_{L^{p(\cdot)}} \le C_N$. $\square$

Proof of Theorem 23.2. Take a non-negative and radial decreasing function $\psi \in C^\infty_{\rm comp}(\mathbb{R}^n)$ so that $\|\psi\|_{L^1} = 1$. By Lemma 23.5, it is enough to prove that

$$\lim_{t\to 0} \|f - \psi_t * f\|_{L^{p(\cdot)}_s} = 0, \tag{23.6}$$

for all $f \in L^{p(\cdot)}_s(\mathbb{R}^n)$ with compact support. Since

$$D^\alpha(\psi_t * f)(x) = \int_{\mathbb{R}^n} (D^\alpha f)(x-y)\psi_t(y)\,dy$$

for every $\alpha \in {\mathbb{N}_0}^n$ with $|\alpha| \le s$, it is also enough to prove that

$$\lim_{t\to 0} \|f - \psi_t * f\|_{L^{p(\cdot)}} = 0 \tag{23.7}$$

for all $f \in L^{p(\cdot)}(\mathbb{R}^n)$ with compact support.

Now, let $f \in L^{p(\cdot)}(\mathbb{R}^n)$ and $\operatorname{supp} f \subset B(0,N)$. Since $L^\infty_{\rm comp}(\mathbb{R}^n)$ is dense in $L^{p(\cdot)}(\mathbb{R}^n)$ by Lemma 23.6, for $\epsilon > 0$ we can take a function $g \in L^\infty_{\rm comp}(\mathbb{R}^n)$ such that $\|f-g\|_{L^{p(\cdot)}(\mathbb{R}^n)} < \epsilon/(2(C_N+1))$, where C_N is the constant in Lemma 23.7. In this case we may assume that $\operatorname{supp}(f-g) \subset B(0,N)$. Then, using Lemma 23.7, we have that, for $t \in (0,1]$,

$$\begin{aligned}
\|\psi_t * f - f\|_{L^{p(\cdot)}} &\le \|\psi_t * f - \psi_t * g\|_{L^{p(\cdot)}} + \|\psi_t * g - g\|_{L^{p(\cdot)}} + \|g - f\|_{L^{p(\cdot)}}\\
&\le C_N\|f-g\|_{L^{p(\cdot)}} + \|\psi_t * g - g\|_{L^{p(\cdot)}} + \|g-f\|_{L^{p(\cdot)}}\\
&\le \epsilon/2 + \|\psi_t * g - g\|_{L^{p(\cdot)}}.
\end{aligned}$$

We note that $\psi_t * g(x) \to g(x)$ a.e. x as $t \to 0$. From $g \in L^\infty_{\rm comp}(\mathbb{R}^n)$ it follows that $\|\psi_t * g\|_{L^\infty} \le \|g\|_{L^\infty}$ and that $\operatorname{supp} \psi_t * g$ is included in $B(0,N+2)$ for $0<t<1$. Hence the Lebesgue dominated convergence theorem gives us $\lim_{t\to 0} \rho_p(g - \psi_t * g) = 0$. Consequently we can take $0 < t_\varepsilon < 1$ so that $\|f - \psi_t * f\|_{L^{p(\cdot)}} < \varepsilon$ holds whenever $0 < t \le t_\varepsilon$. □

Remark 23.1. In the same way as (23.6), one can prove

$$\lim_{t\to 0} \|\psi_t * (\psi_{t^{-1}} \cdot f) - f\|_{L^{p(\cdot)}} = 0.$$

This enables us to use [166, p. 217, Theorem]. By using Remark 23.1 and [166, p. 217, Theorem], Rabinovich and S. Samko [168, Proposition 2.2] proved the following result;

Proposition 23.9. *Let $p_j(\cdot) \in L^\infty(\mathbb{R}^n;[1,\infty))$ and $p_j(\cdot) \in LH$ $(j=1,2)$. Let $\theta \in (0,1)$ and define $p_\theta(x)$ by*

$$\frac{1}{p_\theta(x)} = \frac{1-\theta}{p_1(x)} + \frac{\theta}{p_2(x)}.$$

Let $A : L^{p_1(\cdot)}(\mathbb{R}^n) + L^{p_2(\cdot)}(\mathbb{R}^n) \to L^{p_1(\cdot)}(\mathbb{R}^n) + L^{p_2(\cdot)}(\mathbb{R}^n)$ be a linear operator such that

$$A_{|L^{p_1(\cdot)}(\mathbb{R}^n)} : L^{p_1(\cdot)}(\mathbb{R}^n) \to L^{p_1(\cdot)}(\mathbb{R}^n)$$

is a compact operator and that

$$A_{|L^{p_2(\cdot)}(\mathbb{R}^n)} : L^{p_2(\cdot)}(\mathbb{R}^n) \to L^{p_2(\cdot)}(\mathbb{R}^n)$$

is a bounded operator, then

$$A_{|L^{p_\theta(\cdot)}(\mathbb{R}^n)} : L^{p_\theta(\cdot)}(\mathbb{R}^n) \to L^{p_\theta(\cdot)}(\mathbb{R}^n)$$

is a compact operator.

In the above, for two exponents $p(\cdot)$ and $q(\cdot)$, we define

$$L^{p(\cdot)}(\mathbb{R}^n) + L^{q(\cdot)}(\mathbb{R}^n) := \{f + g \,:\, f \in L^{p(\cdot)}(\mathbb{R}^n), g \in L^{q(\cdot)}(\mathbb{R}^n)\}$$

and the norm for $h \in L^{p(\cdot)}(\mathbb{R}^n) + L^{q(\cdot)}(\mathbb{R}^n)$ is given by

$$\inf\{\|f\|_{L^{p(\cdot)}} + \|g\|_{L^{q(\cdot)}} \,:\, h = f + g\}.$$

23.3 Spaces of potentials S. Samko [186] proved the uniform boundedness of convolution dilation operators $\frac{1}{\varepsilon^n} k\left(\frac{x}{\varepsilon}\right) * f$ in $L^{p(\cdot)}(\mathbb{R}^n)$, for a class of kernels $k(x)$, which was used in this paper to prove the density of $C_0^\infty(\mathbb{R}^n)$ in the generalized Sobolev spaces $W^{m,p(\cdot)}(\mathbb{R}^n)$.

Almeida and S. Samko [4] introduced the spaces of Riesz and Bessel potentials with densities in Lebesgue spaces with variable exponents. As a consequence of this characterization, there was described a relation between the spaces of Riesz or Bessel potentials and the Sobolev spaces with variable exponents. Rafeiro and S. Samko [174] characterized Bessel potential space $\mathfrak{B}^\alpha\left[L^{p(\cdot)}(\mathbb{R}^n)\right]$ in terms of the rate of convergence of the Poisson semigroup P_t. Define

$$\mathbb{D}^\alpha f \equiv \lim_{t \downarrow 0} \frac{1}{t^\alpha}(1 - P_t)^\alpha f.$$

It was shown that the existence of the Riesz fractional derivative $\mathbb{D}^\alpha f$ in the space $L^{p(\cdot)}(\mathbb{R}^n)$ is equivalent to the existence of the limit $\frac{1}{\varepsilon^\alpha}(I - P_\varepsilon)^\alpha f$; if one of these exists, then the other exists and two quantities coincide. In the pre-limiting case $\sup_x p(x) < \frac{n}{\alpha}$ it is shown that the Bessel potential space is characterized by the condition $\|(I - P_\varepsilon)^\alpha f\|_{L^{p(\cdot)}} \le C\varepsilon^\alpha$.

In the case of a bounded open set Ω with Lipschitz boundary, Almeida and S. Samko [5] proved the pointwise estimate

$$|f(x) - f(y)| \le \frac{c}{\min[p(x), p(y)] - n} \, \| \, |\nabla f| \, \|_{L^{p(\cdot)}(\Omega)} \, |x - y|^{1 - \frac{n}{\min[p(x), p(y)]}}$$

for all $f \in W^{1,p(\cdot)}(\Omega)$ and x, y with $|x - y| < 1$ such that $p(x) > n$ and $p(y) > n$. This estimate is used to study the behaviour of hypersingular integrals of Sobolev functions with variable exponents and to prove embedding of Sobolev spaces with variable exponents into Hölder spaces of variable order.

24 Fractional integral operators Let I_α be the fractional integral operator of order $\alpha \in (0, n)$, that is,

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x - y|^{n-\alpha}} \, dy.$$

Then it is known as the Hardy-Littlewood-Sobolev theorem that

$$I_\alpha \in B(L^p(\mathbb{R}^n), L^q(\mathbb{R}^n)),$$

if $\alpha \in (0, n)$, $p, q \in (1, \infty)$ and $-n/p + \alpha = -n/q$. See [148] for related results on the operator I_α.

24.1 Fractional integral operator of order α In the setting of variable exponent, the following is fundamental:

Theorem 24.1 (Diening [37] (2004))**.** *Let $\alpha \in (0, \infty)$ be a constant. Let $p(\cdot) \in LH$ be such that $(-n/p(\cdot) + \alpha)_+ < 0$ and that $1 < p_- \le p_+ < \infty$. Define an exponent $q(\cdot)$ by $-n/q(\cdot) = -n/p(\cdot) + \alpha$. Then $I_\alpha \in B(L^{p(\cdot)}(\mathbb{R}^n), L^{q(\cdot)}(\mathbb{R}^n))$.*

This theorem is proven by using the following pointwise estimate and the boundedness of M on $L^{p(\cdot)}(\mathbb{R}^n)$:

Theorem 24.2 (Diening [37] (2004)). *Let $\alpha \in (0,\infty)$. If $p(\cdot), q(\cdot) \in LH$ satisfies $1 \le p_- \le p_+ < n/\alpha$ and $-n/q(x) = -n/p(x) + \alpha$ for all $x \in \mathbb{R}^n$. Then there exists a positive constant C, dependent only on n and $p(\cdot)$, such that, for all measurable functions f with $\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le 1$,*

$$|I_\alpha f(x)|^{q(x)} \le C(Mf(x)^{p(x)} + (e+|x|)^{-np_-}) \quad \textit{for all } x \in \mathbb{R}^n.$$

S. Samko and Vakulov [200] showed the boundedness of I_α on weighted Lebesgue spaces with variable exponents and two-parametrical power weights. Let $\rho(x) = \rho_{\gamma_0,\gamma_\infty}(x) = |x|^{\gamma_0}(1+|x|)^{\gamma_\infty-\gamma_0}$, and define the space $L^{p(\cdot)}_\rho(\mathbb{R}^n)$ by Definition 22.1. Assume that $p(\cdot) \in LH$ and $1 < p_- \le p_+ < n/\alpha$. Then the Riesz potential operator I_α of constant order α is bounded from $L^{p(\cdot)}_{\rho_{\gamma_0,\gamma_\infty}}(\mathbb{R}^n)$ to $L^{q(\cdot)}_{\rho_{\mu_0,\mu_\infty}}(\mathbb{R}^n)$, where

$$\mu_0 = \frac{q(0)}{p(0)}\gamma_0 \quad \text{and} \quad \mu_\infty = \frac{q_\infty}{p_\infty}\gamma_\infty,$$

if

$$\alpha p(0) - n < \gamma_0 < n(p(0)-1), \quad \alpha p_\infty - n < \gamma_\infty < n(p_\infty - 1),$$

and also

$$\frac{q(0)}{p(0)}\gamma_0 + \frac{q_\infty}{p_\infty}\gamma_\infty = \frac{q_\infty}{p_\infty}((n+\alpha)p_\infty - 2n);$$

the latter condition was removed in the later publication [199].

24.2 Fractional integral operator of variable order $\alpha(\cdot)$ For $\alpha(\cdot) \in L^0(\mathbb{R}^n;(0,n))$, define a generalized fractional integral operator $I_{\alpha(x)}$ with variable order defined by

$$I_{\alpha(x)}f(x) := \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha(x)}}\, dy.$$

S. Samko [184, p. 277] (1998) proved the boundedness of $I_{\alpha(x)}$ on generalized Lebesgue spaces on bounded domain with variable exponents and power weights. Mizuta and Shimomura [134] extended Theorem 24.1 as the following:

Theorem 24.3 (Mizuta and Shimomura [134] (2012)). *Let $\alpha(\cdot) \in L^0(\mathbb{R}^n;(0,n))$ and $0 < \alpha_- \le \alpha_+ < n$. Let $p(\cdot) \in LH$, $1 < p_- \le p_+ < \infty$ and define $q(\cdot)$ by $-n/q(x) = -n/p(x) + \alpha(x)$ for all $x \in \mathbb{R}^n$. Assume that*

$$(-n/p(\cdot) + \alpha(\cdot))_+ < 0, \quad (-n/p_\infty + \alpha(\cdot))_+ < 0.$$

Then $I_{\alpha(x)} \in B(L^{p(\cdot)}(\mathbb{R}^n), L^{q(\cdot)}(\mathbb{R}^n))$.

It is worth noting that $\alpha(\cdot)$ and $q(\cdot)$ need not be continuous. The condition $(-n/p_\infty + \alpha(\cdot))_+ < 0$ is necessary for the boundedness of $I_{\alpha(x)}$, see the following example:

Example 24.1 (Hästö [67] (2009)). Let $R > 2$. Let $\alpha(\cdot) \in L^0(\mathbb{R}^n;(0,n)))$ and $p(\cdot) \in L^0(\mathbb{R}^n;(1,\infty))$ be Lipschitz continuous with

$$\alpha(\cdot)|_{B(0,1)} \equiv \alpha_0, \quad \alpha(\cdot)|_{\mathbb{R}^n\setminus B(0,R)} \equiv \alpha_\infty,$$
$$p(\cdot)|_{B(0,1)} \equiv p_0, \quad p(\cdot)|_{\mathbb{R}^n\setminus B(0,R)} \equiv p_\infty.$$

If $-n/p_\infty + \alpha_0 > 0$, then take β such that $n/p_\infty < \beta \le \alpha_0$. Set

$$f(x) = |x|^{-\beta}\chi_{\mathbb{R}^n \setminus B(0,R)}(x).$$

Then

$$\|f\|_{L^{p(\cdot)}} = \left(\int_{\mathbb{R}^n \setminus B(0,R)} |x|^{-\beta p_\infty}\,dx\right)^{1/p_\infty} \sim R^{-\beta+n/p_\infty} < \infty.$$

On the other hand, for $x \in B(0,1)$,

$$I_{\alpha(x)}f(x) = \int_{\mathbb{R}^n \setminus B(0,R)} \frac{|y|^{-\beta}}{|x-y|^{n-\alpha_0}}\,dy \sim \int_{\mathbb{R}^n \setminus B(0,R)} |y|^{-\beta-n+\alpha_0}\,dy = \infty.$$

24.3 Riemann-Liouville fractional integral In 1995 S. Samko [182] considered the Riemann-Liouville fractional integration operator with variable order on $L^p(\mathbb{R})$ when $n = 1$. Recall that the Riemann-Liouville operator I^α_{a+} is given by

$$I^\alpha_{a+}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)\,dt$$

for $\alpha \in \mathbb{C}$ satisfying $\Re(\alpha) > 1$ and $a \ge -\infty$. We can generalize the Riemann-Liouville fractional integration and differentiation to the case of variable order $\alpha(\cdot)$ directly:

$$I^{\alpha(x)}_{a+}f(x) = \frac{1}{\Gamma(\alpha(x))}\int_a^x (x-t)^{\alpha(x)-1}f(t)\,dt, \quad \Re(\alpha(x)) > 0,$$

and

$$D^{\alpha(x)}_{a+}f(x) = \frac{1}{\Gamma(1-\alpha(x))}\frac{d}{dx}\int_a^x (x-t)^{-\alpha(x)}f(t)\,dt, \quad 0 < \Re(\alpha(x)) < 1,$$

respectively, $a \ge -\infty$, see S. Samko and Ross [198] (1993) and S. Samko [182] (1995). Note that $D^{\alpha(x)}_{a+}$ does not provide the left inverse operator to $I^{\alpha(x)}_{a+}$, that is, $D^{\alpha(x)}_{a+}I^{\alpha(x)}_{a+}f \neq f$. S. Samko [182] dealt with the case $a = -\infty$ and $\alpha(x) \in \mathbb{R}$:

$$I^{\alpha(x)}_{+}\varphi(x) = \frac{1}{\Gamma(\alpha(x))}\int_{-\infty}^x \frac{\varphi(y)}{(x-y)^{1-\alpha(x)}}\,dy, \quad \alpha(x) > 0,$$

and

$$D^{\alpha(x)}_{+}f(x) = \lim_{\epsilon\to 0} D^{\alpha(x)}_{+,\epsilon}f(x) \quad 0 < \alpha(x) < 1,$$

where

$$D^{\alpha(x)}_{+,\epsilon}f(x) = \frac{\alpha(x)}{\Gamma(1-\alpha(x))}\int_\epsilon^\infty \frac{f(x)-f(x-t)}{t^{1+\alpha(x)}}\,dt, \quad 0 < \alpha(x) < 1.$$

Theorem 24.4 (Samko [182] (1995))**.** *Let $\Omega_N = (-\infty, N)$, $N < \infty$ and $\lambda > 1$. Assume that $\alpha \in C^1(\Omega_N)$, $0 < \alpha_- \le \alpha_+ < 1$ and that*

$$|\alpha'(x)| \le C(1+|x|)^{-\lambda}, \quad |\alpha(x)-\alpha(x-h)| \le Ch(1+|x|)^{-1}(1+|x-h|)^{-1}.$$

If $1 \le p < 1/\lim_{x\to-\infty} a(x)$, then one has the following expression

$$\lim_{\varepsilon\downarrow 0} D^{\alpha(x)}_{+,\varepsilon}I^{\alpha(x)}_{+}\varphi(x) = \varphi(x) + K\varphi(x), \quad K\varphi(x) = \int_{-\infty}^x K(x,x-y)\varphi(y)\,dy,$$

and the operator K is compact on $L^p(\Omega_N)$.

See [119, Chapter 5] for an excellent account of this operator.

Ross and S. Samko [176] (1995) proved mapping properties of fractional integrals $I_{a+}^{\alpha(x)}$ of order $\alpha(x)$ from Hölder-type spaces $H^{\lambda(\cdot)}([a,b])$ to $H^{\lambda(\cdot)+\alpha(\cdot)}([a,b])$. This is a generalization of the Hardy-Littlewood theorem, well known in the case of constant orders α and λ (Hardy and Littlewood [60]).

We also refer to [185] for a survey on fractional integrals and derivatives of variable order and also on some initial facts for variable exponent Lebesgue spaces, which contain Minkowsky inequality with variable exponent.

See [61, 62, 87, 92, 171, 172, 178, 179, 183, 184, 181, 193, 194, 195, 199, 200, 224, 225, 226] for more related results. In particular, [191] is a survey of this field.

25 Calderón-Zygmund operators

25.1 Definition and boundedness on $L^p(\mathbb{R}^n)$ and $L^{p(\cdot)}(\mathbb{R}^n)$ Following Yabuta [227] (1985), we recall the definition of Calderón-Zygmund operators.

Definition 25.1 (standard kernel)**.** Let ω be a nonnegative nondecreasing function on $(0,\infty)$ satisfying the Dini condition $\int_0^1 \omega(t)t^{-1}dt < \infty$. A continuous function $K(x,y)$ on $\mathbb{R}^n \times \mathbb{R}^n \setminus \{(x,x) \in \mathbb{R}^{2n}\}$ is said to be a standard kernel of type ω if the following conditions are satisfied;

$$|K(x,y)| \le \frac{C}{|x-y|^n} \quad \text{for} \quad x \neq y, \tag{25.1}$$

$$|K(x,y)-K(x,z)|+|K(y,x)-K(z,x)| \le \frac{C}{|x-y|^n}\,\omega\left(\frac{|y-z|}{|x-y|}\right) \quad \text{for} \quad 2|y-z| \le |x-y|. \tag{25.2}$$

Note that (25.1) and (25.2) generalize (29.5) and (29.6), respectively.

Definition 25.2 (Calderón-Zygmund operator)**.** A linear mapping $T : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ is said to be a Calderón-Zygmund operator of type ω, if T is bounded on $L^2(\mathbb{R}^n)$ and there exists a standard kernel K of type ω such that, for $f \in C^\infty_{\mathrm{comp}}(\mathbb{R}^n)$,

$$Tf(x) = \int_{\mathbb{R}^n} K(x,y)f(y)\,dy, \quad x \notin \operatorname{supp} f. \tag{25.3}$$

Let CZO(ω) be the set of all Calderón-Zygmund operators of type ω.

Remark 25.1. If $x \notin \operatorname{supp} f$, then $K(x,y)$ is continuous on $\operatorname{supp} f$ with respect to y. Therefore, if (25.3) holds for $f \in C^\infty_{\mathrm{comp}}(\mathbb{R}^n)$, then we can extend the domain of T to $L^1_{\mathrm{comp}}(\mathbb{R}^n) + L^2(\mathbb{R}^n) + \mathcal{S}(\mathbb{R}^n)$ so that (25.3) holds for $f \in L^1_{\mathrm{comp}}(\mathbb{R}^n) \cup L^2(\mathbb{R}^n)$.

It is known that $\mathrm{CZO}(\omega) \subset \cap_{1<p<\infty} B(L^p(\mathbb{R}^n)) \cap B(L^1(\mathbb{R}^n), L^1_{\mathrm{weak}}(\mathbb{R}^n))$ ([227, Theorem 2.4]).

We invoke the following result by Alvarez and Pérez.

Theorem 25.1 (Alvarez and Pérez [8] (1994))**.** *Let T be an operator associated with a kernel K satisfying the condition that, there exist positive constants A and N such that*

$$\sup_{r>0} \int_{\mathbb{R}^n \setminus B(w,Nr)} |f(y)| D_{B(w,r)}k(y)\,dy \le AMf(w) \tag{25.4}$$

for all $f \in C^\infty_{\mathrm{comp}}(\mathbb{R}^n)$ *and* $w \in \mathbb{R}^n$, *where*

$$D_{B(w,r)}k(y) := \fint_{B(w,r)} \fint_{B(w,r)} |K(z,y) - K(x,y)|\, dx\, dz.$$

Suppose that T *extends to a bounded operator from* $L^1(\mathbb{R}^n)$ *to* $L^1_{\mathrm{weak}}(\mathbb{R}^n)$. *Then, for each* $\delta \in (0,1)$, *there exists a positive constant* C_δ *such that*

$$M^\sharp(|Tf|^\delta)(x)^{1/\delta} \le C_\delta Mf(x)$$

for all $f \in C^\infty_{\mathrm{comp}}(\mathbb{R}^n)$ *and* $x \in \mathbb{R}^n$.

Corollary 25.2. *Under the same condition as Theorem 25.1,* $T \in B(L^p(\mathbb{R}^n))$ *for all* $p \in (1,\infty)$.

Proof. By density of $C^\infty_{\mathrm{comp}}(\mathbb{R}^n)$ in $L^p(\mathbb{R}^n)$, it suffices to prove that

$$\|Tf\|_{L^p} \le C_p \|f\|_{L^p}$$

for all $f \in C^\infty_{\mathrm{comp}}(\mathbb{R}^n)$, where C_p depends only on p and T. We invoke the well-known sharp maximal estimate;

$$\|h\|_{L^p} \le C_p \|M^\sharp h\|_{L^p} \tag{25.5}$$

for all $h \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ with $h^*(\infty) = 0$. Since $f \in C^\infty_{\mathrm{comp}}(\mathbb{R}^n)$, we have $f \in L^2(\mathbb{R}^n)$ and hence $Tf \in L^2(\mathbb{R}^n)$. Since $h^*(\infty) = 0$ for all $h \in L^2(\mathbb{R}^n)$, it follows that (25.5) is applicable to $h = Tf$. Let $0 < \delta < 1$. Then, using Theorem 5.6, we have

$$\|Tf\|_{L^p} = \|\,|Tf|^\delta\,\|^{1/\delta}_{L^{p/\delta}} \lesssim \|M^\sharp(|Tf|^\delta)\|^{1/\delta}_{L^{p/\delta}} \lesssim \|Mf\|_{L^p} \lesssim \|f\|_{L^p}.$$

This is the conclusion. □

Theorem 25.3 (Diening and Růžička [43] (2003))**.** *Let* T *be an operator associated with a kernel* K *satisfying the condition* (25.4). *Suppose that* T *extends to a bounded operator from* $L^1(\mathbb{R}^n)$ *to* $L^1_{\mathrm{weak}}(\mathbb{R}^n)$. *Then* $T \in B(L^{p(\cdot)}(\mathbb{R}^n))$ *for all* $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$.

Remark 25.2. If $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ and $0 < s \le 1$, then $p(\cdot)/s, (p(\cdot)/s)' \in \mathcal{B}(\mathbb{R}^n)$ by Remark 15.1 and Theorem 15.5.

Since standard kernels of type ω satisfy the condition (25.4), we have the following:

Corollary 25.4. *Let* $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ *and* $1 < p_- \le p_+ < \infty$. *Then* $\mathrm{CZO}(\omega) \subset B(L^{p(\cdot)}(\mathbb{R}^n))$.

To prove Theorem 25.3 they used Theorem 25.1 and showed the following:

Theorem 25.5 (Diening and Růžička [43] (2003))**.** *Suppose* $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ *satisfies* $p_+ < \infty$. *Then, for* $f \in L^{p(\cdot)}(\mathbb{R}^n)$,

$$\|f\|_{L^{p(\cdot)}} \lesssim \|f^\sharp\|_{L^{p(\cdot)}} \lesssim \|f\|_{L^{p(\cdot)}}.$$

Once we admit Theorem 25.5, we can go through the same argument as Corollary 25.2. See also [91, Theorem 3.6] for a passage to the weighted case.

25.2 Boundedness on $L_w^{p(\cdot)}(\mathbb{R}^n)$ with Muckenhoupt weights As for Muckenhoupt weights we can prove the following boundedness of Calderón-Zygmund operators T of type ω, using the Lerner decomposition (see (25.7) in Lemma 25.7 below) for Tf and the vector-valued inequality of the operator M in Subsection 27.2.

Theorem 25.6 (Izuki, Nakai and Sawano [79]). *Suppose that $p(\cdot) \in LH(\mathbb{R}^n)$ and that $p(\cdot)$ satisfies $1 < p_- \le p_+ < \infty$. Let T be a Calderón-Zygmund operator and $w \in A_{p(\cdot)}$. Then*

$$\|Tf\|_{L_w^{p(\cdot)}} \le C\|f\|_{L_w^{p(\cdot)}}$$

for all $f \in L_w^{p(\cdot)}(\mathbb{R}^n)$.

For a cube Q, we let $\mathcal{D}(Q)$ be the set of all dyadic cubes with respect to Q. For $R \in \mathcal{Q}$ and a function $f : R \to \mathbb{R}$, define

$$\omega(f;R) := \inf_{c\in\mathbb{R}}((f-c)\chi_R)^*(2^{-n-2}|R|), \tag{25.6}$$

where g^* denotes the non-increasing rearrangement of a measurable function g. The local dyadic sharp maximal function of f is given by

$$M_Q^{\sharp,d}f(x) := \sup_{R\in\mathcal{D}(Q), R\ni x} \omega(f;R).$$

Then the Lerner decomposition is the following:

Lemma 25.7 (Lerner [104, Theorem 4.5]). *Suppose that Q_0 is a cube and that $f : Q_0 \to \mathbb{R}$ is a measurable function. Let $\{m_f(R)\}_{R\in\mathcal{D}(Q_0)} \subset \mathbb{R}$ be a collection such that*

$$|\{x \in R \,:\, f(x) > m_f(R)\}|, |\{x \in R \,:\, f(x) < m_f(R)\}| \le \frac{1}{2}|R|$$

for all $R \in \mathcal{D}(Q_0)$. Then for each $k \in \mathbb{N}$, there exists a collection $J_k \subset \mathcal{D}(Q_0)$ such that

$$\left|\bigcup_{R\in J_{k+1}} R\cap Q\right| \le \frac{1}{2}|Q| \quad (Q \in J_k), \qquad \sum_{R\in J_{k+1}} \chi_R \le \sum_{S\in J_k} \chi_S \le 1$$

for all $k = 1, 2, \dots$ and that

$$|f(x) - m_f(Q_0)| \le 4M_{Q_0}^{\sharp,d}f(x) + 2\sum_{k=1}^{\infty}\sum_{R\in J_k} \omega(f;R)\chi_R(x) \tag{25.7}$$

for almost all $x \in Q_0$.

25.3 Pseudo-differential operator Recall that a Fredholm operator is a bounded linear operator from a Banach space X to another Banach space Y whose kernel and cokernel are finite-dimensional. A typical example is a compact perturbation of identity; if I denotes the identity operator on a Banach space X and T is a compact operator from X to itself, then $I - T$ is known as the Fredholm operator. Rabinovich and S. Samko [168] proved the boundedness of a certain class of singular type operators with variable kernels $K(x, x-y)$ in weighted variable exponent spaces $L_w^{p(\cdot)}(\mathbb{R}^n)$ with a power type weight w. From this result they derived the boundedness of pseudo-differential operators of Hörmander class $S_{1,0}^0$ in such spaces. The latter result is applied to obtain a necessary and sufficient condition for a class of pseudo-differential operators with symbols slowly oscillating at infinity, to be Fredholm within the frameworks of weighted Sobolev spaces $H^{s,p(\cdot)}_w(\mathbb{R}^n)$ with variable exponent $p(\cdot)$ and exponential weight w. As is remarked above, the key idea is that a compact perturbation of identity yields a Fredholm operator. Recall that S^m is the class of all $C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$-functions such that it satisfies

$$\sup_{x,\xi \in \mathbb{R}^n} (1+|\xi|)^{-|\alpha|+m} |\partial_\xi^\alpha \partial_x^\beta a(x,\xi)| < \infty$$

for all α, β. A well-known result is that pseudo-differential operators with symbol in S^0 is $L^p(\mathbb{R}^n)$-bounded for all $1 < p < \infty$. Likewise, SO^m is the subclass of S^m which is made up of all elements a satisfying

$$\lim_{|x|\to\infty} \left(\sup_{\xi \in \mathbb{R}^n} (1+|\xi|)^{-|\alpha|+m} |\partial_\xi^\alpha \partial_x^\beta a(x,\xi)| \right) = 0$$

for all α, β with $\beta \neq 0$. In the above, if the above relation is valid for all α, β, then a is said to belong to the class SO^m_0. It is known that the pseudo-differential operators with symbol in SO^{-1}_0 is a compact operator in $L^2(\mathbb{R}^n)$ [167, Chapter 4]. By using the complex interpolation and the approximation to the identity described in Remark 23.1, Rabinovich and S. Samko proved an interpolation result [168, Proposition 2.2] and applied this interpolation result to the Fredholmness of pseudo-differential operators; see Proposition 23.9.

We now consider the relation between the Mellin pseudo-differential operators. Recall the following classes of pseudo-differential operators of Mellin class.

Definition 25.3.

(i) Let l_1, l_2 be non-negative integers. For an $n \times n$-matrix-valued C^∞-function $a = \{a_{ij}\}_{i,j=1}^n$, define

$$|a|_{l_1,l_2} := \max_{1\le i,j\le n} \left(\sup_{r,\xi\in(0,\infty)\times\mathbb{R}} \sum_{\alpha=1}^{l_1} \sum_{\beta=1}^{l_2} |(r\partial_r)^\beta \partial_\xi^\alpha a_{ij}(r,\xi)| \right).$$

(ii) The space $\mathcal{E}(n)$ is the set of all $n \times n$-matrix-valued C^∞-functions for which the quantity $|a|_{l_1,l_2}$ is finite for all non-negative integers l_1 and l_2. Define

$$(\mathrm{Op}(a)u)(r) := \frac{1}{2\pi} \int_0^\infty \left(\int_0^\infty a(r,\xi)(r\rho^{-1})^{i\xi} u(\rho)\rho^{-1} d\rho \right) d\xi$$

for $a \in \mathcal{E}(n)$.

(iii) Let l_1, l_2, l_3 be non-negative integers. For an $n \times n$-matrix-valued C^∞-function $a = \{a_{ij}\}_{i,j=1}^n$, define

$$|a|_{l_1,l_2,l_3} := \max_{1\le i,j\le n} \sup_{r,\xi\in(0,\infty)\times\mathbb{R}} \sum_{\alpha=1}^{l_1} \sum_{\beta=1}^{l_2} \sum_{\gamma=1}^{l_3} |(r\partial_r)^\alpha \partial_\xi^\beta (\rho\partial_\rho)^\gamma a_{ij}(r,\rho,\xi)|.$$

(iv) The space $\mathcal{E}_d(n)$ is the set of all $n \times n$-matrix-valued C^∞-functions for which the quantity $|a|_{l_1,l_2,l_3}$ is finite for all non-negative integers l_1, l_2 and l_3. Define

$$(\mathrm{Op}(a)u)(r,\xi) := \frac{1}{2\pi} \int_0^\infty \left(\int_0^\infty a(r,\rho,\xi)(r\rho^{-1})^{i\xi} u(\rho)\rho^{-1} d\rho \right) d\xi.$$

Rabinovich and S. Samko [169] developed the variable exponent Lebesgue theory with application to singular integral equations and pseudo-differential operators. The main results are as follows:

(i) The boundedness of singular integral operators in the variable exponent Lebesgue spaces $L^{p(\cdot)}_w(\Gamma)$ on a class of composed Carleson curves Γ where the weights w have a finite set of oscillating singularities. The proof is based on the boundedness of Mellin pseudo-differential operators on the spaces $L^{p(\cdot)}(\mathbb{R}_+, d\mu)$ where $d\mu = \frac{dt}{t}$ on $\mathbb{R}_+ = \{r \in \mathbb{R} : r > 0\}$.

(ii) Criterion of local invertibility of singular integral operators with piecewise slowly oscillating coefficients acting on $L^{p(\cdot)}_w(\Gamma)$ spaces. This criterion is derived from the criteria of local invertibility at the point 0 of Mellin pseudo-differential operators on $\mathbb{R}_+$ and local invertibility of singular integral operators on $\mathbb{R}$.

(iii) Criterion of Fredholmness of singular integral operators in the variable exponent Lebesgue spaces $L^{p(\cdot)}_w(\Gamma)$ where Γ belongs to a class of composed Carleson curves slowly oscillating at the nodes, and the weight w has a finite set of slowly oscillating singularities.

26 Hardy operators Diening and S. Samko [44] proved that any convolution operator on $\mathbb{R}^n$ with the kernel admitting the estimate $|k(x)| \le c(1+|x|)^{-\nu}$ when ν is sufficiently large, say, $\nu > n\left(1 - \frac{1}{p(\infty)} + \frac{1}{q(\infty)}\right)$, is bounded in $L^{p(\cdot)}(\mathbb{R}^n)$ without local log-condition on $p(\cdot)$, only under the decay log-condition at infinity. By means of this fact, Diening and S. Samko proved the Hardy inequality

$$\left\| x^{\alpha(x)+\mu(x)-1} \int_0^x \frac{f(y)\,dy}{y^{\alpha(y)}} \right\|_{L^{q(\cdot)}(\mathbb{R}^1_+)} \le C\,\|f\|_{L^{p(\cdot)}(\mathbb{R}^1_+)}$$

and a similar inequality for the dual Hardy operator for variable exponent Lebesgue spaces, where $0 \le \mu(0) < \frac{1}{p(0)}, 0 \le \mu(\infty) < \frac{1}{p(\infty)}, \frac{1}{q(0)} = \frac{1}{p(0)} - \mu(0), \quad \frac{1}{q(\infty)} = \frac{1}{p(\infty)} - \mu(\infty)$, and $\alpha(0) < \frac{1}{p'(0)}, \quad \alpha(\infty) < \frac{1}{p'(\infty)}, \quad \beta(0) > -\frac{1}{p(0)}, \quad \beta(\infty) > -\frac{1}{p(\infty)}$, not requiring local log-condition on $\mathbb{R}^1_+$, but supposing that only the decay condition holds for $\alpha(x), \mu(x)$ and $p(x)$ only at the points $x = 0$ and $x = \infty$.

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $x_0 \in \overline{\Omega}$. S. Samko [187] proved the Hardy type inequality

$$\left\| |x - x_0|^{\beta-\alpha} \int\limits_\Omega \frac{f(y)\,dy}{|y - x_0|^\beta |x - y|^{n-\alpha}} \right\|_{L^{p(\cdot)}(\Omega)} \le C\|f\|_{L^{p(\cdot)}(\Omega)}, \quad 0 < \alpha < n,$$

where

$$-\frac{n}{p(x_0)} < \beta < \frac{n}{q(x_0)}.$$

S. Samko [188] proved a Hardy-Stein-Weiss type inequality:

$$\|I^{\alpha(\cdot)} f\|_{L^{q(\cdot)}(\Omega, |x-x_0|^\mu)} \le C\|ft\|_{L^{p(\cdot)}(\Omega, |x-x_0|^\gamma)},$$

where

$$\frac{1}{q(x)} \equiv \frac{1}{p(x)} - \frac{\alpha(x)}{n}, \quad \alpha(x_0)p(x_0) - n < \gamma < n[p(x_0) - 1], \quad \mu = \frac{q(x_0)}{p(x_0)}\,\gamma.$$

Rafeiro and S. Samko [173] proved the variable exponent Hardy type inequality

$$\left\|\frac{1}{\delta(x)^\alpha}\int_\Omega \frac{\varphi(y)}{|x-y|^{n-\alpha}}\,dy\right\|_{L^{p(\cdot)}(\Omega)} \le C\|\varphi\|_{L^{p(\cdot)}(\Omega)}, \quad 0<\alpha<\min\left(1,\frac{n}{p_+}\right)$$

where $\delta(x)=\operatorname{dist}(x,\partial\Omega)$ and Ω is a bounded domain in $\mathbb{R}^n$ with the property that $\mathbb{R}^n\setminus\overline{\Omega}$ has the cone property, is equivalent to a certain property of the domain Ω expressed in terms of α and χ_Ω.

For Hardy type inequalities with variable exponents, see also Harjulehto, Hästö and Koskenoja [64], Hästö [67], Mizuta, Nakai, Ohno and Shimomura [122], etc.

27 Vector-valued inequalities All the results in this section are vector-valued extensions of the results taken up earlier. We consider mainly the operator $M:\{f_j\}_{j=1}^\infty\mapsto\{Mf_j\}_{j=1}^\infty$. We may assume that $f_j=0$ if $j\ge j_0\gg 1$. But in the course of the proof, any constant does not depend on j_0.

27.1 Weighted norm inequalities and extrapolation Here by using the following Theorem 27.1 we shall prove the following inequality (27.1). The following result is due to Andersen and John [9]:

Theorem 27.1 ([9]). *Let $1<p<\infty$, $1<q\le\infty$. If $w\in A_p$, then*

$$\int_{\mathbb{R}^n}\left(\sum_{j=1}^\infty Mf_j(x)^q\right)^{\frac{p}{q}} w(x)\,dx \lesssim \int_{\mathbb{R}^n}\left(\sum_{j=1}^\infty |f_j(x)|^q\right)^{\frac{p}{q}} w(x)\,dx.$$

Remark that in the course of Proposition 21.2, we prove that there exists $\eta>0$ such that $p(\cdot)/(1+\eta)\in\mathcal{B}(\mathbb{R}^n)$.

Using the above theorem, we prove the following theorem:

Theorem 27.2 ([25, Corollary 2.1]). *Let $\delta>0$. Suppose that the exponent $p(\cdot)$ is such that M is bounded on $L^{(p(\cdot)/(1+\delta))'}(\mathbb{R}^n)$ and that $1<p_-\le p_+<\infty$. Then, for $\{f_j\}_{j=1}^\infty\subset L^{p(\cdot)}(\mathbb{R}^n)$,*

$$\left\|\left(\sum_{j=1}^\infty Mf_j{}^q\right)^{\frac1q}\right\|_{L^{p(\cdot)}} \lesssim \left\|\left(\sum_{j=1}^\infty |f_j|^q\right)^{\frac1q}\right\|_{L^{p(\cdot)}}. \tag{27.1}$$

Proof. Let $0<\eta\ll 1$ be such that M is bounded on $L^{(p(\cdot)/(1+\delta))'/(1+\eta)}(\mathbb{R}^n)$. By Theorem 9.2, we can find $g\in L^{(p(\cdot)/(1+\delta))'}(\mathbb{R}^n)$ with norm 1 such that

$$\begin{aligned}
\left\|\left(\sum_{j=1}^\infty Mf_j{}^q\right)^{\frac1q}\right\|_{L^{p(\cdot)}} &= \left(\left\|\left(\sum_{j=1}^\infty Mf_j{}^q\right)^{\frac{1+\delta}{q}}\right\|_{L^{p(\cdot)/(1+\delta)}}\right)^{1/(1+\delta)}\\
&= \left(\int_{\mathbb{R}^n}|g(x)|\left(\sum_{j=1}^\infty Mf_j(x)^q\right)^{\frac{1+\delta}{q}}dx\right)^{1/(1+\delta)}\\
&\le \left(\int_{\mathbb{R}^n}M[|g|^{1+\eta}](x)^{\frac{1}{1+\eta}}\left(\sum_{j=1}^\infty Mf_j(x)^q\right)^{\frac{1+\delta}{q}}dx\right)^{1/(1+\delta)}
\end{aligned}$$

by virtue of the Lebesgue differential theorem. Note that $c_\eta \ge A_1(M[|g|^{1+\eta}]^{\frac{1}{1+\eta}}) \ge A_{1+\delta}(M[|g|^{1+\eta}]^{\frac{1}{1+\eta}})$. Thus, by Theorem 27.1 we obtain

$$\left\| \left(\sum_{j=1}^{\infty} M f_j{}^q \right)^{\frac{1}{q}} \right\|_{L^{p(\cdot)}} \le \left(\int_{\mathbb{R}^n} M[|g|^{1+\eta}](x)^{\frac{1}{1+\eta}} \left(\sum_{j=1}^{\infty} |f_j(x)|^q \right)^{\frac{1+\delta}{q}} dx \right)^{1/(1+\delta)}.$$

By Theorem 9.2 again, we have

$$\left\| \left(\sum_{j=1}^{\infty} M f_j{}^q \right)^{\frac{1}{q}} \right\|_{L^{p(\cdot)}} \lesssim \|M[|g|^{1+\eta}]\|_{L^{(p(\cdot)/(1+\delta)')/(1+\eta)}} \left\| \left(\sum_{j=1}^{\infty} |f_j|^q \right)^{\frac{1}{q}} \right\|_{L^{p(\cdot)}}.$$

By assumption $p(\cdot)$ is such that M is bounded on $L^{(p(\cdot)/(1+\delta))'}(\mathbb{R}^n)$. Thus,

$$\left\| \left(\sum_{j=1}^{\infty} M f_j{}^q \right)^{\frac{1}{q}} \right\|_{L^{p(\cdot)}} \lesssim \left\| \left(\sum_{j=1}^{\infty} |f_j|^q \right)^{\frac{1}{q}} \right\|_{L^{p(\cdot)}}.$$

The proof is therefore complete. □

Remark 27.1.

(i) By using a weighted Littlewood-Paley estimate

$$\|f\|_{L^p_w} \sim \left\| \left(\sum_{j=-\infty}^{\infty} |\mathcal{F}^{-1}[\varphi(2^{-j}\cdot)\mathcal{F}f]|^2 \right)^{1/2} \right\|_{L^p_w},$$

we can prove (23.1). However, in [154], we used the boundedness of singular integral operator by using the atomic decomposition.

(ii) The same can be said for Calderón-Zygmund operators;

$$\left\| \left(\sum_{j=1}^{\infty} |T f_j|^q \right)^{\frac{1}{q}} \right\|_{L^{p(\cdot)}} \lesssim \left\| \left(\sum_{j=1}^{\infty} |f_j|^q \right)^{\frac{1}{q}} \right\|_{L^{p(\cdot)}}. \tag{27.2}$$

This result can be found in [25].

(iii) [42, p. 1746] We cannot extend (27.1) to the version where the exponent q is a variable exponent as well. In the theory of vector-valued inequalities such as

$$\left\| \left(\sum_{j=1}^{\infty} M f_j{}^q \right)^{\frac{1}{q}} \right\|_{L^p} \lesssim \left\| \left(\sum_{j=1}^{\infty} |f_j|^q \right)^{\frac{1}{q}} \right\|_{L^p},$$

the simplest case is $p = q$. Indeed, in this case the result is just a matter of combining the monotone convergence theorem and the $L^p(\mathbb{R}^n)$-boundedness of the Hardy-Littlewood maximal operator. We are thus tempted to consider

$$\left\| \left(\sum_{j=1}^{\infty} M f_j{}^{p(\cdot)} \right)^{\frac{1}{p(\cdot)}} \right\|_{L^{p(\cdot)}} \lesssim \left\| \left(\sum_{j=1}^{\infty} |f_j|^{p(\cdot)} \right)^{\frac{1}{p(\cdot)}} \right\|_{L^{p(\cdot)}}. \tag{27.3}$$

Suppose that (27.3) holds even when $p(\cdot)$ is a continuous function which is not constant on $\mathbb{R}^n$. Then we can choose open sets U and V such that $m = \sup_U p < \inf_V p = M$. Let $x_0 \in V$. Choose $r > 0$ so that $B(x_0, r) \subset U$. Then, substitute $f_j = j^{-\frac{2}{m+M}} \chi_{B(x_0,r)}$ to (27.3). Then we would have

$$\sum_{j=1}^{\infty} j^{-\frac{2m}{m+M}} \lesssim \left\| \left(\sum_{j=1}^{\infty} M f_j{}^{p(\cdot)} \right)^{\frac{1}{p(\cdot)}} \right\|_{L^{p(\cdot)}} \lesssim \left\| \left(\sum_{j=1}^{\infty} |f_j|^{p(\cdot)} \right)^{\frac{1}{p(\cdot)}} \right\|_{L^{p(\cdot)}} \lesssim \sum_{j=1}^{\infty} j^{-\frac{2M}{m+M}}.$$

This is a contradiction, since $m < M$.

A similar technique works and we obtain the following result:

Theorem 27.3. *Let $\delta > 0$. Suppose that the exponent $p(\cdot)$ is such that M is bounded on $L^{(p(\cdot)/(1+\delta))'}(\mathbb{R}^n)$ and that $1 < p_- \le p_+ < \infty$. Then, for $\{f_j\}_{j=1}^{\infty} \subset L^{p(\cdot)}(\mathbb{R}^n)$,*

$$\left\| \left(\sum_{j=1}^{\infty} M f_j{}^q \right)^{\frac{1}{q}} \right\|_{L^{p(\cdot)}} \lesssim \left\| \left(\sum_{j=1}^{\infty} M^{\sharp} f_j{}^q \right)^{\frac{1}{q}} \right\|_{L^{p(\cdot)}}. \tag{27.4}$$

The proof hinges upon [28] heavily.

Proof. Let

$$\mathfrak{M} := \left\| \left(\sum_{j=1}^{\infty} M^{\sharp} f_j{}^q \right)^{\frac{1}{q}} \right\|_{L^{p(\cdot)}}, \quad H(x) := \frac{1}{\mathfrak{M}} \left(\sum_{j=1}^{\infty} M^{\sharp} f_j(x)^q \right)^{\frac{1}{q}}.$$

We may assume $\mathfrak{M} \neq 0$. Otherwise $f_j = 0$ for all j and in this case there is nothing to prove.

By the duality (see Theorem 9.2), we obtain

$$\left\| \left(\sum_{j=1}^{\infty} M f_j{}^q \right)^{\frac{1}{q}} \right\|_{L^{p(\cdot)}} = \int_{\mathbb{R}^n} \left(\sum_{j=1}^{\infty} M f_j{}^q \right)^{\frac{1}{q}} g(x)\, dx$$

for some non-negative $g \in L^{p'(\cdot)}(\mathbb{R}^n)$ with norm 1.

By the Hölder inequality, we obtain

$$\begin{aligned} &\int_{\mathbb{R}^n} \left(\sum_{j=1}^{\infty} M f_j(x)^q \right)^{\frac{1}{q}} g(x)\, dx \\ &\le \left(\int_{\mathbb{R}^n} \left(\sum_{j=1}^{\infty} M f_j(x)^q \right) \left(M[H^{1/\theta}](x)^{\theta} \right)^{1-q} M[g^{1/\theta}](x)^{\theta}\, dx \right)^{1/q} \\ &\quad \times \left(\int_{\mathbb{R}^n} M[H^{1/\theta}](x)^{\theta} M[g^{1/\theta}](x)^{\theta}\, dx \right)^{1/q'}. \end{aligned}$$

Note first that

$$\int_{\mathbb{R}^n} M[H^{1/\theta}](x)^\theta M[g^{1/\theta}](x)^\theta \, dx \le C \|M[H^{1/\theta}]^\theta\|_{L^{p(\cdot)}} \|M[g^{1/\theta}]^\theta\|_{L^{p'(\cdot)}} \le C_\theta.$$

Thus,

$$\begin{aligned}&\int_{\mathbb{R}^n} \left(\sum_{j=1}^\infty Mf_j(x)^q\right)^{\frac{1}{q}} g(x)\, dx \\ &\lesssim \left(\int_{\mathbb{R}^n} \left(\sum_{j=1}^\infty Mf_j(x)^q\right) \left(M[H^{1/\theta}](x)^\theta\right)^{1-q} M[g^{1/\theta}](x)^\theta \, dx\right)^{1/q}.\end{aligned}$$

Next, observe that

$$[\left(M[H^{1/\theta}]^\theta\right)^{1-q} M[g^{1/\theta}]^\theta]_{A_q} \le C_\theta < \infty.$$

Consequently,

$$\begin{aligned}&\int_{\mathbb{R}^n} \left(\sum_{j=1}^\infty Mf_j(x)^q\right)^{\frac{1}{q}} g(x)\, dx \\ &\lesssim \left(\int_{\mathbb{R}^n} \left(\sum_{j=1}^\infty M^\sharp f_j(x)^q\right) \left(M[H^{1/\theta}](x)^\theta\right)^{1-q} M[g^{1/\theta}](x)^\theta \, dx\right)^{1/q}.\end{aligned}$$

In view of the definition of H, we obtain

$$\int_{\mathbb{R}^n} \left(\sum_{j=1}^\infty Mf_j(x)^q\right)^{\frac{1}{q}} g(x)\, dx \lesssim \mathfrak{M} \left(\int_{\mathbb{R}^n} M[H^{1/\theta}](x)^\theta M[g^{1/\theta}](x)^\theta \, dx\right)^{1/q} \lesssim \mathfrak{M}.$$

This is the desired result. □

27.2 Vector-valued weighted norm inequalities In this subsection we shall take up vector-valued inequalities for weighted Lebesgue spaces $L^{p(\cdot)}_w(\mathbb{R}^n)$ with variable exponents. We shall extend the boundedness (C2) in Theorem 22.1 to the following vector-valued inequality:

Theorem 27.4 (Izuki, Nakai and Sawano [79]). *Let $p(\cdot) : \mathbb{R}^n \to (1, \infty)$. Suppose that $p(\cdot) \in LH(\mathbb{R}^n)$ and $1 < p_- \le p_+ < \infty$. Suppose in addition that $r \in (1, \infty]$ and $w \in A_{p(\cdot)}$. Then*

$$\left\| \left(\sum_{j=1}^\infty (Mf_j)^r\right)^{\frac{1}{r}} \right\|_{L^{p(\cdot)}_w} \le C \left\| \left(\sum_{j=1}^\infty |f_j|^r\right)^{\frac{1}{r}} \right\|_{L^{p(\cdot)}_w} \tag{27.5}$$

for all sequences $\{f_j\}_{j=1}^\infty$ of measurable functions. When $r = \infty$, (27.5) reads

$$\left\| \sup_{j\in\mathbb{N}} Mf_j \right\|_{L^{p(\cdot)}_w} \le C \left\| \sup_{j\in\mathbb{N}} |f_j| \right\|_{L^{p(\cdot)}_w}.$$

To prove this vector-valued inequality we need the following lemmas:

Lemma 27.5 ([79]). *If the Hardy-Littlewood maximal operator M is bounded on $L_w^{p(\cdot)}(\mathbb{R}^n)$, then there exists $\eta > 1$ such that M is bounded on $L_w^{p(\cdot)/\eta}(\mathbb{R}^n)$.*

Lemma 27.6 ([79]). *Let $p(\cdot) : \mathbb{R}^n \to (1, \infty)$. Suppose that $p \in LH(\mathbb{R}^n)$ and $1 < p_- \le p_+ < \infty$. Suppose in addition that $r \in (1, \infty)$ and $w \in A_{p(\cdot)}$. Then*

$$\left\| \sum_{j=1}^{\infty} (Mf_j)^r \right\|_{L_w^{p(\cdot)}} \le C \left\| \sum_{j=1}^{\infty} |f_j|^r \right\|_{L_w^{p(\cdot)}} \tag{27.6}$$

for all sequences $\{f_j\}_{j=1}^{\infty}$ of measurable functions.

Usign these lemmas we prove Theorem 27.4.

Proof of Theorem 27.4. If $r = \infty$, then we can resort to Theorem 22.1 and a trivial inequality

$$\sup_{j \in \mathbb{N}} Mf_j \le M \left[\sup_{j \in \mathbb{N}} |f_j| \right].$$

Next, by Lemma 27.5 there exists $\eta > 1$ such that $w \in A_{p(\cdot)/\eta}$. Let $1 < r < \min(\eta, p_-)$. Then $p(\cdot)/r \in LH$, $1 < p_-/r \le p_+/r < \infty$ and $w \in A_{p(\cdot)/\eta} \subset A_{p(\cdot)/r}$. By the definition of the weighted norm and Lemma 27.6, we have

$$\begin{aligned} \left\| \left(\sum_{j=1}^{\infty} (Mf_j)^r \right)^{\frac{1}{r}} \right\|_{L_w^{p(\cdot)}} &= \left(\left\| \sum_{j=1}^{\infty} (Mf_j)^r \right\|_{L_w^{p(\cdot)/r}} \right)^{\frac{1}{r}} \\ &\le C \left(\left\| \sum_{j=1}^{\infty} |f_j|^r \right\|_{L_w^{p(\cdot)/r}} \right)^{\frac{1}{r}} = C \left\| \left(\sum_{j=1}^{\infty} |f_j|^r \right)^{\frac{1}{r}} \right\|_{L_w^{p(\cdot)}}. \end{aligned}$$

So we are done in the case where r is small enough.

For the remaining case, we consider a linear operator

$$U : \{f_j\}_{j=1}^{\infty} \mapsto \left\{ \frac{1}{r_j(\cdot)^n} \int_{B(\cdot, r_j(\cdot))} f_j(y)\, dy \right\}_{j=1}^{\infty},$$

where each $r_j(\cdot)$ is a positive measurable function. We know that U is bounded from $L_w^{p(\cdot)}(\ell^r)$ to itself when $r = \infty$ or r is sufficiently close to 1. Thus, we are in the position of using the interpolation theorem [54, Theorem 3.4 (page 492)] to conclude that the vector-valued inequality is valid for all $1 < r \le \infty$. $\square$

Part V
Several function spaces with variable exponents

There are many function spaces describing the smoothness and integrability. With much information on the Hardy-Littlewood maximal operator M, we were led to investigate other function spaces. In this part, we mainly describe recent results on (generalized) Morrey spaces, Campanato spaces and Hardy spaces with variable exponents.

To begin with, we introduce some notions on functions on $(0, \infty)$ and $(0, \infty) \times \mathbb{R}^n$.

By a growth function we mean any function from $(0, \infty)$ to itself. By a variable growth function we mean any function from $\mathbb{R}^n \times (0, \infty)$ to $(0, \infty)$.

For a variable growth function $\phi : \mathbb{R}^n \times (0, \infty) \to (0, \infty)$ and a ball $B = B(x, r)$, we write $\phi(B) = \phi(x, r)$.

(i) For functions $\theta, \kappa : \mathbb{R}^n \times (0, \infty) \to (0, \infty)$, we denote $\theta \sim \kappa$ if there exists a constant $C > 0$ such that

$$C^{-1}\theta(x, r) \le \kappa(x, r) \le C\theta(x, r) \quad \text{for} \quad x \in \mathbb{R}^n,\ r > 0.$$

(ii) A function $\theta : \mathbb{R}^n \times (0, \infty) \to (0, \infty)$ is said to satisfy the doubling condition if there exists a constant $C > 0$ such that

$$C^{-1} \le \frac{\theta(x, r)}{\theta(x, s)} \le C \quad \text{for} \quad x \in \mathbb{R}^n,\ \frac{1}{2} \le \frac{r}{s} \le 2. \tag{27.7}$$

Or equivalently, a function $\theta : \mathbb{R}^n \times (0, \infty) \to (0, \infty)$ is said to satisfy the doubling condition if $\theta(\cdot_1, t\cdot_2) \sim \theta$ for all $t \in [1, 2]$.

(iii) A function $\theta : \mathbb{R}^n \times (0, \infty) \to (0, \infty)$ is said to be almost increasing (almost decreasing) if there exists a constant $C > 0$ such that

$$\theta(x, r) \le C\theta(x, s) \quad (\theta(x, r) \ge C\theta(x, s)) \quad \text{for} \quad x \in \mathbb{R}^n,\ r \le s.$$

(iv) A function $\phi : (0, \infty) \times \mathbb{R}^n$ is said to belong to $\mathbb{Z}^1$, if there exists a constant $C > 0$ such that

$$\int_0^r \frac{\phi(x, t)}{t}\, dt \le C\phi(x, r), \quad (x \in \mathbb{R}^n,\ r > 0).$$

(v) A function $\phi : (0, \infty) \times \mathbb{R}^n$ is said to belong to $\mathbb{Z}_1$, if there exists a constant $C > 0$ such that

$$\int_r^\infty \frac{\phi(x, t)}{t}\, dt \le C\phi(x, r).$$

In this part many results for Morrey, Campanato and Hölder spaces hold in the setting of spaces of homogeneous type $(\mathcal{X}, d, \mu)$.

28 Morrey and Campanato spaces with general growth condition

28.1 Definitions We start with a fundamental and classical definition of generalized Morrey spaces, generalized Campanato spaces and generalized Lipschitz spaces:

Definition 28.1. For a constant $p \in [1, \infty)$ and a growth function $\phi : (0, \infty) \to (0, \infty)$, let $\mathcal{L}_{p,\phi}(\mathbb{R}^n)$, $L_{p,\phi}(\mathbb{R}^n)$ and $\Lambda_\phi(\mathbb{R}^n)$ be the sets of all f such that

$$\|f\|_{\mathcal{L}_{p,\phi}} := \sup_{B=B(z,r)} \frac{1}{\phi(r)} \left(\fint_B |f(x) - f_B|^p\, dx \right)^{1/p} < \infty,$$

$$\|f\|_{L_{p,\phi}} := \sup_{B=B(z,r)} \frac{1}{\phi(r)} \left(\fint_B |f(x)|^p\, dx \right)^{1/p} < \infty,$$

$$\|f\|_{\Lambda_\phi} := \sup_{x,y\in\mathbb{R}^n,\ x\ne y} \frac{|f(x) - f(y)|}{\phi(|x - y|)} < \infty,$$

respectively. In the above, the first two supremums are taken over all balls $B = B(z, r)$ in $\mathbb{R}^n$ and in the definition of $\|f\|_{\mathcal{L}_{p,\phi}}$ f is tacitly assumed to belong to $L^1_{\mathrm{loc}}(\mathbb{R}^n)$.

Definition 28.2 (Morrey spaces, Campanato spaces, Lipschitz spaces)**.** Let $1 \le p < \infty$. For $\phi(r) = r^\lambda$, $\lambda \in \mathbb{R}$, we denote $\mathcal{L}_{p,\phi}(\mathbb{R}^n)$ and $L_{p,\phi}(\mathbb{R}^n)$ by $\mathcal{L}_{p,\lambda}(\mathbb{R}^n)$ and $L_{p,\lambda}(\mathbb{R}^n)$, respectively. For $\phi(r) = r^\alpha$, $\alpha > 0$, we denote $\Lambda_\phi(\mathbb{R}^n)$ by $\mathrm{Lip}_\alpha(\mathbb{R}^n)$.

If ϕ is almost increasing, then $\mathcal{L}_{p,\phi}(\mathbb{R}^n) = \mathcal{L}_{1,\phi}(\mathbb{R}^n)$ and $\|f\|_{\mathcal{L}_{p,\phi}} \sim \|f\|_{\mathcal{L}_{1,\phi}}$ for $1 < p < \infty$. We denote this function space by $\mathrm{BMO}_\phi(\mathbb{R}^n)$ (see Spanne [211] (1965) and Janson [81] (1976)). If $\phi \equiv 1$, then $\mathrm{BMO}_\phi(\mathbb{R}^n) = \mathrm{BMO}(\mathbb{R}^n)$. If $\phi(r) = r^\alpha$ $(0 < \alpha \le 1)$, then

$$\mathrm{BMO}_\phi(\mathbb{R}^n) = \mathrm{Lip}_\alpha(\mathbb{R}^n)$$

(see Meyers [118] (1964) and Nakai [140] (1984)). Guliyev, Hasanov, and S. Samko [57] introduced variable exponent local "complementary" generalized Morrey spaces. For Morrey spaces, see Morrey [136]. For Campanato spaces, see Campanato [13] (1963), [14] (1964), and, Peetre [164] (1966), [165] (1969). Remark that Morrey type spaces with variable exponents can be found in [133, p.110].

28.2 Maximal and fractional integral operators

Theorem 28.1 (Chiarenza and Frasca [18] (1987))**.** *If $p \in (1, \infty)$ and $\lambda \in [-n/p, 0]$, then the operator M is bounded from $L_{p,\lambda}(\mathbb{R}^n)$ to itself, that is, there exists a positive constant C such that*

$$\|Mf\|_{L_{p,\lambda}} \le C\|f\|_{L_{p,\lambda}}, \quad f \in L_{p,\lambda}(\mathbb{R}^n).$$

The Hardy-Littlewood-Sobolev theorem is extended to Morrey spaces as the following:

Theorem 28.2 (Adams [1] (1975))**.** *Let $0 < \alpha < n$, $1 < p < \infty$ and $-n/p + \alpha \le \lambda + \alpha = \mu < 0$. If $q = (\lambda/\mu)p$, then I_α is bounded from $L_{p,\lambda}(\mathbb{R}^n)$ to $L_{q,\mu}(\mathbb{R}^n)$, that is, there exists a positive constant C such that*

$$\|I_\alpha f\|_{L_{q,\mu}} \le C\|f\|_{L_{p,\lambda}}, \quad f \in L_{p,\lambda}(\mathbb{R}^n).$$

For a measurable function $\rho : (0, \infty) \to (0, \infty)$, define the generalized fractional integral operator by

$$I_\rho f(x) = \int_{\mathbb{R}^n} \frac{\rho(|x-y|)}{|x-y|^n} f(y)\, dy.$$

We consider the following conditions on ρ:

(i) (Dini condition) $\displaystyle\int_0^1 \frac{\rho(t)}{t}\, dt < \infty$.

(ii) (Doubling condition) There exists a constant $C > 0$ such that $\dfrac{1}{C} \le \dfrac{\rho(r)}{\rho(s)} \le C$ for $\dfrac{1}{2} \le \dfrac{r}{s} \le 2$.

(iii) There exists a constant $C > 0$ such that

$$\frac{\rho(r)}{r^n} \ge C\frac{\rho(s)}{s^n} \tag{28.1}$$

for $0 < r < s < \infty$.

The doubling function satisfying (28.1) is called a $\mathcal{G}_1$-function.

If $\rho(r) = r^\alpha$, $0 < \alpha < n$, then I_ρ is the usual fractional integral operator I_α. The operators I_ρ with $\rho : (0,\infty) \to (0,\infty)$ are studied in [144, 145, 146, 149].

Theorem 28.2 was extended to the following:

Theorem 28.3 (Gunawan [59] (2003))**.** *Let $1 < p < q < \infty$. Let $\phi \in \mathbb{Z}_1$ and $\rho : (0,\infty) \to (0,\infty)$ satisfy the doubling condition, Assume that there exists constant $C > 0$ such that*

$$\left(\int_0^r \frac{\rho(t)}{t}\,dt\right)\phi(r) + \int_r^\infty \frac{\rho(t)\phi(t)}{t}\,dt \le C\phi(r)^{p/q} \quad \text{for} \quad r > 0.$$

Then I_ρ is bounded from $L_{p,\phi}(\mathbb{R}^n)$ to $L_{q,\phi^{p/q}}(\mathbb{R}^n)$.

To consider the supercritical case (see Corollary 28.5), we modify the definition of I_ρ above. We also define the modified generalized fractional integral operator $\tilde{I}_\rho$ by

$$\tilde{I}_\rho f(x) = \int_{\mathbb{R}^n} f(y)\left(\frac{\rho(|x-y|)}{|x-y|^n} - \frac{\rho(|y|)(1-\chi_{B_0}(y))}{|y|^n}\right)dy.$$

In this case we also assume

$$\left|\frac{\rho(r)}{r^n} - \frac{\rho(s)}{s^n}\right| \le A_3|r-s|\frac{\rho(r)}{r^{n+1}} \quad \text{for} \quad \frac{1}{2} \le \frac{s}{r} \le 2. \tag{28.2}$$

We denote $\tilde{I}_\rho$ by $\tilde{I}_\alpha$ if $\rho(r) = r^\alpha$.

Theorem 28.4 (Nakai [146], 2002)**.** *If measurable functions $\phi, \rho : (0,\infty) \to (0,\infty)$ satisfy*

$$\phi(r)\int_0^r \frac{\rho(t)}{t}\,dt \le C\psi(r), \qquad \int_r^\infty \frac{\rho(t)\phi(t)}{t^2}\,dt \le C\frac{\psi(r)}{r} \quad (r > 0)$$

as well as (28.2), then $\tilde{I}_\rho$ is bounded from $L_{1,\phi}(\mathbb{R}^n)$ to $\mathcal{L}_{1,\psi}(\mathbb{R}^n)$.

Corollary 28.5. *Let $0 < \alpha < n$, $1 < p < \infty$ and $0 < -n/p + \alpha = \beta < 1$. Then $\tilde{I}_\alpha$ is bounded from $L_{1,-n/p}(\mathbb{R}^n)$ to $\mathrm{Lip}_\beta(\mathbb{R}^n)$.*

Remark 28.1. Note that $L^p_{\mathrm{weak}}(\mathbb{R}^n) \subsetneqq L_{1,-n/p}(\mathbb{R}^n)$ for $1 < p < \infty$. For this inclusion, see [146, Theorem 3.4], for example. To see that the inclusion is strict, let $n = 1$ for simplicity and take a positive sequence $\{a_k\}_{k=1}^\infty$ such that $a_k + 1 < a_{k+1} - 1$ and that a_k is comparable to $k^{p'}$ with $1/p + 1/p' = 1$, and let

$$f := \sum_{k=1}^\infty \chi_{[a_k, a_k+1]}.$$

In this case, for any interval $B = (a-r, a+r)$, if $r \le 1/2$, then B intersects at most one interval $[a_k, a_k+1]$. Hence

$$\frac{1}{|B|^{1-1/p}}\int_B |f(x)|\,dx \le |B|^{1/p} \le 1.$$

If $r > 1/2$, then the number of k satisfying $B \cap [a_k, a_k+1] \ne \emptyset$ is less than some constant times $r^{1/p'}$. Hence

$$\frac{1}{|B|^{1-1/p}}\int_B |f(x)|\,dx \lesssim \frac{r^{1/p'}}{|B|^{1-1/p}} \lesssim 1.$$

That is, $f \in L_{1,-n/p}(\mathbb{R})$. On the other hand, $f \notin L^p_{\mathrm{weak}}(\mathbb{R})$, since $|\{|f| > 1/2\}| = \infty$.

29 Campanato spaces with variable growth condition

29.1 Generalized Campanato, Morrey and Hölder spaces The idea of the following definition came from the pointwise multipliers on $\mathrm{BMO}(\mathbb{R}^n)$ in Nakai and Yabuta [157], see Subsection 5.4. We abuse a notation: For a function $\phi : \mathbb{R}^n \times (0,\infty)$, we write $\phi(B(x,r)) = \phi(x,r)$.

Definition 29.1 ([147, p. 2])**.** For a constant $p \in [1,\infty)$ and a variable growth function ϕ, let $\mathcal{L}_{p,\phi}(\mathbb{R}^n)$, $L_{p,\phi}(\mathbb{R}^n)$ and $\Lambda_\phi(\mathbb{R}^n)$ be the sets of all f such that

$$\begin{aligned}
\|f\|_{\mathcal{L}_{p,\phi}} &:= \sup_B \frac{1}{\phi(B)} \left(⨍_B |f(x) - f_B|^p \, dx \right)^{1/p} < \infty, \\
\|f\|_{L_{p,\phi}} &:= \sup_B \frac{1}{\phi(B)} \left(⨍_B |f(x)|^p \, dx \right)^{1/p} < \infty, \\
\|f\|_{\Lambda_\phi} &:= \sup_{x,y\in\mathbb{R}^n,\, x\neq y} \frac{2|f(x)-f(y)|}{\phi(x,|x-y|)+\phi(y,|x-y|)} < \infty,
\end{aligned}$$

respectively. In the above, the first two supremums are taken over all balls B in $\mathbb{R}^n$ and in the definition of $\|f\|_{\mathcal{L}_{p,\phi}}$ f is assumed to be locally integrable.

If $p = 1$, then $\mathcal{L}_{1,\phi}(\mathbb{R}^n) = \mathrm{BMO}_\phi(\mathbb{R}^n)$. If $\phi \equiv 1$, then $\mathcal{L}_{1,\phi}(\mathbb{R}^n) = \mathrm{BMO}(\mathbb{R}^n)$. If $\phi(B) = |B|^{-1/p}$, then $L_{p,\phi}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$. If $\phi \equiv 1$, then $L_{p,\phi}(\mathbb{R}^n) = L^\infty(\mathbb{R}^n)$. See [80] for a passage of $\|f\|_{\mathcal{L}_{p,\phi}}$ to the variable exponent case which turns out to be a characterization of the BMO norm.

We regard $\mathcal{L}_{p,\phi}(\mathbb{R}^n)$ and $L_{p,\phi}(\mathbb{R}^n)$ as spaces of functions modulo null-functions, and $\Lambda_\phi(\mathbb{R}^n)$ as a space of functions defined at all $x \in \mathbb{R}^n$. Then $\mathcal{L}_{p,\phi}(\mathbb{R}^n)/\mathcal{C}$, $L_{p,\phi}(\mathbb{R}^n)$ and $\Lambda_\phi(\mathbb{R}^n)/\mathcal{C}$ are Banach spaces with the norm $\|f\|_{\mathcal{L}_{p,\phi}}$, $\|f\|_{L_{p,\phi}}$ and $\|f\|_{\Lambda_\phi}$, respectively, where $\mathcal{C}$ is the space of all constant functions. For any fixed ball B_0 and for any fixed point x_0, $\|f\|_{\mathcal{L}_{p,\phi}} + |f_{B_0}|$ and $\|f\|_{\Lambda_\phi} + |f(x_0)|$ are norms on $\mathcal{L}_{p,\phi}(\mathbb{R}^n)$ and $\Lambda_\phi(\mathbb{R}^n)$, respectively. Thereby $\mathcal{L}_{p,\phi}(\mathbb{R}^n)$ and $\Lambda_\phi(\mathbb{R}^n)$ are Banach spaces. We note that for each ball B_1 and for each point $x_1 \in \mathbb{R}^n$, $\|f\|_{\mathcal{L}_{p,\phi}} + |f_{B_0}| \sim \|f\|_{\mathcal{L}_{p,\phi}} + |f_{B_1}|$ and $\|f\|_{\Lambda_\phi} + |f(x_0)| \sim \|f\|_{\Lambda_\phi} + |f(x_1)|$. If $\mu(\mathbb{R}^n) < \infty$, then $\|f\|_{\mathcal{L}_{p,\phi}} + |f_{B_0}| \sim \|f\|_{\mathcal{L}_{p,\phi}} + \|f\|_{L^p}$. If $\sup_B \phi(B) < \infty$, then $\|f\|_{\Lambda_\phi} + |f(x_0)| \sim \|f\|_{\Lambda_\phi} + \|f\|_{L^\infty}$. The theory of generalized Campanato spaces is not a mere quest to generality. One of the prominent examples is Theorem 5.7.

We consider the following conditions on $\phi : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$.:

$$\text{(29.1)} \qquad \text{doubling} \qquad \frac{1}{A_1} \le \frac{\phi(x,s)}{\phi(x,r)} \le A_1, \qquad \frac{1}{2} \le \frac{s}{r} \le 2,$$

$$\text{(29.2)} \qquad \text{compatibility} \qquad \frac{1}{A_2} \le \frac{\phi(x,r)}{\phi(y,r)} \le A_2, \qquad |x-y| \le r,$$

$$\text{(29.3)} \qquad \text{almost increasing} \qquad \phi(x,r) \le A_3\phi(x,s), \quad 0 < r < s < \infty,$$

where A_i, $i = 1,2,3$, are positive constants independent of $x, y \in \mathbb{R}^n$, $r, s > 0$. Note that (29.2) and (29.3) imply that there exists a positive constant C such that

$$\phi(x,r) \le C\phi(y,s) \quad \text{for} \quad B(x,r) \subset B(y,s),$$

where the constant C is independent of balls $B(x,r)$ and $B(y,s)$.

The following three theorems are known:

Theorem 29.1 ([147, Theorem 2.1], [164])**.** *Let* $1 \le p < \infty$. *If* $\phi \in \mathbb{Z}_1$, *then* $\mathcal{L}_{p,\phi}(\mathbb{R}^n)/\mathcal{C} = L_{p,\phi}(\mathbb{R}^n)$. *Moreover, if* $\phi(B) = |B|^{-1/p}$ *also, then* $\mathcal{L}_{p,\phi}(\mathbb{R}^n)/\mathcal{C} = L_{p,\phi}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ *with equivalent norms.*

Theorem 29.2 ([150, Theorem 3.1])**.** *If* ϕ *fulfills* (29.1)–(29.3), *then* $\mathcal{L}_{p,\phi}(\mathbb{R}^n) = \mathcal{L}_{1,\phi}(\mathbb{R}^n)$ *and* $\mathcal{L}_{p,\phi}(\mathbb{R}^n)/\mathcal{C} = \mathcal{L}_{1,\phi}(\mathbb{R}^n)/\mathcal{C}$ *with equivalent norms for every* $1 \le p < \infty$.

Theorem 29.3 ([147])**.** *If* $\phi \in \mathbb{Z}^1$ *satisfies* (29.1)–(29.3), *then* $\mathcal{L}_{p,\phi}(\mathbb{R}^n) = \Lambda_\phi(\mathbb{R}^n)$ *and* $\mathcal{L}_{p,\phi}(\mathbb{R}^n)/\mathcal{C} = \Lambda_\phi(\mathbb{R}^n)/\mathcal{C}$ *with equivalent norms for every* $1 \le p < \infty$.

Morrey spaces with variable exponents fall under the scope of generalized Morrey spaces in Definition 29.1 and they are related as follows:

Theorem 29.4 ([151, Theorem 3.8])**.** *Let* $p(\cdot) : \mathbb{R}^n \to (1, \infty)$ *be a variable exponent such that* $p'(\cdot)$ *is log-Hölder continuous. Define*

$$\phi(x,r) := \begin{cases} r^{-n/p(x)}, & 0 < r < 1/2, \\ r^{-n/p_+}, & 1/2 \le r < \infty. \end{cases}$$

Then $L^{p(\cdot)}(\mathbb{R}^n) \subset L_{1,\phi}(\mathbb{R}^n) \subset \mathcal{L}_{1,\phi}(\mathbb{R}^n)$ *and there exist positive constants* C_1 *and* C_2 *such that*

$$\|f\|_{\mathcal{L}_{1,\phi}} + |f_{B_0}| \le C_1 \|f\|_{L_{1,\phi}} \le C_2 \|f\|_{L^{p(\cdot)}}.$$

Here B_0 *stands for the unit ball in* $\mathbb{R}^n$.

29.2 Lipschitz spaces with variable exponents There are two types of Lipschitz spaces with variable exponents:

Definition 29.2 ([151, Definition 2.2])**.** Let α_* be a constant in $[0, \infty)$ and let $\alpha(\cdot) : \mathbb{R}^n \to [0, \infty)$ be a measurable function with $0 \le \alpha_- \le \alpha_+ < \infty$.

(i) For $\phi(x,r) = r^{\alpha(x)}$, denote $\Lambda_\phi(\mathbb{R}^n)$ by $\mathrm{Lip}_{\alpha(\cdot)}(\mathbb{R}^n)$. In this case,

$$\|f\|_{\mathrm{Lip}_{\alpha(\cdot)}} = \sup_{x,y\in\mathbb{R}^n,\, x\ne y} \frac{2|f(x)-f(y)|}{|x-y|^{\alpha(x)} + |y-x|^{\alpha(y)}}.$$

(ii) For

$$\phi(x,r) := \begin{cases} r^{\alpha(x)}, & 0 < r < 1/2, \\ r^{\alpha_*}, & 1/2 \le r < \infty, \end{cases} \tag{29.4}$$

denote $\Lambda_\phi(\mathbb{R}^n)$ by $\mathrm{Lip}^{\alpha_*}_{\alpha(\cdot)}(\mathbb{R}^n)$. In this case,

$$\|f\|_{\mathrm{Lip}^{\alpha_*}_{\alpha(\cdot)}} = \max\left\{ \sup_{0<|x-y|<1/2} \frac{2|f(x)-f(y)|}{|x-y|^{\alpha(x)} + |x-y|^{\alpha(y)}}, \sup_{|x-y|\ge 1/2} \frac{|f(x)-f(y)|}{|x-y|^{\alpha_*}} \right\}.$$

An expression equivalent to $\|f\|_{\mathrm{Lip}^{\alpha_*}_{\alpha(\cdot)}}$ is

$$\sup_{x,y\in\mathbb{R}^n,\, x\ne y} \frac{|f(x)-f(y)|}{\|\chi_{B(x,|x-y|)}\|_{L^{n/\alpha(\cdot)}}}$$

if $\alpha \in LH$ and $\alpha_\infty = \alpha_*$. See Section 19.

Remark 29.1. (i) Let ϕ be as in (29.4). For any $r_1 > 0$, let

$$\phi_1(x,r) := \begin{cases} r^{\alpha(x)}, & 0 < r < r_1, \\ r^{\alpha_*}, & r_1 \le r < \infty. \end{cases}$$

Then $\phi \sim \phi_1$ and $\mathcal{L}_{p,\phi}(\mathbb{R}^n) = \mathcal{L}_{p,\phi_1}(\mathbb{R}^n)$ with equivalent norms.

(ii) If we consider function spaces on the torus $\mathbb{T}^n$, or equivalently, assume that the functions are 2π-periodic, then we have $\mathrm{Lip}_{\alpha(\cdot)}^{\alpha_*}(\mathbb{T}^n) = \mathrm{Lip}_{\alpha(\cdot)}(\mathbb{T}^n)$ and $\|f\|_{\mathrm{Lip}_{\alpha(\cdot)}(\mathbb{T}^n)} + |f(x_0)| \sim \|f\|_{\mathrm{Lip}_{\alpha(\cdot)}(\mathbb{T}^n)} + \|f\|_{L^\infty(\mathbb{T}^n)}$.

We recall that a function $\alpha(\cdot) : \mathbb{R}^n \to (-\infty, \infty)$ is log-Hölder continuous if there exists a constant $C_{\alpha(\cdot)} > 0$ such that

$$|\alpha(x) - \alpha(y)| \le \frac{C_{\alpha(\cdot)}}{\log(1/|x-y|)} \quad \text{for} \quad 0 < |x-y| < 1/2.$$

The set of all such functions is denoted by LH_0.

Of interest is the following example:

Example 29.1 ([151, Proposition 3.3])**.** Let $\alpha_* \in (-\infty, \infty)$ be a real constant and let $\alpha(\cdot) \in LH_0 \cap L^\infty(\mathbb{R}^n)$. Define

$$\phi(x,r) := \begin{cases} r^{\alpha(x)}, & 0 < r < 1/2, \\ r^{\alpha_*}, & 1/2 \le r < \infty. \end{cases}$$

Then ϕ satisfies (29.1) and (29.2).

As corollaries of Example 29.1 and Theorems 29.2 and 29.3 we have the following:

Example 29.2. Let $\alpha_* \in [0, \infty)$ be a real constant and let $\alpha(\cdot) \in LH_0$ with $0 \le \alpha_- \le \alpha_+ < \infty$.

$$\phi(x,r) := \begin{cases} r^{\alpha(x)}, & 0 < r < 1/2, \\ r^{\alpha_*}, & 1/2 \le r < \infty. \end{cases}$$

Then ϕ satisfies (29.1), (29.2) and (29.3). In this case, $\mathcal{L}_{p,\phi}(\mathbb{R}^n)/\mathcal{C} = \mathcal{L}_{1,\phi}(\mathbb{R}^n)/\mathcal{C}$ and $\mathcal{L}_{p,\phi}(\mathbb{R}^n) = \mathcal{L}_{1,\phi}(\mathbb{R}^n)$ with equivalent norms for every $1 \le p < \infty$.

Example 29.3. Let $\alpha_* \in [0, \infty)$ be a real constant and let $\alpha(\cdot) \in LH_0$ with $0 < \alpha_- \le \alpha_+ < \infty$. Define

$$\phi(x,r) := \begin{cases} r^{\alpha(x)}, & 0 < r < 1/2, \\ r^{\alpha_*}, & 1/2 \le r < \infty. \end{cases}$$

Then ϕ belongs to $\mathbb{Z}^1$ and it satisfies (29.1), (29.2) and (29.3). In this case, $\mathcal{L}_{p,\phi}(\mathbb{R}^n)/\mathcal{C} = \mathrm{Lip}_{\alpha(\cdot)}^{\alpha_*}(\mathbb{R}^n)/\mathcal{C}$ and $\mathcal{L}_{p,\phi}(\mathbb{R}^n) = \mathrm{Lip}_{\alpha(\cdot)}^{\alpha_*}(\mathbb{R}^n)$ with equivalent norms for every $1 \le p < \infty$.

Variable exponent Hölder type spaces have appeared in various papers [83, 84, 175, 224, 225]; see also the surveys [190, 193]. In [175], variable exponent Campanato spaces $\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\mathcal{X})$ are defined by Rafeiro and S. Samko in the setting of doubling quasimetric measure spaces $(\mathcal{X}, d, \mu)$, and they investigated function spaces whose smoothness order is less than or equal to 1 in [175]. To define the variable Campanato space $\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\mathcal{X})$, we use the functional

$$\mathcal{I}^{p(\cdot),\lambda(\cdot)}(f) := \sup_{x \in \mathcal{X}, r > 0} \frac{1}{\mu(B(x,r))^{\lambda(x)}} \int_{B(x,r)} |f(y) - f_{B(x,r)}|^{p(y)} \, d\mu(y).$$

See [175, Section 3] for more details.

29.3 Singular integrals on Morrey and Campanato spaces About the boundedness of (sub)linear operator, the following result is of fundamental importance. See [142].

Theorem 29.5 ([142]). *Let $1 < p < \infty$. Assume that a variable growth function $\phi \in \mathbb{Z}^1$ satisfies the doubling condition. Let $T : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ be a bounded linear operator associated with a kernel K satisfying the size condition*

$$|K(x, y)| \le \frac{C}{|x - y|^n}, \quad x \ne y.$$

If T is bounded on $L^p(\mathbb{R}^n)$, then T can be extended to a bounded operator on $L_{p,\phi}(\mathbb{R}^n)$.

Let $0 < \kappa \le 1$. We shall consider a singular integral operator T with measurable kernel K on $\mathbb{R}^n \times \mathbb{R}^n$ satisfying the following properties:

$$|K(x, y)| \le \frac{C}{|x - y|^n} \quad \text{for} \quad x \ne y, \tag{29.5}$$

$$|K(x, y) - K(z, y)| + |K(y, x) - K(y, z)| \le \frac{C}{|x - y|^n}\left(\frac{|x - z|}{|x - y|}\right)^{\kappa} \quad \text{for} \quad |x - y| \ge 2K_1|x - z|, \tag{29.6}$$

$$\int_{r \le |x-y| < R} K(x, y)\, dy = \int_{r \le |x-y| < R} K(y, x)\, dy = 0 \quad \text{for } 0 < r < R < \infty \text{and } x \in \mathbb{R}^n, \tag{29.7}$$

where C is a positive constant independent of $x, y, z \in \mathbb{R}^n$. For $\eta > 0$, let

$$T_\eta f(x) := \int_{|x-y| \ge \eta} K(x, y) f(y)\, dy.$$

Then $T_\eta f(x)$ is well defined for $f \in L^p_{\text{comp}}(\mathbb{R}^n)$, $1 < p < \infty$. We assume that, for all $1 < p < \infty$, there exists positive constant C_p independently $\eta > 0$ such that,

$$\|T_\eta f\|_p \le C_p \|f\|_p \quad \text{for} \quad f \in L^p_{\text{comp}}(\mathbb{R}^n),$$

and $T_\eta f$ converges to Tf in $L^p(\mathbb{R}^n)$ as $\eta \to 0$. By this assumption, the operator T can be extended as a continuous linear operator on $L^p(\mathbb{R}^n)$. We shall say the operator T satisfying (29.5)–(29.7) is a singular integral operator of type κ.

Now, to define T for functions $f \in \mathcal{L}_{p,\phi}(\mathbb{R}^n)$, we first define the modified version of T_η as follows. For a fixed ball $B_0 = B(x_0, r_0)$, let

$$\tilde{T}_\eta f(x) = \int_{|x-y| \ge \eta} f(y)\big[K(x, y) - K(x_0, y)(1 - \chi_{B_0}(y))\big]\, dy. \tag{29.8}$$

Then we can show that the integral in the definition above converges absolutely for all x and that $\tilde{T}_\eta f$ converges in $L^p(B)$ as $\eta \to 0$ for all balls B (see [151, the proof of Theorem 29.6]). We denote the limit by $\tilde{T} f$. Then, changing B_0 in the definition above results in adding a constant.

Theorem 29.6 ([151, Theorem 4.1]). *Let $0 < \kappa \le 1$ and $1 < p < \infty$. Assume that ϕ satisfies* (29.1) *and that there exists a constant $A > 0$ such that, for all $x \in \mathbb{R}^n$ and $r > 0$,*

$$r^\kappa \int_r^\infty \frac{\phi(x, t)}{t^{1+\kappa}}\, dt \le A\phi(x, r). \tag{29.9}$$

If T is a singular integral operator of type κ, then $\tilde{T}$ is bounded on $\mathcal{L}_{p,\phi}(\mathbb{R}^n)/\mathcal{C}$ and on $\mathcal{L}_{p,\phi}(\mathbb{R}^n)$, that is, there exist constants $C_1 > 0$ and $C_2 > 0$ such that

$$\|\tilde{T}f\|_{\mathcal{L}_{p,\phi}} \le C_1 \|f\|_{\mathcal{L}_{p,\phi}}, \quad f \in \mathcal{L}_{p,\phi}(\mathbb{R}^n)/\mathcal{C},$$

and that

$$\|\tilde{T}f\|_{\mathcal{L}_{p,\phi}} + \left|(\tilde{T}f)_{B_0}\right| \le C_2 \left(\|f\|_{\mathcal{L}_{p,\phi}} + |f_{B_0}|\right), \quad f \in \mathcal{L}_{p,\phi}(\mathbb{R}^n).$$

Moreover, if ϕ satisfies (29.2) *and* (29.3) *also, then $\tilde{T}$ is bounded on $\mathcal{L}_{1,\phi}(\mathbb{R}^n)/\mathcal{C}$ and on $\mathcal{L}_{1,\phi}(\mathbb{R}^n)$.*

If we combine Theorems 29.3 and 29.6, we obtain the following boundedness.

Corollary 29.7. *In addition to the assumption in Theorem 29.6, if ϕ belongs to $\mathbb{Z}^1$ and it satisfies* (29.2) *and* (29.3)*, then $\tilde{T}$ is bounded on $\Lambda_\phi(\mathbb{R}^n)/\mathcal{C}$ and on $\Lambda_\phi(\mathbb{R}^n)$.*

As a special case of Corollary 29.3 we have the following:

Corollary 29.8 ([151, Corollary 4.2])**.** *Let $\kappa \in (0,1]$ and $\alpha_* \in (0,\kappa)$. Let $\alpha(\cdot) \in LH_0$ satisfy $0 < \alpha_- \le \alpha_+ < \kappa$. If T is a singular integral operator of type κ, then $\tilde{T}$ is bounded on $\mathrm{Lip}_{\alpha(\cdot)}^{\alpha_*}(\mathbb{R}^n)/\mathcal{C}$ and on $\mathrm{Lip}_{\alpha(\cdot)}^{\alpha_*}(\mathbb{R}^n)$.*

29.4 Fractional integrals on Campanato spaces We consider the fractional integral operator I_α, $0 < \alpha < n$, and its modified version $\tilde{I}_\alpha$ defined by

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\, dy,$$

$$\tilde{I}_\alpha f(x) = \int_{\mathbb{R}^n} f(y) \left(\frac{1}{|x-y|^{n-\alpha}} - \frac{1-\chi_{B(0,1)}(y)}{|x_0-y|^{n-\alpha}} \right) dy. \tag{29.10}$$

It is well known as the Hardy-Littlewood-Sobolev theorem that I_α is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ if $1 < p < q < \infty$ and $-n/p + \alpha = -1/q$. Note that, if $0 < \alpha < 1$, then

$$\frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|z-y|^{n-\alpha}}$$

is integrable on $\mathbb{R}^n$ as a function of y and for every choice of x and z, and

$$\int_{\mathbb{R}^n} \left(\frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|z-y|^{n-\alpha}} \right) dy = 0 \quad \text{for any} x, z \ \text{in} \mathbb{R}^n. \tag{29.11}$$

We can show that, for $f \in \mathcal{L}_{p,\phi}(\mathbb{R}^n)$, the integral in (29.10) converges absolutely for all x (see the proof of [151, Theorem 29.9]) and therefore changing $B(0,1)$ to another ball in the definition of $\tilde{I}_\alpha f(x)$ above results in adding a constant.

Theorem 29.9 ([151, Theorem 5.1])**.** *Let $0 < \alpha < 1$, $1 \le p < \infty$ and let ϕ and ψ satisfy* (29.1)*. Assume that there exists a constant $A > 0$ such that, for all $x \in \mathbb{R}^n$ and $0 < r < \infty$,*

$$r \int_r^\infty \frac{t^\alpha \phi(x,t)}{t^2}\, dt \le A \psi(x,r). \tag{29.12}$$

Assume also that p and q satisfy one of the following conditions:

(i) *$p = 1$ and $1 \le q < n/(n-\alpha)$;*

(ii) $1 < p < n/\alpha$ *and* $1 \le q \le pn/(n-p\alpha)$;

(iii) $n/\alpha \le p < \infty$ *and* $1 \le q < \infty$.

Then $\tilde{I}_\alpha$ *is a bounded linear operator from* $\mathcal{L}_{p,\phi}(\mathbb{R}^n)/\mathcal{C}$ *to* $\mathcal{L}_{q,\psi}(\mathbb{R}^n)/\mathcal{C}$ *and from* $\mathcal{L}_{p,\phi}(\mathbb{R}^n)$ *to* $\mathcal{L}_{q,\psi}(\mathbb{R}^n)$, *namely, there exist constants* $C_1 > 0$ *and* $C_2 > 0$ *such that*

$$\|\tilde{I}_\alpha f\|_{\mathcal{L}_{q,\psi}} \le C_1 \|f\|_{\mathcal{L}_{p,\phi}}, \quad f \in \mathcal{L}_{p,\phi}(\mathbb{R}^n)/\mathcal{C},$$

and

$$\|\tilde{I}_\alpha f\|_{\mathcal{L}_{q,\psi}} + \left|(\tilde{I}_\alpha f)_{B(0,1)}\right| \le C_2 \left(\|f\|_{\mathcal{L}_{p,\phi}} + |f_{B(0,1)}|\right), \quad f \in \mathcal{L}_{p,\phi}(\mathbb{R}^n).$$

Corollary 29.10 ([151, Corollary 5.2]). *Let* $\alpha, \beta_*, \gamma_* \in (0,1)$. *Let* $\beta(\cdot), \gamma(\cdot) \in LH_0$ *such that* $\beta(x), \gamma(x) \in (0,1)$ *for all* $x \in \mathbb{R}^n$. *Assume that* (29.11) *holds. If* $\gamma(\cdot) = \alpha + \beta(\cdot)$, $0 < \beta_- < \gamma_+ < 1$ *and* $\gamma_* = \alpha + \beta_*$, *then* $\tilde{I}_\alpha$ *is bounded from* $\mathrm{Lip}^{\beta_*}_{\beta(\cdot)}(\mathbb{R}^n)/\mathcal{C}$ *to* $\mathrm{Lip}^{\gamma_*}_{\gamma(\cdot)}(\mathbb{R}^n)/\mathcal{C}$ *and from* $\mathrm{Lip}^{\beta_*}_{\beta(\cdot)}(\mathbb{R}^n)$ *to* $\mathrm{Lip}^{\gamma_*}_{\gamma(\cdot)}(\mathbb{R}^n)$.

For the boundedness of $\tilde{I}_\alpha$ from $L_{p,\phi}(\mathbb{R}^n)$ to $\mathcal{L}_{q,\psi}(\mathbb{R}^n)$, the assumption $0 < \alpha < 1$ in Theorem 29.9 can be relaxed into $0 < \alpha < n$.

Theorem 29.11. *Let* $0 < \alpha < n$, $1 \le p < \infty$ *and let* ϕ *and* ψ *satisfy* (29.1). *Assume that there exists a constant* $A > 0$ *such that, for all* $x \in \mathbb{R}^n$ *and* $0 < r < \infty$,

$$r \int_r^\infty \frac{t^\alpha \phi(x,t)}{t^2}\, dt \le A\psi(x,r). \tag{29.13}$$

Assume that p *and* q *satisfy one of the following conditions:*

(i) $p = 1$ *and* $1 \le q < n/(n-\alpha)$;

(ii) $1 < p < n/\alpha$ *and* $1 \le q \le pn/(n-p\alpha)$;

(iii) $n/\alpha \le p < \infty$ *and* $1 \le q < \infty$.

Then $\tilde{I}_\alpha$ *is bounded from* $L_{p,\phi}(\mathbb{R}^n)$ *to* $\mathcal{L}_{q,\psi}(\mathbb{R}^n)$, *that is, there exists a constants* $C > 0$ *such that*

$$\|\tilde{I}_\alpha f\|_{\mathcal{L}_{q,\psi}} + \left|(\tilde{I}_\alpha f)_{B(0,1)}\right| \le C\|f\|_{L_{p,\phi}}, \quad f \in L_{p,\phi}(\mathbb{R}^n).$$

By Theorems 29.4, 29.11 and Corollary 29.3 we have the following, which is a generalization of the boundedness of $\tilde{I}_\alpha$ from $L^{n/\alpha}(\mathbb{R}^n)$ to $\mathrm{BMO}(\mathbb{R}^n)$ and from $L^p(\mathbb{R}^n)$ to $\mathrm{Lip}_{\alpha-n/p}(\mathbb{R}^n)$ for $0 < \alpha - n/p < 1$.

Corollary 29.12 ([151, Corollary 5.4]). *Let* $0 < \alpha < n$, $1 \le q < n/(n-\alpha)$, $p(\cdot) : \mathbb{R}^n \to (1,\infty)$ *and* $p(\cdot)/(p(\cdot)-1)$ *be log-Hölder continuous. Assume that* $p_+ < n/(\alpha-1)$ *if* $\alpha > 1$. *Define*

$$\psi(x,r) := \begin{cases} r^{\alpha-n/p(x)}, & 0 < r < 1/2, \\ r^{\alpha-n/p_+}, & 1/2 \le r < \infty. \end{cases}$$

Then $\tilde{I}_\alpha$ *is bounded from* $L^{p(\cdot)}(\mathbb{R}^n)$ *to* $\mathcal{L}_{q,\psi}(\mathbb{R}^n)$, *that is*

$$\|\tilde{I}_\alpha f\|_{\mathcal{L}_{q,\psi}} + \left|(\tilde{I}_\alpha f)_{B(0,1)}\right| \le C\|f\|_{L^{p(\cdot)}}, \quad f \in L^{p(\cdot)}(\mathbb{R}^n).$$

If $n/\alpha < p_-$ *also, then* $\tilde{I}_\alpha$ *is bounded from* $L^{p(\cdot)}(\mathbb{R}^n)$ *to* $\mathrm{Lip}^{\alpha-n/p_+}_{\alpha-n/p(\cdot)}(\mathbb{R}^n)$, *that is*

$$\|\tilde{I}_\alpha f\|_{\mathrm{Lip}^{\alpha-n/p_+}_{\alpha-n/p(\cdot)}} + \left|\tilde{I}_\alpha f(0)\right| \le C\|f\|_{L^{p(\cdot)}}, \quad f \in L^{p(\cdot)}(\mathbb{R}^n).$$

When ϕ and ψ are power functions, we can restate the above corollary as follows:

Corollary 29.13. *Let $0 < \alpha < n$, $1 \le p < \infty$ and $-n/p + \alpha \le \lambda + \alpha = \mu < 1$. Assume that p and q satisfy one of the following conditions:*

(i) $p = 1$ *and* $1 \le q < n/(n-\alpha)$;

(ii) $1 < p < n/\alpha$ *and* $1 \le q \le pn/(n - p\alpha)$;

(iii) $\mu \ge 0, n/\alpha \le p < \infty$ *and* $1 \le q < \infty$.

Then $\tilde{I}_\alpha$ is bounded from $L_{p,\lambda}(\mathbb{R}^n)$ to $\mathcal{L}_{q,\mu}(\mathbb{R}^n)$, that is, there exists a positive constant C such that

$$\|\tilde{I}_\alpha f\|_{\mathcal{L}_{q,\mu}} + |(\tilde{I}_\alpha f)_{B(0,1)}| \le C\|f\|_{L_{p,\lambda}}, \quad f \in L_{p,\lambda}(\mathbb{R}^n).$$

Corollary 29.14. *Let $0 < \alpha < 1$, $1 \le p < \infty$ and $-n/p + \alpha \le \lambda + \alpha = \mu < 1$. Assume that p and q satisfy one of the following conditions:*

(i) $p = 1$ *and* $1 \le q < n/(n-\alpha)$;

(ii) $1 < p < n/\alpha$ *and* $1 \le q \le pn/(n - p\alpha)$;

(iii) $\mu \ge 0$, $n/\alpha \le p < \infty$ *and* $1 \le q < \infty$.

Then $\tilde{I}_\alpha$ is bounded from $\mathcal{L}_{p,\lambda}(\mathbb{R}^n)/\mathcal{C}$ to $\mathcal{L}_{q,\mu}(\mathbb{R}^n)/\mathcal{C}$ and from $\mathcal{L}_{p,\lambda}(\mathbb{R}^n)$ to $\mathcal{L}_{q,\mu}(\mathbb{R}^n)$, that is, there exist positive constants C_1 and C_2 such that

$$\|\tilde{I}_\alpha f\|_{\mathcal{L}_{q,\mu}} \le C_1\|f\|_{\mathcal{L}_{p,\lambda}}, \quad f \in \mathcal{L}_{p,\lambda}(\mathbb{R}^n)/\mathcal{C},$$

and

$$\|\tilde{I}_\alpha f\|_{\mathcal{L}_{q,\mu}} + |(\tilde{I}_\alpha f)_{B(0,1)}| \le C_2 \left(\|f\|_{\mathcal{L}_{p,\lambda}} + |f_{B(0,1)}|\right), \quad f \in \mathcal{L}_{p,\lambda}(\mathbb{R}^n),$$

respectively.

The above results still hold in the setting of spaces of homogeneous type $(\mathcal{X}, d, \mu)$. N. Samko, S. Samko, and Vakulov [180] proved theorems on mapping properties of potential operators of variable order $\alpha(\cdot)$ in variable exponent Hölder spaces $H^{\lambda(\cdot)}(\mathcal{X})$ of functions f on a quasi-metric measure space $(\mathcal{X}, d, \mu)$. It is shown that they act from Hölder space with the exponent $\lambda(\cdot)$ to another one with a better exponent $\lambda(\cdot) + \alpha(\cdot)$, and similar mapping properties of hypersingular integrals of variable order $\alpha(\cdot)$ from such a space into the space with the worse exponent $\lambda(\cdot) - \alpha(\cdot)$ in the case $\alpha(\cdot) < \lambda(\cdot)$.

See [204, 205] for the results in the non-doubling setting. In [204], an example showing that a modification adapted to non-doubling measures can be found.

29.5 Campanato, Morrey and Hölder spaces on metric measure spaces Let $\lambda : \mathcal{X} \to (0, 1]$ be a function on a doubling metric measure space $(\mathcal{X}, d, \mu)$. The Hölder space $H^{\lambda(\cdot)}(\mathcal{X})$ of variable exponent $\lambda(\cdot)$ is the set of all bounded continuous functions f on $\mathcal{X}$ for which the quantity

$$\|f\|_{H^{\lambda(\cdot)}(\mathcal{X})} = \|f\|_{L^\infty(\mathcal{X})} + [f]_{\lambda(\cdot)}$$

is finite. Remark that the $L^\infty(\mathcal{X})$-norm comes into play in the above definition. Here, we defined

$$[f]_{\lambda(\cdot)} := \sup_{x,y\in\mathcal{X}, 0<d(x,y)\le 1} \frac{|f(x) - f(y)|}{d(x,y)^{\max(\lambda(x),\lambda(y))}}.$$

A function $f \in L^{p(\cdot)}(\mathcal{X})$ is said to belong to the variable exponent Hajłasz-Sobolev space $M^{1,p(\cdot)}(\mathcal{X})$ if there exists $g \in L^{p(\cdot)}(\mathcal{X})$ such that

$$|f(x) - f(y)| \le d(x,y)(g(x) + g(y)) \tag{29.14}$$

for μ-a.e. $x, y \in \mathcal{X}$. The function g satisfying (29.14) is called the generalized gradient of f. Define the generalized Sobolev space $\mathcal{M}^{1,p(\cdot)}(\mathcal{X})$ by;

$$\|f\|_{\mathcal{M}^{1,p(\cdot)}(\mathcal{X})} := \|f\|_{L^{p(\cdot)}(\mathcal{X})} + \inf_g \|g\|_{L^{p(\cdot)}(\mathcal{X})},$$

where g runs over all μ-measurable functions satisfying (29.14).

Example 29.15. ([5, Lemma 1]) *Let $\alpha(\cdot), \beta(\cdot) : \mathcal{X} \to (0,\infty)$ be bounded measurable functions. Define*

$$\mathcal{M}^{\sharp}_{\alpha(\cdot)} f(x) = \sup_{r>0} \frac{r^{-\alpha(x)}}{\mu(B(x,r))} \int_{B(x,r)} |f(y) - f_{B(x,r)}| \, d\mu(y).$$

Suppose $0 < \alpha_- \le \alpha_+ < \infty$ and $0 < \beta_- \le \beta_+ < \infty$. Then

$$|f(x) - f(y)| \le C(\mu, \alpha, \beta)(d(x,y)^{\alpha(x)} \mathcal{M}^{\sharp}_{\alpha(\cdot)} f(x) + d(x,y)^{\beta(x)} \mathcal{M}^{\sharp}_{\beta(\cdot)} f(x)).$$

Thus, when $\alpha \equiv \beta \equiv 1$, $\mathcal{M}^{\sharp}_{\alpha(\cdot)} f$ is a generalized gradient.

Example 29.16. ([5]) *Let $\alpha(\cdot), \beta(\cdot) : \mathcal{X} \to (0,\infty)$ be bounded measurable functions. Suppose $0 < \alpha_- \le \alpha_+ < 1$ and $0 < \beta_- \le \beta_+ < 1$. If g is the generalized gradient of f, then*

$$|f(x) - f(y)| \le C(\mu, \alpha, \beta)(d(x,y)^{1-\alpha(x)} \mathcal{M}_{\alpha(\cdot)} g(x) + d(x,y)^{1-\beta(x)} \mathcal{M}_{\beta(\cdot)} g(x)).$$

Almeida and S. Samko [6] extended the results of the paper on Euclidean spaces [5] to a more general setting. Namely, Almeida and S. Samko obtained embeddings of variable exponent Hajłasz-Sobolev spaces into Hölder classes of variable order on a bounded quasi-metric measure space $(\mathcal{X}, d, \mu)$ with doubling condition. the proofs are based on the estimation of Sobolev functions through maximal functions.

Theorem 29.17. ([5]) *If $\mathcal{X}$ is a bounded doubling metric measure space and $p(\cdot)$ satisfies the log-Hölder condition and $p_- > \log_2 C_\mu$, where C_μ is the doubling constant, then*

$$\mathcal{M}^{1,p(\cdot)}(\mathcal{X}) \hookrightarrow H^{1 - \frac{\log_2 C_\mu}{p(\cdot)}}(\mathcal{X}).$$

See [7, 85, 170, 190] and [123] for more related results.

30 Morrey spaces with variable exponents and variable growth function In this section we introduce Morrey spaces with variable exponents and variable growth function. The results in this section show that the smoothing effect of I_α is local by considering Morrey spaces with variable exponents. These spaces can be generalized to Musielak-Orlicz-Morrey spaces. See Mizuta, Nakai, Ohno and Shimomura [123, 125] and Maeda, Mizuta, Ohno and Shimomura [115]. See also [110, 124, 126].

Definition 30.1. For a variable exponent $p(\cdot) : \mathbb{R}^n \to [1, \infty)$, and a variable growth function $\varphi : \mathbb{R}^n \times (0, \infty) \to (0, \infty)$, let $L^{(p,\varphi)}(\mathbb{R}^n)$ be the set of all measurable functions f such that

$$\|f\|_{L^{(p,\varphi)}} := \sup_B \|f\|_{p,\varphi,B},$$

where the supremum is taken over all balls B in $\mathbb{R}^n$ and

$$\|f\|_{p,\varphi,B} := \inf \left\{ \lambda > 0 : \frac{1}{\varphi(B)} \fint_B \left(\frac{|f(y)|}{\lambda} \right)^{p(y)} dy \le 1 \right\}.$$

The next remark shows that the definition is a natural extension of $L^{p(\cdot)}(\mathbb{R}^n)$ considered here.

Remark 30.1. Let $\|f\|^*_{L^{(p,\varphi)}} := \inf \{\lambda > 0 : I_{p,\varphi}(f/\lambda) \le 1\}$ for $f \in L^0(\mathbb{R}^n)$, where

$$I_{p,\varphi}(f) := \sup_B \frac{1}{\varphi(B)} \fint_B |f(y)|^{p(y)} \, dy.$$

Then $\|f\|_{L^{(p,\varphi)}} = \|f\|^*_{L^{(p,\varphi)}}$. Actually, if

$$\lambda_0 := \|f\|_{L^{(p,\varphi)}} = \sup_B \|f\|_{p,\varphi,B},$$

then, for all $\varepsilon \in (0, \lambda_0)$, there exists a ball B_0 such that $\lambda_0 \ge \|f\|_{p,\varphi,B_0} > \lambda_0 - \varepsilon$, that is,

$$\sup_B \frac{1}{\varphi(B)} \fint_B \left(\frac{|f(y)|}{\lambda_0} \right)^{p(y)} dy \le 1 < \frac{1}{\varphi(B_0)} \fint_{B_0} \left(\frac{|f(y)|}{\lambda_0 - \varepsilon} \right)^{p(y)} dy.$$

This implies

$$I_{p,\varphi}\left(\frac{f}{\lambda_0} \right) \le 1 < I_{p,\varphi}\left(\frac{f}{\lambda_0 - \varepsilon} \right), \quad \textit{i.e.} \quad \lambda_0 \ge \|f\|^*_{L^{(p,\varphi)}} > \lambda_0 - \varepsilon,$$

showing $\|f\|_{L^{(p,\varphi)}} = \|f\|^*_{L^{(p,\varphi)}}$.

30.1 Maximal function on $L^{(p,\varphi)}(\mathbb{R}^n)$ Mimicking the proof of the theorems in Part III, especially the boundedness of M on $L^{p(\cdot)}(\mathbb{R}^n)$, we have the boundedness of M on $L^{(p,\varphi)}(\mathbb{R}^n)$.

Theorem 30.1 ([152, Theorem 2.3])**.** *Let $p \in (1, \infty)$ be a constant and $\varphi : \mathbb{R}^n \times (0, \infty) \to (0, \infty)$ be an almost decreasing function. Then the Hardy-Littlewood maximal operator M is bounded from $L^{(p,\varphi)}(\mathbb{R}^n)$ to itself.*

For example, let $\varphi(x, r) = r^{\lambda(x)}$ with $\lambda_+ \le 0$. Then φ is an almost decreasing function.
The following theorem can be proven in the same way as Theorem 15.2:

Theorem 30.2 ([152, Theorem 2.7])**.** *Let $p(\cdot) \in LH$ with $1 \le p_- \le p_+ < \infty$ and let $\varphi : \mathbb{R}^n \times (0, \infty) \to (0, \infty)$ be an almost decreasing function. Assume that there exist positive constants C and N such that*

$$\sup_{x \in \mathbb{R}^n} \varphi(x, r) \le C(r^{-N} + 1) \quad \textit{for} \quad r > 0. \tag{30.1}$$

Then

$$Mf(x)^{p(x)} \le C(M(|f|^{p(\cdot)/p_-})(x)^{p_-} + (e + |x|)^{-np_-}), \quad x \in \mathbb{R}^n,$$

for all measurable functions $f \in L^0(\mathbb{R}^n)$ with $\|f\|_{L^{(p,\varphi)}} \le 1$.

Using Theorem 30.2, we have the following:

Theorem 30.3 ([152, Theorem 2.7]). *Let $p(\cdot) \in LH$ with $1 < p_- \le p_+ < \infty$ and let $\varphi : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$ be an almost decreasing function such that*

$$C^{-1}(1+r)^{-n} \le \varphi(x,r) \le C(r^{-N}+1) \quad \text{for} \quad x \in \mathbb{R}^n, \ r > 0$$

for some positive constants C and N. Then the Hardy-Littlewood maximal operator M is bounded from $L^{(p,\varphi)}(\mathbb{R}^n)$ to itself.

30.2 Fractional integral operators Let $p \in [1,\infty)$ be a constant and

$$\varphi_\lambda(x,r) = r^\lambda$$

with $\lambda \in [-n,0]$. Then $L^{(p,\varphi_\lambda)}(\mathbb{R}^n)$ is the classical Morrey space. In this notation we have the following fundamental result (cf. Theorem 28.2):

Theorem 30.4 (Adams [1] (1975)). *If $\alpha \in (0,n)$, $p,q \in (1,\infty)$, $\lambda \in [-n,0)$ and $\lambda/p + \alpha = \lambda/q$, then I_α is bounded from $L^{(p,\varphi_\lambda)}(\mathbb{R}^n)$ to $L^{(q,\varphi_\lambda)}(\mathbb{R}^n)$.*

Let ρ be a function from $\mathbb{R}^n \times (0,\infty)$ to $(0,\infty)$. We always assume a Dini type condition:

$$\int_0^1 \frac{\rho(x,t)}{t}\, dt < \infty \quad \text{for each } x \in \mathbb{R}^n, \tag{30.2}$$

and that there exist constants $C > 0$ and $0 < k_1 < k_2 < \infty$ such that

$$\sup_{r/2 \le t \le r} \rho(x,t) \le C \int_{k_1 r}^{k_2 r} \frac{\rho(x,t)}{t}\, dt \quad \text{for all } x \in \mathbb{R}^n \text{ and } r > 0. \tag{30.3}$$

We consider generalized fractional integral operators I_ρ defined by

$$I_\rho f(x) = \int_{\mathbb{R}^n} \frac{\rho(x,|x-y|)}{|x-y|^n} f(y)\, dy. \tag{30.4}$$

In view of [139, Lemma 2.5], we see that $(1-\Delta)^{-\alpha/2}$ falls under the scope of our setting. Indeed, Nagayasu and Wadade showed that the kernel ρ which corresponds to $(1-\Delta)^{-\alpha/2}$ satisfies

$$\rho(r) \sim r^\alpha \quad (0 < r < 1), \quad \rho(r) \lesssim e^{-r} \quad (r \ge 1).$$

This means that we have (30.3) with $k_1 = 1/16$ and $k_2 = 1$.

If ρ satisfies the doubling condition, that is, there exists a positive constant C such that

$$\frac{1}{C} \le \frac{\rho(x,r)}{\rho(x,s)} \le C \quad \text{for all } x \in \mathbb{R}^n \text{ and } \frac{1}{2} \le \frac{r}{s} \le 2, \tag{30.5}$$

then ρ satisfies the condition (30.3). If $\rho(x,r) = r^\alpha$, then I_ρ is the usual fractional integral operator I_α. If $\alpha(\cdot) : \mathbb{R}^n \to (0,n)$ and $\rho(x,r) = r^{\alpha(x)}$, then I_ρ is a generalized fractional integral operator $I_{\alpha(x)}$ with variable order defined by

$$I_{\alpha(x)} f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha(x)}}\, dy.$$

The generalized fractional integral operators I_ρ with $\rho : (0,\infty) \to (0,\infty)$ are studied in [144, 145, 146, 149, 206, 207] and the spaces $L^{(p,\varphi)}(\mathbb{R}^n)$ with constant p and $\varphi : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$ are studied in [142, 147, 151, 157].

Theorem 28.2 was extended to the following fundamental theorem of I_ρ (cf. Theorem 28.3):

Theorem 30.5 (Gunawan [59] (2003)). *Let $1 < p < q < \infty$. Let also $\varphi \in \mathbb{Z}_1$ satisfy $\lim_{r\to 0}\varphi(r) = \infty$ and $\lim_{r\to\infty}\varphi(r) = 0$ and let $\rho : (0,\infty) \to (0,\infty)$ satisfy the doubling condition. Assume that there exists constant $C > 0$ such that*

$$\left(\int_0^r \frac{\rho(t)}{t}\,dt\right)\varphi(r)^{1/p} + \int_r^\infty \frac{\rho(t)\varphi(t)^{1/p}}{t}\,dt \le C\varphi(r)^{1/q} \quad \textit{for} \quad r > 0.$$

Then I_ρ is bounded from $L^{(p,\varphi)}(\mathbb{R}^n)$ to $L^{(q,\varphi)}(\mathbb{R}^n)$.

We can extend Theorems 28.2 and 28.3 to generalized Morrey spaces $L^{(p,\varphi)}(\mathbb{R}^n)$ with variable exponents $p : \mathbb{R}^n \to [1,\infty)$ and variable growth function $\varphi : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$ by using generalized fractional integral operators I_ρ.

To state results on generalized fractional integral operators I_ρ, we define classes of growth functions:

Definition 30.2. (i) Let $\mathcal{G}$ be the set of all almost decreasing functions $\varphi : \mathbb{R}^n\times(0,\infty) \to (0,\infty)$ such that, for $x \in \mathbb{R}^n$ and for $0 < r \le s$,

$$\varphi(x,r)r^n \le C_1\varphi(x,s)s^n, \quad C_2 \le \varphi(x,1) \le C_3,$$

where C_i $(i = 1,2,3)$ are positive constants independent of x, r, s.

(ii) Let $\mathcal{G}_*$ be the set of all $\varphi \in \mathcal{G}$ such that, for each $x \in \mathbb{R}^n$,

$$\lim_{r\to 0}\varphi(x,r) = \infty, \quad \lim_{r\to\infty}\varphi(x,r) = 0.$$

(iii) Let $\tilde{\mathcal{G}}$ (resp. $\tilde{\mathcal{G}}_*$) be the set of all $\varphi \in \mathcal{G}$ (resp. $\varphi \in \mathcal{G}_*$) such that $\varphi(x,1) \equiv 1$ for all $x \in \mathbb{R}^n$ and that $\varphi(x,\cdot)$ is continuous and strictly decreasing for each $x \in \mathbb{R}^n$.

The next lemma justifies why we introduced the classes $\mathcal{G}, \tilde{\mathcal{G}}_*$.

Lemma 30.6 (cf. [149, Lemma 3.4] (2008)).

(i) *Let $\varphi \in \mathcal{G}$. Then there exist $\tilde{\varphi} \in \tilde{\mathcal{G}}$ such that $\varphi \sim \tilde{\varphi}$.*

(ii) *If $\varphi \in \mathcal{G}_*$, then there exists $\tilde{\varphi} \in \tilde{\mathcal{G}}_*$ such that $\varphi \sim \tilde{\varphi}$. We can even arrange that $\tilde{\varphi}(x,\cdot)$ be bijective for each x.*

We can extend Theorem 30.4 due to Adams to the variable setting.

Theorem 30.7 ([152, Theorem 2.11]). *Let $\varphi \in \mathcal{G}_*$, $p(\cdot) \in LH$ and $1 < p_- \le p_+ < \infty$. Let $q(\cdot) : \mathbb{R}^n \to (1,\infty)$, $q_+ < \infty$, and $p_0 \in (1, p_-]$ be a constant. Assume that*

$$\left(\int_0^r \frac{\rho(x,t)}{t}\,dt\right)\varphi(x,r)^{1/p(x)} + \int_r^\infty \frac{\rho(x,t)\varphi(x,t)^{1/p(x)}}{t}\,dt \le C\varphi(x,r)^{1/q(x)} \quad \textit{for} \quad x \in \mathbb{R}^n,\ r > 0, \tag{30.6}$$

and that

$$\int_{\tilde{\varphi}^{-1}(x,(1+|x|)^{-np_\infty})}^\infty \frac{\rho(x,t)\varphi(x,t)^{1/p_\infty}}{t}\,dt \le C(1+|x|)^{-np_0/q(x)} \quad \textit{for} \quad x \in \mathbb{R}^n, \tag{30.7}$$

where $\tilde{\varphi}$ is a function in $\tilde{\mathcal{G}}_*$ such that $\varphi \sim \tilde{\varphi}$. Then the operator I_ρ is bounded form $L^{(p,\varphi)}(\mathbb{R}^n)$ to $L^{(q,\varphi)}(\mathbb{R}^n)$.

Fractional maximal operators in the limiting case are investigated in [131, Theorem 4.10] and [205, Theorem 2.1].

Theorem 30.7 is proved by using the next pointwise estimate and Theorem 30.3, which

Theorem 30.8 ([152, Theorem 2.12])**.** *Under the same condition as Theorem 30.7, there exists a positive constant C such that*

$$|I_\rho f(x)|^{q(x)} \leq C(Mf(x)^{p(x)} + (e+|x|)^{-np_0}),$$

for all $x \in \mathbb{R}^n$ and $f \in L^{(p,\varphi)}(\mathbb{R}^n)$ satisfying $\|f\|_{L^{(p,\varphi)}} \leq 1$.

This is an improvement of [131, Lemma 4.6], where the Hardy operator appeared. There are many variants of Theorem 30.8; see [122, Lemma 2.5].

For Herz-Morrey spaces with variable exponents, see [128]. One dimensional fractional integral operator is investigated in [33, 34, 88].

30.3 Morrey spaces with variable exponents on bounded domains In connection with applications to the study of classical operators of harmonic analysis in variable exponents Morrey spaces, S. Samko [192] proved estimates of weighted variable exponent norms

$$\||\cdot - x|^{-\beta(x)}\chi_{B(x,r)}\|_{L^{p(\cdot)}}$$

of potential kernels truncated to balls, where weights are of radial type. Conditions on the validity of estimates are given in terms of Zygmund type inequalities on weight or in terms of their Matuszewska-Orlicz indices.

Let $\Omega \subsetneq \mathbb{R}^n$ be a bounded domain. Almeida, Hasanov, and S. Samko [2] introduced variable Morrey spaces $L^{p(\cdot),\lambda(\cdot)}(\Omega)$ and proved equivalence of several different norms. Define temporarily

$$\|f\|_{L^{p(\cdot),\lambda(\cdot)}(\Omega)} := \sup_{x\in\Omega, t>0} t^{-\frac{\lambda(x)}{p(x)}} \|f\chi_{B(x,t)\cap\Omega}\|_{L^{p(\cdot)}(\Omega)},$$

where $p(\cdot)$ satisfies the log-Hölder condition as well as $1 < p_- \leq p_+ < \infty$ and λ is a bounded measurable function satisfying $0 \leq \lambda_- \leq \lambda_+ < n$. They showed that M is bounded on $L^{p(\cdot),\lambda(\cdot)}(\Omega)$.

In the case of bounded sets Ω, Almeida, Hasanov, and S. Samko proved embeddings between such spaces and the boundedness of the maximal operator and a Sobolev-Adams type $L^{p(\cdot),\lambda(\cdot)} \to L^{q(\cdot),\lambda(\cdot)}$-theorem for the potential operators $I^{\alpha(\cdot)}$, also of variable order, where $\frac{1}{q(x)} = \frac{1}{p(x)} - \frac{\alpha(x)}{n-\lambda(x)}$. In the case of constant α, the limiting case $\lambda(x) + \alpha p(x) \equiv n$ is also studied when the potential operator I^α acts into $\mathrm{BMO}(\mathbb{R}^n)$.

Guliyev, Hasanov, and S. Samko [56] introduced generalized Morrey spaces $\mathcal{M}^{p(\cdot),\omega(\cdot)}(\Omega)$ with variable exponent $p(\cdot)$ and a general function $\omega(x,r)$ defining the Morrey-type norm, which we recall now. Let $p(\cdot) : \Omega \to [1,\infty)$ be a measurable function and let $\omega : \Omega\times[0,\infty) \to (0,\infty)$ be a function. Define

$$\|f\|_{\mathcal{M}^{p(\cdot),\omega(\cdot)}(\Omega)} = \sup_{x\in\Omega, r>0} \frac{r^{-n/p(x)}}{\omega(x,r)} \|f\|_{L^{p(\cdot)}(B(x,r)\cap\Omega)}.$$

In case of bounded sets $\Omega \subset \mathbb{R}^n$, in such spaces, Guliyev, Hasanov, and S. Samko proved theorems on the boundedness of the maximal operator and Calderon-Zygmund singular operators with standard kernel, in such spaces and a Sobolev-Adams type $\mathcal{M}^{p(\cdot),\omega(\cdot)}(\Omega) \to \mathcal{M}^{q(\cdot),\omega(\cdot)}(\Omega)$-theorem for the potential operators $I^{\alpha(\cdot)}$, also of variable order. The conditions for the boundedness are given it terms of Zygmund-type integral inequalities on $\omega(x,r)$, which do not assume any assumption on monotonicity of $\omega(x,r)$ in r.

Guliyev, Hasanov, and S. Samko proposed alternative generalization $\mathcal{M}^{p(\cdot),\theta(\cdot),\omega(\cdot)}(\Omega)$, $\Omega \subset \mathbb{R}^n$, of Morrey spaces with variable exponents $p(x)$, $\theta(r)$ and a general function $\omega(x,r)$ defining the Morrey-type norm in [55], where L^∞-norm in r is replaced by $L^{\theta(\cdot)}$-norm,

$$1 \le \theta_- \le \theta(r) \le \theta_+ < \infty.$$

Namely,

$$\|f\|_{\mathcal{M}^{p(\cdot),\theta(\cdot),\omega(\cdot)}} = \sup_{x\in\Omega} \left\| \frac{\omega(x,r)}{r^{n/p(x)}} \|f\|_{L^{p(\cdot)}(B(x,r)\cap\Omega)} \right\|_{L^{\theta(\cdot)}(0,\mathrm{diam}(\Omega))}.$$

In case of bounded sets Ω, in such spaces Guliyev, Hasanov, and S. Samko proved theorems on the boundedness of the maximal operator and Calderon-Zygmund singular integral operators with standard kernel, and a Sobolev-Adams type $\mathcal{M}^{p(\cdot),\theta_1(\cdot),\omega_1(\cdot)}(\Omega) \to \mathcal{M}^{q(\cdot),\theta_2(\cdot),\omega_2(\cdot)}(\Omega)$-theorem for the potential operator $I^{\alpha(\cdot)}$, also of variable order. In all the cases the conditions for the boundedness are given in terms of Zygmund-type integral inequalities on $\omega(x,r)$ without any assumption on monotonicity of $\omega(x,r)$ in r.

31 Campanato spaces with variable growth conditions of higher order When the functions belongs to the class LH, we can consider the higher order spaces. We follow [154]. See [111] for Musielak-Orlicz-Campanato spaces.

31.1 Definition and examples Recall that we let $L^q_{\mathrm{comp}}(\mathbb{R}^n)$ be the set of all $L^q(\mathbb{R}^n)$-functions having compact support. Given a nonnegative integer d, let

$$L^{q,d}_{\mathrm{comp}}(\mathbb{R}^n) := \left\{ f \in L^q_{\mathrm{comp}}(\mathbb{R}^n) : \int_{\mathbb{R}^n} f(x)x^\alpha\,dx = 0,\ |\alpha| \le d \right\}.$$

Likewise if Q is a cube, then we write

$$L^{q,d}(Q) := \left\{ f \in L^q(Q) : \int_Q f(x)x^\alpha\,dx = 0,\ |\alpha| \le d \right\}.$$

We define $\mathcal{P}_d(\mathbb{R}^n)$ to be the set of all polynomials having degree at most d. For a locally integrable function f, a cube Q and a nonnegative integer d, there exists a unique polynomial $P \in \mathcal{P}_d(\mathbb{R}^n)$ such that, for all $q \in \mathcal{P}_d(\mathbb{R}^n)$,

$$\int_Q (f(x) - P(x))q(x)\,dx = 0.$$

Denote this unique polynomial P by $P^d_Q f$. It follows immediately from the definition that $P^d_Q g = g$ if $g \in \mathcal{P}_d(\mathbb{R}^n)$. Recall that $\mathcal{Q}$ denotes the set of all open cubes whose edges are parallel to the coordinate axes.

Definition 31.1 ([154, Definition 6.1], $\mathcal{L}_{q,\phi,d}(\mathbb{R}^n)$)**.** Let $1 \le q \le \infty$. Let $\phi : \mathcal{Q} \to (0,\infty)$ be a function and $f \in L^q_{\mathrm{loc}}(\mathbb{R}^n)$. One denotes

$$\|f\|_{\mathcal{L}_{q,\phi,d}} := \sup_{Q\in\mathcal{Q}} \frac{1}{\phi(Q)} \left(\frac{1}{|Q|} \int_Q |f(x) - P^d_Q f(x)|^q\,dx \right)^{1/q},$$

when $q < \infty$ and

$$\|f\|_{\mathcal{L}_{q,\phi,d}} := \sup_{Q\in\mathcal{Q}} \frac{1}{\phi(Q)} \|f - P^d_Q f\|_{L^\infty(Q)}.$$

when $q = \infty$. Then the Campanato space $\mathcal{L}_{q,\phi,d}(\mathbb{R}^n)$ is defined to be the sets of all $f \in L^q_{\rm loc}(\mathbb{R}^n)$ such that the quantity $\|f\|_{\mathcal{L}_{q,\phi,d}}$ is finite. One considers elements in $\mathcal{L}_{q,\phi,d}(\mathbb{R}^n)$ modulo polynomials of degree d so that $\mathcal{L}_{q,\phi,d}(\mathbb{R}^n)$ is a Banach space. When one writes $f \in \mathcal{L}_{q,\phi,d}(\mathbb{R}^n)$, then f stands for the representative of the set

$$\{f + P \,:\, P \text{ is a polynomial of degree } d\}.$$

Here and below we abuse notation slightly. We write $\phi(x,r) := \phi(Q(x,r))$ for $x \in \mathbb{R}^n$ and $r > 0$.

Remark 31.1. For $Q \in \mathcal{Q}$ and $f \in L^q(Q)$, since two Banach spaces of dimension $d+1$ are isomorphic, we have

$$\|P^d_Q f\|_{L^\infty(Q)} \lesssim \left(\fint_Q |f(x)|^q \, dx\right)^{\frac{1}{q}}, \tag{31.1}$$

where the implicit constant in $\lesssim$ does not depend on $Q \in \mathcal{Q}$ and $f \in L^q(Q)$. Hence we see

$$\|f\|_{\mathcal{L}_{q,\phi,d}} \sim \sup_{Q\in\mathcal{Q}} \left\{ \inf_{P\in\mathcal{P}_d(\mathbb{R}^n)} \frac{1}{\phi(Q)} \left(\frac{1}{|Q|}\int_Q |f(x) - P(x)|^q \, dx\right)^{1/q} \right\}.$$

Here are some typical examples of the function ϕ we envisage.

Example 31.1 ([154, Example 6.3]). *Let u be a real number in $(0,\infty)$.*

(1) $\phi_1(Q) = |Q|^{\frac{1}{u}-1}$. *In this case $\mathcal{L}_{p,\phi_1,d}(\mathbb{R}^n)$ is known to be the Lipschitz space when $u < 1$ and the* $\mathrm{BMO}(\mathbb{R}^d)$ *space when $u = 1$.*

(2) $\phi_2(Q) = \dfrac{|Q|^{\frac{1}{u}} + |Q|}{|Q|} = \phi_1(Q) + 1$.

(3) $\phi_3(Q) = \dfrac{\|\chi_Q\|_{L^{p(\cdot)}}}{|Q|}$.

Despite Example 31.1 (1) and (2), we can consider the function space $\mathcal{L}_{q,\phi,d}(\mathbb{R}^n)$ in a wide generality. It often turns out that the doubling condition and the compatibility condition suffice. Apart from the doubling condition, we shall show in Proposition 31.8 that compatibility condition is a natural condition as well.

In view of our variable setting, the following example is fundamental:

Example 31.2 ([154, Example 6.4]). *If $p(\cdot)$ satisfies $0 < p_- \le p_+ < \infty$,* (14.1) *and* (14.2), *then ϕ_3 taken up in Example* 31.1 *does satisfy the doubling condition and the compatibility condition.*

Here are examples of the calculation of the norm of the functions in $\mathcal{L}_{q,\phi,d}(\mathbb{R}^n)$.

Proposition 31.3 ([154, Example 6.7]). *Assume that $p(\cdot)$ satisfies* (14.1), (14.2) *and $0 < p_- \le p_+ \le 1$ and that $d \ge d_{p(\cdot)}$. Let $\phi_3(Q) = \dfrac{\|\chi_Q\|_{L^{p(\cdot)}}}{|Q|}$ for $Q \in \mathcal{Q}$. Let $\psi \in \mathcal{S}(\mathbb{R}^n) \setminus \{0\}$ be supported on $Q(0,1)$.*

(i) $(-\Delta)^L\psi$ *is not a zero function.*

(ii) *Let us set*

$$\psi^{Q(z,r)}(x) := \phi_3(z,r) \cdot \left((-\Delta)^L \psi\right) \left(\frac{x-z}{r}\right), \quad L = \left[\frac{d}{2}+1\right]$$

for $Q = Q(z,r) \in \mathcal{Q}$. We claim that

$$\|\psi^{Q(z,r)}\|_{\mathcal{L}_{q,\phi,d}} \sim 1 \tag{31.2}$$

for all $1 \le q \le \infty$, where implicit constants in $\sim$ depend only on d, q and $p(\cdot)$.

Proposition 31.4 ([154, Example 6.9])**.** *Let $1 \le q < \infty$ and $d = 0$. Assume that $\phi : \mathcal{Q} \to (0,\infty)$ satisfies the conditions the doubling condition, the compatibility condition and*

$$\int_0^r \frac{\phi(x,t)t^{n/q}}{t}\,dt \lesssim \phi(x,r)r^{n/q}, \quad \phi(x,sr) \lesssim s\,\phi(x,r) \tag{31.3}$$

for all $x \in \mathbb{R}^n$, $r \in (0,\infty)$, $s \in [1,\infty)$. For $z \in \mathbb{R}^n$, define

$$h^z(x) := \int_{|x-z|}^1 \frac{\phi(z,t)}{t}\,dt, \quad x \in \mathbb{R}^n.$$

Then $\{h^z\}_{z\in\mathbb{R}^n}$ forms a bounded set in $\mathcal{L}_{q,\phi,0}(\mathbb{R}^n)$ [141, Lemma 3.1]*. In this case, if we set*

$$f^{(z,r)}(x) := \int_{\min(|x-z|,r/2)}^{r/2} \frac{\phi(z,t)}{t}\,dt = \max\left(0, h^z(x) - \int_{r/2}^1 \frac{\phi(z,t)}{t}\,dt\right),$$

then $\{f^{(z,r)}\}_{z\in\mathbb{R}^n,\, r>0}$ also forms a bounded subset by [141, Lemma 2.2]*. Moreover, we have*

$$\left(\frac{1}{r^n}\int_{Q(z,r)} |f^{(z,r)}(x) - P^0_{Q(z,r)} f^{(z,r)}(x)|^q\,dx\right)^{\frac{1}{q}} \gtrsim \phi(z,r). \tag{31.4}$$

31.2 Fundamental structure of the space $\mathcal{L}_{q,\phi,d}(\mathbb{R}^n)$ Before we investigate the duality, we make a preliminary observation of $\mathcal{L}_{q,\phi,d}(\mathbb{R}^n)$.

Let us say that $\phi : \mathcal{Q} \to (0,\infty)$ is a nice function, if there exists $b \in (0,1)$ such that, for all cubes $Q \in \mathcal{Q}$, we have

$$\frac{1}{\phi(Q)}\left(⨍_Q |f(x) - P^d_Q f(x)|^q\,dx\right)^{\frac{1}{q}} > b$$

for some $f \in \mathcal{L}_{q,\phi,d}(\mathbb{R}^n)$ with norm 1.

Here are some examples of nice functions.

Example 31.5 ([154, Example 6.10])**.**

(i) *A direct consequence of Example* 31.2 *is that the function $\phi_3(Q) = \dfrac{\|\chi_Q\|_{L^{p(\cdot)}}}{|Q|}$ is a nice function, when $d \ge d_{p(\cdot)}$, where $p(\cdot)$ satisfies* (14.1)*,* (14.2) *and $0 < p_- \le p_+ \le 1$.*

(ii) *Another example is the function ϕ in Example* 31.4*. Moreover, assume that $p(\cdot)$ satisfies* (14.1)*,* (14.2) *and $n/(n+1) < p_- \le p_+ < \infty$. Let $d = 0$ and $q \ge 1$ be a real number such that $p_+ < q' = q/(q-1)$. Also, set*

$$\phi_3(Q) := \frac{\|\chi_Q\|_{L^{p(\cdot)}}}{|Q|} \quad (Q \in \mathcal{Q}).$$

Then ϕ_3 satisfies the doubling condition, the compatibility condition and (31.3)*, and it is also a nice function by* (31.4)*.*

In the next lemma, we claim that we have automatically an additional condition on the ϕ. So we can limit the class of ϕ. This lemma corresponds to a fact about generalized Morrey spaces described in [143].

Lemma 31.6 ([154, Lemma 6.11]). *For any $\phi : \mathcal{Q} \to (0,\infty)$, there exists a nice function $\phi' : \mathcal{Q} \to (0,\infty)$ such that $\mathcal{L}_{q,\phi,d}(\mathbb{R}^n)$ and $\mathcal{L}_{q,\phi',d}(\mathbb{R}^n)$ are isomorphic with norm coincidence.*

The next proposition shows the diversity of the function spaces $\mathcal{L}_{q,\phi,d}(\mathbb{R}^n)$.

Proposition 31.7 ([154, Proposition 6.12]). *Assume that $\phi_1, \phi_2 : \mathcal{Q} \to (0,\infty)$ are nice functions. Then the function space $\mathcal{L}_{q,\phi_1,d}(\mathbb{R}^n)$ is continuously embedded into $\mathcal{L}_{q,\phi_2,d}(\mathbb{R}^n)$ if and only if we have $\phi_1(Q) \lesssim \phi_2(Q)$ for all $Q \in \mathcal{Q}$. In particular, the spaces $\mathcal{L}_{q,\phi_1,d}(\mathbb{R}^n)$ and $\mathcal{L}_{q,\phi_2,d}(\mathbb{R}^n)$ are isomorphic if and only if $\phi_1 \sim \phi_2$.*

Next, we justify that the compatibility condition is natural; see [143, p.455] as well.

Proposition 31.8 ([154, Proposition 6.13]). *Let $\phi : \mathcal{Q} \to (0,\infty)$ be a function satisfying the doubling condition. Suppose that ϕ is not always a nice function and that ϕ does not always satisfy the compatibility condition.*

(i) *If we define*

$$\tilde{\phi}(x,r) := \inf\{\phi(y,r) \,:\, y \in \mathbb{R}^n,\, r > 0,\, |x-y| \le r\}, \tag{31.5}$$

then the function spaces $\mathcal{L}_{q,\phi,d}(\mathbb{R}^n)$ and $\mathcal{L}_{q,\tilde{\phi},d}(\mathbb{R}^n)$ are isomorphic.

(ii) *If we assume that ϕ is a nice function, then so does $\tilde{\phi}$ defined by* (31.5).

From Proposition 31.8 we see that the compatibility condition is a natural condition.

Next, we consider the continuity property of the functions in $\mathcal{L}_{q,\phi,d}(\mathbb{R}^n)$ very crudely. We invoke the following result from [35, p.23 Lemma 4.1]:

Lemma 31.9 ([35, p.23 Lemma 4.1]). *Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Then we have*

$$\lim_{r\downarrow 0} P^d_{Q(x,r)} f(x) = f(x)$$

for almost every $x \in \mathbb{R}^n$. If f is continuous at a point x, then the above equality holds.

Here by using Lemma 31.9 we can prove;

Proposition 31.10 ([154, Proposition 6.15]). *Let E be an open set in $\mathbb{R}^n$. Let $\phi : \mathcal{Q} \to (0,\infty)$ satisfy the doubling condition. Assume further that*

$$\lim_{r\downarrow 0}\left(\sup_{y\in Q(x,r)} \int_0^r \frac{\phi(y,t)}{t}\,dt\right) = 0 \tag{31.6}$$

for each $x \in E$. If $f \in \mathcal{L}_{q,\phi,d}(\mathbb{R}^n)$, there exists a continuous function $g : E \to \mathbb{C}$ that is equal almost everywhere to f on E. In this sense, any $f \in \mathcal{L}_{q,\phi,d}(\mathbb{R}^n)$ has a representative continuous on E.

The condition (31.6) is not so strong as the following example shows:

Example 31.11 ([154, Example 6.16]). *The condition* (31.6) *is satisfied on $\mathbb{R}^n$ as long as $p(\cdot)$ satisfies* (14.1), (14.2) *and $0 < p_- \le p_+ < 1$ and $\phi(Q)$ is given by $\phi(Q) = \dfrac{\|\chi_Q\|_{L^{p(\cdot)}}}{|Q|}$ for $Q \in \mathcal{Q}$.*

31.3 Hölder-Zygmund spaces of higher order with variable exponents We define Δ_h^k to be a difference operator, which is defined inductively by

$$\Delta_h^1 f = \Delta_h f := f(\cdot + h) - f, \quad \Delta_h^k := \Delta_h^1 \circ \Delta_h^{k-1}, \quad k \geq 2. \tag{31.7}$$

Definition 31.2 ([154, Definition 8.1], $\Lambda_{\phi,d}(\mathbb{R}^n)$)**.** Let $\phi : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$ be a function and $d \in \mathbb{N} \cup \{0\}$. Then $\Lambda_{\phi,d}(\mathbb{R}^n)$, the Hölder space with variable exponent $p(\cdot)$, is defined to be the set of all continuous functions f such that $\|f\|_{\Lambda_{\phi,d}} < \infty$, where

$$\|f\|_{\Lambda_{\phi,d}} := \sup_{x \in \mathbb{R}^n,\, h \neq 0} \frac{1}{\phi(x,|h|)} \left|\Delta_h^{d+1} f(x)\right|.$$

One considers elements in $\Lambda_{\phi,d}(\mathbb{R}^n)$ modulo polynomials of degree d so that $\Lambda_{\phi,d}(\mathbb{R}^n)$ is a Banach space. When one writes $f \in \Lambda_{\phi,d}(\mathbb{R}^n)$, then f again stands for the representative of $\{f + P : P \text{ is a polynomial of degree } d\}$.

Several helpful remarks may be in order.

Remark 31.2 ([154, Remark 8.2]).

(i) Assume that there exists a constant $\mu > 0$ such that $\phi(Q) \lesssim |Q|^\mu$ for all Q with $|Q| \geq 1$. If a continuous function f satisfies $\|f\|_{\Lambda_{\phi,d}} < \infty$, then f is of polynomial order. In particular the representative of such a function f can be regarded as an element in $\mathcal{S}'(\mathbb{R}^n)$. Actually, since f is assumed continuous, f is bounded on a neighborhood $Q(0,1)$. Using $\|f\|_{\Lambda_{\phi,d}} < \infty$, inductively on $k \in \mathbb{N} \cup \{0\}$, we can show that $|f(x)| \lesssim (k+1)^{d+\mu+1}$ for all $x \in \mathbb{R}^n$ with $k \leq |x| \leq k+1$.

(ii) It is absolutely necessary to assume that f is a continuous function, when $d \geq 1$. We remark that there exists a discontinuous function f such that $\Delta_h^{d+1} f(x) = 0$ for all $x, h \in \mathbb{R}^n$. See [135] for such an example.

(iii) The function space $\Lambda_{\phi,d}(\mathbb{R}^n)$ is used to measure the Hölder continuity uniformly, when ϕ does not depend on x. Such an attempt can be found in [130].

As for $\Lambda_{\phi,d}(\mathbb{R}^n)$, we have the following equivalence:

Theorem 31.12 ([154, Theorem 8.4])**.** *Assume that $\phi : \mathcal{Q} \to (0,\infty)$ satisfies the doubling condition, the compatibility condition and the $\mathbb{Z}^1$ condition. Then the function spaces $\Lambda_{\phi,d}(\mathbb{R}^n)$ and $\mathcal{L}_{q,\phi,d}(\mathbb{R}^n)$ are isomorphic. Speaking more precisely, we have the following* :

(i) *For any $f \in \Lambda_{\phi,d}(\mathbb{R}^n)$ we have $\|f\|_{\mathcal{L}_{q,\phi,d}} \lesssim \|f\|_{\Lambda_{\phi,d}}$.*

(ii) *Any element in $\mathcal{L}_{q,\phi,d}(\mathbb{R}^n)$ has a continuous representative. Moreover, whenever $f \in \mathcal{L}_{q,\phi,d}(\mathbb{R}^n) \cap C(\mathbb{R}^n)$, then $f \in \Lambda_{\phi,d}(\mathbb{R}^n)$ and we have $\|f\|_{\Lambda_{\phi,d}} \lesssim \|f\|_{\mathcal{L}_{q,\phi,d}}$.*

Here is an example of the function ϕ we envisage.

Example 31.13 ([154, Example 8.4])**.** *Suppose $p(\cdot)$ satisfies* (14.1), (14.2) *and $0 < p_- \leq p_+ < 1$. Then the function $\phi_3(Q) = \dfrac{\|\chi_Q\|_{L^{p(\cdot)}}}{|Q|}$, $Q \in \mathcal{Q}$ satisfies the doubling condition, the compatibility condition and the $\mathbb{Z}^1$ condition. Indeed, the doubling condition and the compatibility condition are verified by using Lemma* 19.1 *and* [151, Proposition 3.3]. *Let us check the $\mathbb{Z}^1$ condition. To this end, first let us suppose that Q can be written $Q = Q(x,r)$ with $r < 1$. Then by Lemma* 19.1, *we have*

$$\int_0^r \frac{\phi_3(x,t)}{t}\,dt \lesssim \int_0^r \frac{t^{n/p(x)-n}}{t}\,dt = \frac{r^{n/p(x)-n}}{n/p(x)-n} \sim \phi_3(Q).$$

If we assume that Q can be written $Q = Q(x,r)$ with $r \geq 1$, then again by Lemma 19.1 *we have*

$$\int_0^r \frac{\phi_3(x,t)}{t}\,dt \lesssim 1 + \int_1^r \frac{t^{n/p_\infty - n}}{t}\,dt = 1 + \frac{r^{n/p_\infty - n} - 1}{n/p_\infty - n} \sim \phi_3(Q).$$

Observe that the we can take implicit constants uniformly over x in the chain of inequalities above because we are assuming that $p_+ < 1$.

Also, a direct consequence of the $\mathbb{Z}^1$-condition is the following;

Lemma 31.14 ([154, Lemma 8.5])**.** *If we assume that*

$$\phi(x,r) \leq C_0\phi(x,t), \quad \frac{r}{2} \leq t \leq r, \quad x \in \mathbb{R}^n \tag{31.8}$$

and

$$\int_0^r \frac{\phi(x,t)}{t}\,dt \leq C_1\phi(x,r), \quad r > 0,\, x \in \mathbb{R}^n.$$

Then for all $\varepsilon \in [0, {C_1}^{-1})$ we have

$$\frac{\phi(x,r)}{r^\varepsilon} \leq \frac{C_0\varepsilon}{2^\varepsilon - 1} \cdot \frac{C_1}{1-\varepsilon C_1} \cdot \frac{\phi(x,s)}{s^\varepsilon}, \quad 0 < r \leq s < \infty,\, x \in \mathbb{R}^n. \tag{31.9}$$

In this section we also deal with the function spaces of Besov type. Let us fix an even function $\psi \in \mathcal{S}(\mathbb{R}^n)$ so that $\chi_{Q(0,1)} \leq \psi \leq \chi_{Q(0,2)}$. As usual, let us set $\varphi_j = \psi(2^{-j}\cdot) - \psi(2^{-j+1}\cdot)$ for $j \in \mathbb{Z}$. For $f \in \mathcal{S}'(\mathbb{R}^n)$ and $\rho \in \mathcal{S}(\mathbb{R}^n)$, we write $\rho(D)f := \mathcal{F}^{-1}[\rho \cdot \mathcal{F}f]$.

We now seek to a result similar to the one due to Taibleson and Grevholm [58, 216].

Definition 31.3 ([154, Definition 8.6], $\mathcal{L}^D_{q,\phi}(\mathbb{R}^n)$)**.** Let $1 \leq q \leq \infty$ and $\phi : \mathcal{Q} \to (0,\infty)$ a function. A function $f \in \mathcal{S}'(\mathbb{R}^n)$ is said to belong $\mathcal{L}^D_{q,\phi}(\mathbb{R}^n)$, if

$$\|f\|_{\mathcal{L}^D_{q,\phi}} := \sup_{\substack{x\in\mathbb{R}^n\\ j\in\mathbb{Z}}} \frac{1}{\phi(x,2^{-j})} \left(\fint_{Q(x,2^{-j})} |\varphi_j(D)f(y)|^q\,dy \right)^{\frac{1}{q}} < \infty.$$

From the definition of $\mathcal{L}^D_{q,\phi}(\mathbb{R}^n)$, the space seems to depend on the admissible choices of ψ. However, as an example of ϕ, we envisage the function $\phi_3(Q) = \dfrac{\|\chi_Q\|_{L^{p(\cdot)}}}{|Q|}$ as in Example 31.1(3) and in this case we shall show that $\mathcal{L}^D_{q,\phi_3}(\mathbb{R}^n)$ does not depend on the admissible choices of ψ. Note that this space is a homogeneous counterpart for the local space defined [3, Definition 5.2] when $q = \infty$ in this special case. Returning to the general theory, more precisely, we can prove the following theorem:

Theorem 31.15 ([154, Theorem 8.7])**.** *Suppose that ϕ satisfies the doubling condition, the compatibility condition and $\mathbb{Z}^1$-condition. Assume in addition that ϕ fulfils for some integer $d \geq 0$ and that*

$$\sup_{x\in\mathbb{R}^n} \phi(x,1) < \infty.$$

Then we have;

(i) *The spaces $\mathcal{L}^D_{q,\phi}(\mathbb{R}^n)$ and $\mathcal{L}_{q,\phi,d}(\mathbb{R}^n)$ are isomorphic. More precisely, we have the following :*

(a) *Let* $f \in \mathcal{L}^D_{q,\phi}(\mathbb{R}^n)$. *Then,* f *can be represented by an* $L^q_{\rm loc}(\mathbb{R}^n)$*-function and there exists* $P \in \mathcal{P}(\mathbb{R}^n)$ *such that* $f - P \in \mathcal{L}_{q,\phi,d}(\mathbb{R}^n)$. *In this case we have* $\|f-P\|_{\mathcal{L}_{q,\phi,d}} \lesssim \|f\|_{\mathcal{L}^D_{q,\phi}}$

(b) *If* $f \in \mathcal{L}_{q,\phi,d}(\mathbb{R}^n)$, *then*

$$f \in \mathcal{L}^D_{q,\phi}(\mathbb{R}^n) \text{ and} \|f\|_{\mathcal{L}^D_{q,\phi}} \lesssim \|f\|_{\mathcal{L}_{q,\phi,d}}. \tag{31.10}$$

In particular, the definition of the function space $\mathcal{L}^D_{q,\phi}(\mathbb{R}^n)$ *does not depend on the admissible choices of* ψ : *Any* ψ *will do in the definition of* $\mathcal{L}^D_{q,\phi}(\mathbb{R}^n)$ *as long as* $\chi_{Q(0,1)} \le \psi \le \chi_{Q(0,2)}$.

(ii) *The function space* $\mathcal{L}^D_{q,\phi}(\mathbb{R}^n)$ *does not depend on* q.

Remark that any representative of the element in $\mathcal{L}_{q,\phi,d}(\mathbb{R}^n)$ belongs to $\mathcal{S}'(\mathbb{R}^n)$. The following is the dual of Lemma 31.14:

Lemma 31.16 ([154, Lemma 8.8]). *If we assume the* $\mathbb{Z}_d$*-condition for some integer* $d \ge 0$, *then there exists* $0 < \varepsilon' < 1$ *such that*

$$\frac{\phi(x,r)}{r^{d+1-\varepsilon'}} \lesssim \frac{\phi(x,s)}{s^{d+1-\varepsilon'}} \quad (0 < s \le r < \infty).$$

As examples of ϕ and d satisfying the condition of Theorem 31.15, we can list the following:

Example 31.17 ([154, Example 8.9]).

(i) *Assume that* $p(\cdot) : \mathbb{R}^n \to (0,\infty)$ *satisfies* $0 < p_- \le p_+ < 1$, (14.1) *and* (14.2) *and let* $d \ge d_{p(\cdot)} = \min\{d' \in \mathbb{N} \cup \{0\} \,:\, p_-(n+d'+1) > n\}$. *Then* $\phi_3(Q) = \dfrac{\|\chi_Q\|_{L^{p(\cdot)}}}{|Q|}$ *does satisfy the requirements of Theorem* 31.15.

(ii) *Let* $p_1, p_2 \in (0,1)$ *be constants. Assume that* $d \in \mathbb{Z}$ *satisfies* $p_1(n+d+1) > n$ *and* $p_2(n+d+1) > n$. *Then*

$$\phi_4(Q) = |Q|^{1/p_1-1} + |Q|^{1/p_2-1}$$

also satisfies the requirements of Theorem 31.15.

The following propositions and corollary will be useful for later considerations. See [11] for Propositions 31.18 and see [216, Theorem 4] and Grevholm [58, Lemma 2.1] for Proposition 31.19 respectively.

Proposition 31.18 ([154, Proposition 8.10]). *Let* $d \in \mathbb{N} \cup \{0\}$. *Suppose that we are given a sequence* $\{f_j\}_{j=1}^\infty$ *of* $\mathcal{S}'(\mathbb{R}^n)$ *such that* $\{\partial^\alpha f_j\}_{j=1}^\infty$ *is convergent for each* α *with* $|\alpha| = d+1$. *Then there exists a sequence of polynomials* $\{P_j\}_{j=1}^\infty \subset \mathcal{P}_d$ *such that* $\{f_j + P_j\}_{j=1}^\infty$ *is convergent in* $\mathcal{S}'(\mathbb{R}^n)$.

Proposition 31.19 ([154, Proposition 8.11]). *Let* $s > 0$ *and* $f \in \mathcal{S}'(\mathbb{R}^n)$. *Then*

$$\sup_{j \in \mathbb{N}} 2^{js} \|\varphi_j(D) f\|_{L^\infty} < \infty \tag{31.11}$$

and

$$\sup_{j \in \mathbb{Z}\setminus\mathbb{N}} 2^{js} \|\varphi_j(D) f\|_{L^\infty} < \infty \tag{31.12}$$

if and only if there exists a polynomial P such that

$$\sup_{x\in\mathbb{R}^n,\, h\in\mathbb{R}^n\setminus\{0\}} \frac{|\Delta_h^{[s+1]}(f-P)(x)|}{|h|^s} < \infty. \tag{31.13}$$

If this is the case, we have

$$\sup_{x\in\mathbb{R}^n,\, h\in\mathbb{R}^n\setminus\{0\}} \frac{|\Delta_h^{[s+1]}(f-P)(x)|}{|h|^s} \lesssim \sup_{j\in\mathbb{Z}} 2^{js}\|\varphi_j(D)f\|_{L^\infty}.$$

Furthermore, if $0 \notin \mathrm{supp}(\mathcal{F}f)$, that is, $\mathcal{F}f$ vanishes near the origin, then we can take $P := 0$.

Inequalities (31.12) and (31.13) read $\sup_{j\in\mathbb{Z}} 2^{js}\|\varphi_j(D)f\|_{L^\infty} < \infty$. However, in the setting of our variable continuity, our argument works because (31.12) approximately corresponds to the local case and (31.13) to the global case and this intuitive argument matches Lemma 19.1.

From Propositions 31.18 and 31.19, we have the following:

Corollary 31.20 ([154, Corollary 8.12])**.** *Let $s > 0$. Assume $f \in \mathcal{S}'(\mathbb{R}^n)$ satisfies* (31.12)*. Set $d := [s]$.*

(i) *The sum*

$$\sum_{j=-\infty}^{0} \partial^\alpha \varphi_j(D)f$$

is convergent uniformly whenever α is a multiindex with length $d+1$.

(ii) *There exists a sequence of polynomials $\{P_j\}_{j=-\infty}^{0} \subset \mathcal{P}_d$ such that*

$$f_{\mathrm{low}} = \sum_{j=-\infty}^{0} (\varphi_j(D)f + P_j)$$

is convergent in the topology of $\mathcal{S}'(\mathbb{R}^n)$. The distribution f_{low} is actually a continuous function satisfying (31.13) *with $P = 0$.*

(iii) *The limit*

$$f_{\mathrm{high}} = \sum_{j=1}^{\infty} \varphi_j(D)f$$

exists in the topology of $\mathcal{S}'(\mathbb{R}^n)$.

(iv) *The distribution $f - f_{\mathrm{high}} - f_{\mathrm{low}}$ is a polynomial P.*

(v) *If $0 \notin \mathrm{supp}(\mathcal{F}f)$, that is, $\mathcal{F}f$ vanishes near the origin, then we can take this polynomial P in (iv) to be 0.*

32 Lorentz and Herz spaces with variable exponents

32.1 Lorentz spaces with variable exponents In 2003 and 2008 Ephremidze, Kokilashvili, and S. Samko [50] generalized the paper [94] and introduced the Lorentz space $\mathcal{L}^{p(\cdot),q(\cdot)}$ with variable exponents $p(t), q(t)$ and proved the boundedness of singular integral and potential type operators, and corresponding ergodic operators in these spaces. The boundedness of these operators in such spaces is possible without the local log-condition on the exponents, typical for the variable exponent Lebesgue spaces; instead the exponents $p(s)$ and $q(s)$ should only satisfy decay conditions of log-type as $s \to 0$ and $|s| \to \infty$. The proofs are based on the use of results for Hardy inequalities in variable exponent Lebesgue spaces obtained in [44]. Kokilashvili and S. Samko [94] introduced a version of variable exponent Lebesgue spaces in terms of Lorentz type spaces, and proved the boundedness of singular integral and potential type operators in the Euclidean setting, including the weighted case with power weights, and also of the Cauchy singular integral on Lyapunov curves.

32.2 Herz spaces with variable exponents S. Samko [197, 196] introduced a new type of variable exponent function spaces $\dot{H}^{p(\cdot),q(\cdot),\alpha(\cdot)}(\mathbb{R}^n)$ and $H^{p(\cdot),q(\cdot),\alpha(\cdot)}(\mathbb{R}^n)$ of Herz type, homogeneous and non-homogeneous versions, where all the three parameters are variable, and Samko compared continuous and discrete approaches to their definitions.

Definition 32.1. Let $0 < \gamma < \delta < \infty$ and $\varepsilon > 0$. Let also $p(\cdot) : \mathbb{R}^n \to [1, \infty)$, $q(\cdot) : [\gamma\nu, \infty) \to [1, \infty)$ and $\alpha(\cdot) : [\gamma\nu, \infty) \to \mathbb{R}$ be variable exponents. Define

$$\|f\|_{H_\nu^{p(\cdot),q(\cdot),\alpha(\cdot)}} \equiv \|f\|_{L^{p(\cdot)}(B(0,\gamma\nu+\varepsilon))} + \|t^{\alpha(t)}\|f\chi_{B(0,\delta t)\setminus B(0,\gamma t)}\|_{L^{p(\cdot)}}\|_{L^{q(\cdot)}((\gamma\nu,\infty),dt/t)}.$$

The Herz space $H_\nu^{p(\cdot),q(\cdot),\alpha(\cdot)}(\mathbb{R}^n)$ is the set of all measurable functions for which

$$\|f\|_{H_\nu^{p(\cdot),q(\cdot),\alpha(\cdot)}} < \infty.$$

When $\nu = 0$, then this space is homogeneous. Otherwise, this space is called inhomogeneous. In [196] under the only assumption that the exponents p, q and α are subject to the log-decay condition at infinity, we prove that sublinear operators, satisfying the size condition known for singular integrals and bounded in $L^{p(\cdot)}(\mathbb{R}^n)$, are also bounded in the nonhomogeneous version of the introduced spaces, which includes the case maximal and Calderón-Zygmund singular operators. See [74, 75] for the case when q is constant.

33 Hardy spaces with variable exponents Now we consider Hardy spaces, which are investigated in two independent works; Nakai and Sawano [154] and Cruz-Uribe and Wang [30].

In 2012, Nakai and Sawano [154] proved the atomic decomposition of Hardy spaces with variable exponents and established the duality. The dual space is a generalized Campanato space with variable growth condition of higher order. See [46] for Herz-type Hardy spaces.

In this section we summarize what we obtained in [154, 203]. Let $p(\cdot) : \mathbb{R}^n \to (0, \infty)$ be an exponent such that $0 < p_- = \inf_{x\in\mathbb{R}^n} p(x) \le p_+ = \sup_{x\in\mathbb{R}^n} p(x) < \infty$. Here and below, for the sake of simplicity, we shall postulate the following conditions (14.1) and (14.2) on $p(\cdot)$ as usual: note that (14.1) and (14.2) are necessary when we consider the property of maximal operators as we have seen in Part III.

33.1 Definition In the celebrated paper [51], by using a suitable family $\mathcal{F}_N$, C. Fefferman and E. Stein defined the Hardy space $H^p(\mathbb{R}^n)$ with the norm given by

$$\|f\|_{H^p} := \left\|\sup_{t>0} \sup_{\varphi\in\mathcal{F}_N} |t^{-n}\varphi(t^{-1}\cdot) * f|\right\|_{L^p}, \quad f \in \mathcal{S}'(\mathbb{R}^n)$$

for $0 < p < \infty$. Here, in this part, we aim to replace $L^p(\mathbb{R}^n)$ with $L^{p(\cdot)}(\mathbb{R}^n)$ and investigate the function space obtained in this way.

The aim of the present paper is to review the definition of Hardy spaces with variable exponents and then to consider and apply the atomic decomposition. As is the case with the classical theory, we choose a suitable subset $\mathcal{F}_N \subset \mathcal{S}(\mathbb{R}^n)$, which we describe.

Definition 33.1.

(i) Topologize $\mathcal{S}(\mathbb{R}^n)$ by the collection of semi-norms $\{p_N\}_{N\in\mathbb{N}}$ given by

$$p_N(\varphi) := \sum_{|\alpha|\le N} \sup_{x\in\mathbb{R}^n} (1+|x|)^N |\partial^\alpha \varphi(x)|$$

for each $N \in \mathbb{N}$. Define

$$\mathcal{F}_N := \{\varphi \in \mathcal{S}(\mathbb{R}^n) : p_N(\varphi) \le 1\}. \tag{33.1}$$

(ii) Let $f \in \mathcal{S}'(\mathbb{R}^n)$. Denote by $\mathcal{M}f$ the grand maximal operator given by

$$\mathcal{M}f(x) := \sup\{|t^{-n}\psi(t^{-1}\cdot) * f(x)| \,:\, t>0, \quad \psi \in \mathcal{F}_N\},$$

where we choose and fix a large integer N.

(iii) The Hardy space $H^{p(\cdot)}(\mathbb{R}^n)$ is the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ for which the quantity

$$\|f\|_{H^{p(\cdot)}} := \|\mathcal{M}f\|_{L^{p(\cdot)}}$$

is finite.

The definition of $\mathcal{F}_N$ dates back to the original work [214].

Suppose that $0 < p_- \le p_+ < \infty$ below. The following theorem about the definition of $H^{p(\cdot)}(\mathbb{R}^n)$ is obtained in [154]:

Theorem 33.1 ([154, Theorem 1.2 and 3.3]). *Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ be a function such that $\displaystyle\int_{\mathbb{R}^n} \varphi(x)\,dx \ne 0$. We define*

$$\|f\|_{H^{p(\cdot)}_{\varphi,*}} := \left\| \sup_{t>0} |t^{-n}\varphi(t^{-1}\cdot) * f| \right\|_{L^{p(\cdot)}}, \quad f \in \mathcal{S}'(\mathbb{R}^n). \tag{33.2}$$

Then the norms $\|\cdot\|_{H^{p(\cdot)}_{\varphi,*}}$ and $\|f\|_{H^{p(\cdot)}}$ are equivalent.

Note that it can happen that $0 < p_- < 1 < p_+ < \infty$ in our setting.

33.2 Atomic decomposition Here is another key result which we shall highlight. To formulate we adopt the following definition of the atomic decomposition:

Definition 33.2 ($(p(\cdot), q)$-atom). Let $p(\cdot) : \mathbb{R}^n \to (0,\infty)$, $0 < p_- \le p_+ < q \le \infty$ and $q \ge 1$. Fix an integer $d \ge d_{p(\cdot)} := \min\{d \in \mathbb{N}\cup\{0\} : p_-(n+d+1) > n\}$. A function a on $\mathbb{R}^n$ is called a $(p(\cdot), q)$-atom if there exists a cube Q such that

(a1) $\mathrm{supp}(a) \subset Q$,

(a2) $\displaystyle\|a\|_{L^q} \le \frac{|Q|^{1/q}}{\|\chi_Q\|_{L^{p(\cdot)}}}$, (a3) $\displaystyle\int_{\mathbb{R}^n} a(x)x^\alpha\,dx = 0$ for $|\alpha| \le d$.

The set of all such pairs (a, Q) will be denoted by $A(p(\cdot), q)$.

Under this definition, we define the atomic Hardy spaces with variable exponents. Here and below we denote

$$\underline{p} := \min(p_-, 1). \tag{33.3}$$

Definition 33.3 (Sequence norm $\mathcal{A}(\{\kappa_j\}_{j=1}^\infty, \{Q_j\}_{j=1}^\infty)$ and $H^{p(\cdot),q}_{\mathrm{atom}}(\mathbb{R}^n)$)**.** Given sequences of nonnegative numbers $\{\kappa_j\}_{j=1}^\infty$ and cubes $\{Q_j\}_{j=1}^\infty$, define

$$\mathcal{A}(\{\kappa_j\}_{j=1}^\infty, \{Q_j\}_{j=1}^\infty) := \inf\left\{\lambda > 0 \,:\, \int_{\mathbb{R}^n}\left\{\sum_{j=1}^\infty\left(\frac{\kappa_j\chi_{Q_j}(x)}{\lambda\|\chi_{Q_j}\|_{L^{p(\cdot)}}}\right)^{\underline{p}}\right\}^{\frac{p(x)}{\underline{p}}} dx \le 1\right\}.$$

The atomic Hardy space $H^{p(\cdot),q}_{\mathrm{atom}}(\mathbb{R}^n)$ is the set of all distributions $f \in \mathcal{S}'(\mathbb{R}^n)$ such that it can be written as

$$f = \sum_{j=1}^\infty \kappa_j a_j \text{ in } \mathcal{S}'(\mathbb{R}^n), \tag{33.4}$$

where $\{\kappa_j\}_{j=1}^\infty$ is a sequence of nonnegative numbers, $\{(a_j, Q_j)\}_{j=1}^\infty \subset \mathcal{A}(p(\cdot), q)$ and $\mathcal{A}(\{\kappa_j\}_{j=1}^\infty, \{Q_j\}_{j=1}^\infty)$ is finite. One defines

$$\|f\|_{H^{p(\cdot),q}_{\mathrm{atom}}} := \inf \mathcal{A}(\{\kappa_j\}_{j=1}^\infty, \{Q_j\}_{j=1}^\infty),$$

where the infimum is taken over all admissible expressions as in (33.4).

Under these definitions, we have the following:

Theorem 33.2 ([203])**.** *The variable Hardy norms given in Theorem* 33.1 *and the ones given by means of atoms are isomorphic as long as*

$$q > p_+ \ge 1 \quad \textit{or} \quad q = 1 > p_+.$$

Remark that we could not specify the condition of q precisely in [154] but as the calculation in [203] shows $q > p_+ \ge 1$ or $q = 1 > p_+$ suffices.

It counts that Hardy spaces with variable exponents coincide Lebesgue spaces with variable exponents if $p_- > 1$ because our results readily yield ones for Lebesgue spaces with variable exponents in such cases.

Lemma 33.3 ([154, Lemma 3.1])**.** *If $p_- > 1$, then the boundedness of the Hardy-Littlewood maximal operator M obtained in Part III and the reflexivity of $L^{p(\cdot)}(\mathbb{R}^n)$ yield $H^{p(\cdot)}(\mathbb{R}^n) \simeq L^{p(\cdot)}(\mathbb{R}^n)$ with equivalent norms.*

Remark 33.1 ([154, Remark 3.5]). The $H^{p(\cdot)}(\mathbb{R}^n)$-norm topology is stronger than the topology of $\mathcal{S}'(\mathbb{R}^n)$; indeed, setting $\check{\varphi}(x) := \varphi(-x)$ for $\varphi \in \mathcal{S}(\mathbb{R}^n)$, we have

$$|\langle f, \varphi\rangle| = |f * \check{\varphi}(0)| \lesssim \|f\|_{H^{p(\cdot)}}.$$

Here, we define an index $d_{p(\cdot)} \in \mathbb{N} \cup \{0\}$ by

$$d_{p(\cdot)} := \min \{d \in \mathbb{N} \cup \{0\} : p_-(n+d+1) > n\}. \tag{33.5}$$

For a nonnegative integer d, let $\mathcal{P}_d(\mathbb{R}^n)$ denote the set of all polynomials having degree at most d.

Let $p(\cdot) : \mathbb{R}^n \to (0,\infty)$, $0 < p_- \le p_+ < q \le \infty$ and $q \ge 1$. Recall that we have defined $(p(\cdot), q)$-atoms in Definition 33.2. In the variable setting as well, we have that atoms have $L^{p(\cdot)}$-norm less than 1. Recall that $A(p(\cdot), q)$ is the set of all pairs (a, Q) such that a is a $(p(\cdot), q)$-atom and that Q is the corresponding cube.

Remark 33.2. (i) Define another variable exponent $\tilde{q}(\cdot)$ by

$$\frac{1}{p(x)} = \frac{1}{q} + \frac{1}{\tilde{q}(x)} \quad (x \in \mathbb{R}^n). \tag{33.6}$$

Then we have

$$\|f \cdot g\|_{L^{p(\cdot)}} \lesssim \|g\|_{L^q} \|f\|_{L^{\tilde{q}(\cdot)}} \tag{33.7}$$

for all measurable functions f and g [127].

(ii) A direct consequence of Lemma 19.1 and (33.7) is that $\|a\|_{L^{p(\cdot)}} \lesssim 1$ whenever $(a, Q) \in A(p(\cdot), q)$.

Of course, as is the case when $p(\cdot)$ is a constant, Remark 33.2 can be extended as follows:

Proposition 33.4. (i) *Let $q > \max(1, p_+)$. If $p(\cdot)$ satisfies $0 < p_- \le p_+ < \infty$ as well as* (14.1) *and* (14.2)*, then we have*

$$\|a\|_{H^{p(\cdot)}} \lesssim 1$$

for any $(a, Q) \in A(p(\cdot), q)$.

(ii) *If $p(\cdot)$ satisfies $0 < p_- \le p_+ < 1$ as well as* (14.1) *and* (14.2)*, then we have*

$$\|a\|_{H^{p(\cdot)}} \lesssim 1$$

for any $(a, Q) \in A(p(\cdot), 1)$.

Remark 33.3. In [154, Proposition 4.2], we assumed only that $q > 1$. However, in order to define an exponent $\tilde{q}(\cdot)$ by

$$\frac{1}{p(x)} \equiv \frac{1}{q} + \frac{1}{\tilde{q}(x)} \quad (x \in \mathbb{R}^n), \tag{33.8}$$

it was necessary that $q > p_+$. Hence, we should have assumed that $q > \max(1, p_+)$. However, all the results in [154] remain true modulo this minor modification.

Here for the sake of completeness, we prove Proposition 33.4.

Proof of Proposition 33.4. In [154, (4.4)], we essentially obtained

$$\mathcal{M}a(x) \lesssim Ma(x)\chi_{2\sqrt{n}Q}(x) + \left(1 + \frac{|x - c_Q|}{\ell(Q)}\right)^{-n-d-1} \chi_{\mathbb{R}^n \setminus 2\sqrt{n}Q}(x).$$

We can treat the second term as we did in [154, Proposition 4.2]. When $q > \max(1, p_+)$, then we can go through the same argument as we did in the proof of [154, Proposition 4.2] after $\tilde{q}(\cdot)$ by (33.8). If $q = 1 > p_+$, then we can use the Kolmogorov inequality to obtain

$$\|\chi_{2\sqrt{n}Q}\mathcal{M}a\|_{L^{p(\cdot)}} \le \|\chi_{2\sqrt{n}Q}Ma\|_{L^{p(\cdot)}} \le \|Ma\|_{L^{p(\cdot)}} \le \|a\|_{L^{p(\cdot)}} \lesssim 1.$$

Thus, the proof is therefore complete. $\square$

The atomic Hardy space $H^{p(\cdot),q}_{\rm atom}(\mathbb{R}^n)$ was defined to be the set of all distributions $f \in \mathcal{S}'(\mathbb{R}^n)$ such that it can be written in the form $f = \sum_{j=1}^{\infty} \kappa_j a_j$ in $\mathcal{S}'(\mathbb{R}^n)$, where

$$\mathcal{A}(\{\kappa_j\}_{j=1}^{\infty}, \{Q_j\}_{j=1}^{\infty}) < \infty$$

and $\{(a_j, Q_j)\}_{j\in\mathbb{N}} \subset A(p(\cdot), q)$. One defines

$$\|f\|_{H^{p(\cdot),q}_{\rm atom}} := \inf \mathcal{A}(\{\kappa_j\}_{j=1}^{\infty}, \{Q_j\}_{j=1}^{\infty}),$$

where the infimum is taken over all expressions as above.

Observe that if $p(\cdot) \equiv p_+ = p_-$, that is, $p(\cdot)$ is a constant function, then we can recover classical Hardy spaces. Unlike the classical case, $(p(\cdot), \infty)$-atoms are not dealt separately. Consequently we have two types of results for $(p(\cdot), \infty)$-atoms.

Definition 33.4 ($H^{p(\cdot),\infty}_{\rm atom,*}(\mathbb{R}^n)$, [154, Definition 4.3])**.** Let $q \geq 1$ be a fixed constant and let $p(\cdot) : \mathbb{R}^n \to (0, \infty)$ be a function such that $0 < p_- \leq p_+ < q \leq \infty$. Then $f \in \mathcal{S}'(\mathbb{R}^n)$ is in $H^{p(\cdot),\infty}_{\rm atom,*}(\mathbb{R}^n)$ if and only if there exist sequences of nonnegative numbers $\{\kappa_j\}_{j=1}^{\infty}$ and $\{(a_j, Q_j)\}_{j=1}^{\infty} \subset A(p(\cdot), \infty)$ such that

$$f = \sum_{j=1}^{\infty} \kappa_j a_j \text{ in } \mathcal{S}'(\mathbb{R}^n), \text{ and that} \sum_j \int_{Q_j} \left(\frac{\kappa_j}{\|\chi_{Q_j}\|_{L^{p(\cdot)}}} \right)^{p(x)} dx < \infty. \tag{33.9}$$

For sequences of nonnegative numbers $\{\kappa_j\}_{j=1}^{\infty}$ and cubes $\{Q_j\}_{j=1}^{\infty}$, define

$$\mathcal{A}^*(\{\kappa_j\}_{j=1}^{\infty}, \{Q_j\}_{j=1}^{\infty}) := \inf \left\{ \lambda > 0 : \int_{Q_j} \sum_j \left(\frac{\kappa_j}{\lambda \|\chi_{Q_j}\|_{L^{p(\cdot)}}} \right)^{p(x)} dx \leq 1 \right\}.$$

Before we proceed further, four helpful remarks may be in order.

Remark 33.4.

(i) A trivial fact that can be deduced from the embedding $\ell^{\underline{p}} = \ell^{\min(p_-,1)} \hookrightarrow \ell^\infty$ is that

$$\mathcal{A}^*(\{\kappa_j\}_{j=1}^{\infty}, \{Q_j\}_{j=1}^{\infty}) \leq \mathcal{A}(\{\kappa_j\}_{j=1}^{\infty}, \{Q_j\}_{j=1}^{\infty}). \tag{33.10}$$

(ii) Let $(a, Q) \in \mathcal{A}(p(\cdot), q)$. Then in view of (33.10), we have

$$\|a\|_{H^{p(\cdot),q}_{\rm atom,*}} \leq \|a\|_{H^{p(\cdot),q}_{\rm atom}} \leq 1. \tag{33.11}$$

(iii) From (33.11) we conclude

$$\|f\|_{H^{p(\cdot),q}_{\rm atom,*}} \leq \|f\|_{H^{p(\cdot),q}_{\rm atom}} \leq \frac{\|\chi_Q\|_{L^{p(\cdot)}}}{|Q|^{\frac{1}{q}}} \|f\|_{L^q} \tag{33.12}$$

whenever $f \in L^q(\mathbb{R}^n)$ is supported on a cube Q and satisfies $\int_{\mathbb{R}^n} x^\alpha f(x)\, dx = 0$ for all multiindices α with length d.

(iv) Assume $p_+ \le 1$ in addition. For sequences of nonnegative numbers $\{\kappa_j\}_{j=1}^\infty$ and $\{(a_j, Q_j)\}_{j=1}^\infty \subset A(p(\cdot), q)$, we have

$$\sum_{j=1}^{\infty} \kappa_j \le \mathcal{A}^*(\{\kappa_j\}_{j=1}^\infty, \{Q_j\}_{j=1}^\infty). \tag{33.13}$$

Now we formulate our atomic decomposition theorem. Let us begin with the space $H^{p(\cdot),q}_{\mathrm{atom},*}(\mathbb{R}^n)$ with $q = \infty$.

Theorem 33.5. [154] *If $p(\cdot)$ satisfies $0 < p_- \le p_+ < \infty$,* (14.1) *and* (14.2)*, then, for all $f \in \mathcal{S}'(\mathbb{R}^n)$,*

$$\|f\|_{H^{p(\cdot)}} \sim \|f\|_{H^{p(\cdot),\infty}_{\mathrm{atom}}} \sim \|f\|_{H^{p(\cdot),\infty}_{\mathrm{atom},*}}.$$

The atomic decomposition for $\mathcal{A}(p(\cdot), q)$ can be also obtained.

Theorem 33.6. [203] *Suppose either* (i) *or* (ii) *holds;*

(i) $0 < p_- \le p_+ < q \le \infty$ *and* $p_+ \ge 1$*;*

(ii) $0 < p_- \le p_+ < 1 \le q \le \infty$.

Assume $p(\cdot)$ satisfies (14.1) *and* (14.2)*. Then, for all $f \in \mathcal{S}'(\mathbb{R}^n)$, $\|f\|_{H^{p(\cdot)}} \sim \|f\|_{H^{p(\cdot),q}_{\mathrm{atom}}}$.*

See [155] for the case of Orlicz-Morrey spaces and [70] for the case of Morrey spaces with constant exponent.

33.3 Duality $H^{p(\cdot)}(\mathbb{R}^n)$**-**$\mathcal{L}_{q,\phi,d}(\mathbb{R}^n)$ In this section, we shall show that the dual spaces of $H^{p(\cdot)}(\mathbb{R}^n)$ are generalized Campanato spaces $\mathcal{L}_{q,\phi,d}(\mathbb{R}^n)$ with variable growth conditions when $0 < p_- \le p_+ \le 1$. For the definition of $\mathcal{L}_{q,\phi,d}(\mathbb{R}^n)$ see Section 31.

If d is as in (33.5), then $L^{q,d}_{\mathrm{comp}}(\mathbb{R}^n)$ is dense in $H^{p(\cdot),q}_{\mathrm{atom}}(\mathbb{R}^n)$. Indeed, it contains all the finite linear combinations of $(p(\cdot), q)$-atoms from the definition of $H^{p(\cdot),q}_{\mathrm{atom}}(\mathbb{R}^n)$.

To show the duality we first consider the dual of $H^{p_0}(\mathbb{R}^n) \cap H^1(\mathbb{R}^n)$, where p_0 is a constant with $0 < p_0 \le 1$. Recall that $\mathrm{bmo}(\mathbb{R}^n)$, the local $\mathrm{BMO}(\mathbb{R}^n)$, is the set of all locally integrable functions f such that

$$\|f\|_{\mathrm{bmo}} := \sup_{\substack{Q \in \mathcal{Q} \\ |Q| \le 1}} \fint_Q \left| f(x) - \fint_Q f(y)\, dy \right| dx + \sup_{\substack{Q \in \mathcal{Q} \\ |Q| = 1}} \fint_Q |f(x)|\, dx < \infty.$$

Then from the definition of the norms $\|\cdot\|_{\mathrm{BMO}}$ and $\|\cdot\|_{\mathrm{bmo}}$ we have $\|f\|_{\mathrm{BMO}} \lesssim \|f\|_{\mathrm{bmo}}$. By the well-known $H^1(\mathbb{R}^n)$-$\mathrm{BMO}(\mathbb{R}^n)$ duality, $\mathrm{bmo}(\mathbb{R}^n)$ is canonically embedded into the dual space of $H^1(\mathbb{R}^n)$. We have the following important conclusions:

Theorem 33.7 ([154, Theorem 7.3])**.** *Let $0 < p_0 \le 1$ and $1 \le q \le \infty$. Set $\phi_1(Q) := |Q|^{\frac{1}{p_0}-1}$ and $\phi_2(Q) := |Q|^{\frac{1}{p_0}-1} + 1$ for $Q \in \mathcal{Q}$. Then we have $\mathcal{L}_{q,\phi_2,d}(\mathbb{R}^n) \hookrightarrow \mathcal{L}_{q,\phi_1,d}(\mathbb{R}^n) + \mathrm{bmo}(\mathbb{R}^n)$ in the sense of continuous embedding. More quantitatively, if we choose $\psi \in \mathcal{S}(\mathbb{R}^n)$ so that $\chi_{Q(0,1)} \le \psi \le \chi_{Q(0,2)}$, then we have*

$$\|\psi(D)g\|_{\mathcal{L}_{q,\phi_1,d}} \lesssim \|g\|_{\mathcal{L}_{q,\phi_2,d}}, \quad \|(1-\psi(D))g\|_{\mathrm{bmo}} \lesssim \|g\|_{\mathcal{L}_{q,\phi_2,d}}.$$

Corollary 33.8 ([154, Corollary 7.4]). *Let $0 < p_0 \le 1$, $1 \le q \le \infty$ and define* $\phi_2(Q) := \dfrac{|Q|^{\frac{1}{p_0}} + |Q|}{|Q|}$. *Denote by q' the conjugate of q : $q' = \dfrac{q}{q-1}$. Any element g in $\mathcal{L}_{q',\phi_2,d}(\mathbb{R}^n)$ defines a continuous linear functional ℓ_g on $H^{p_0}(\mathbb{R}^n) \cap H^1(\mathbb{R}^n)$ such that*

$$\ell_g(a) = \int_{\mathbb{R}^n} g(x)a(x)\,dx, \quad \ell_g(f) = \sum_{j=1}^{\infty} \ell_g(f_j),$$

whenever $a \in L^{q,d}_{\mathrm{comp}}(\mathbb{R}^n)$ and $f = \sum_{j=1}^{\infty} f_j$ in the topology of $H^{p_0}(\mathbb{R}^n) \cap H^1(\mathbb{R}^n)$.

Now we specify the dual of $H^{p(\cdot)}(\mathbb{R}^n)$ with $0 < p_- \le p_+ \le 1$. It follows from the definition of the dual norm that, for all $\ell \in (H^{p(\cdot),q}_{\mathrm{atom}}(\mathbb{R}^n))^*$,

$$\|\ell\|_{(H^{p(\cdot),q}_{\mathrm{atom}}(\mathbb{R}^n))^*} = \sup\left\{|\ell(f)| : \|f\|_{H^{p(\cdot),q}_{\mathrm{atom}}} \le 1\right\}$$

is finite and $\|\ell\|_{(H^{p(\cdot),q}_{\mathrm{atom}}(\mathbb{R}^n))^*}$ is a norm on $(H^{p(\cdot),q}_{\mathrm{atom}}(\mathbb{R}^n))^*$. Then, using the above results, we have the following:

Theorem 33.9 ([154]). *Let $p(\cdot) : \mathbb{R}^n \to (0,\infty)$, $0 < p_- \le p_+ \le 1$, $p_+ < q \le \infty$ and $1/q + 1/q' = 1$. Suppose that the integer d is as in* (33.5). *Define*

$$\phi_3(Q) := \frac{\|\chi_Q\|_{L^{p(\cdot)}}}{|Q|} \quad (Q \in \mathcal{Q}). \tag{33.14}$$

If $p(\cdot)$ satisfies (14.1) *and* (14.2)*, then*

$$(H^{p(\cdot),q}_{\mathrm{atom}}(\mathbb{R}^n))^* \simeq \mathcal{L}_{q',\phi_3,d}(\mathbb{R}^n)$$

with equivalent norms. More precisely, we have the following assertions:

(i) *Let $f \in \mathcal{L}_{q',\phi_3,d}(\mathbb{R}^n)$. Then the functional*

$$\ell_f : a \in L^{q,d}_{\mathrm{comp}}(\mathbb{R}^n) \mapsto \int_{\mathbb{R}^n} a(x)f(x)\,dx \in \mathbb{C}$$

extends to a bounded linear functional on $(H^{p(\cdot),q}_{\mathrm{atom}}(\mathbb{R}^n))^*$ such that

$$\|\ell_f\|_{(H^{p(\cdot),q}_{\mathrm{atom}})^*} \lesssim \|f\|_{\mathcal{L}_{q',\phi_3,d}}.$$

(ii) *Conversely, any linear functional ℓ on $(H^{p(\cdot),q}_{\mathrm{atom}}(\mathbb{R}^n))^*$ can be realized as above with some $f \in \mathcal{L}_{q',\phi_3,d}(\mathbb{R}^n)$ and we have $\|f\|_{\mathcal{L}_{q',\phi_3,d}} \lesssim \|\ell\|_{(H^{p(\cdot),q}_{\mathrm{atom}})^*}$.*

In particular, we have

$$(H^{p(\cdot)}(\mathbb{R}^n))^* \simeq \mathcal{L}_{q',\phi_3,d}(\mathbb{R}^n).$$

Namely, any $f \in \mathcal{L}_{q',\phi_3,d}(\mathbb{R}^n)$ defines a continuous linear functional on $(H^{p(\cdot)}(\mathbb{R}^n))^*$ such that

$$L_f(a) = \int_{\mathbb{R}^n} a(x)f(x)\,dx$$

for any $a \in L^{q,d}_{\mathrm{comp}}(\mathbb{R}^n)$ and any continuous linear functional on $(H^{p(\cdot)}(\mathbb{R}^n))^*$ is realized with some some $f \in \mathcal{L}_{q',\phi_3,d}(\mathbb{R}^n)$.

Note that there was no need to assume $q \gg 1$ in Theorem 33.9, since we refined Theorem 33.6. When $q \gg 1$, this theorem is recorded as [154, Theorem 7.5]. Note also that we can take $q = \infty$ and $q' = 1$ in this theorem, see Theorem 33.12 below. To show this theorem we need the following lemmas:

Lemma 33.10 ([154, Lemma 7.6]). *Let $p(\cdot) : \mathbb{R}^n \to (0, \infty)$, $0 < p_- \le p_+ < q \le \infty$, $q \ge 1$ and $1/q + 1/q' = 1$. Let d be as in* (33.5) *and ϕ as in* (33.14). *Then, for all $g \in \mathcal{L}_{q',\phi_3,d}(\mathbb{R}^n)$ and all $(p(\cdot), q)$-atoms $a(\cdot)$,*

$$\left| \int_{\mathbb{R}^n} a(x) g(x) \, dx \right| \le \|g\|_{\mathcal{L}_{q',\phi_3,d}}. \tag{33.15}$$

In general it is very hard to obtain (33.16). However, with the help of auxiliary space $H^1(\mathbb{R}^n)$, this can be achieved.

Lemma 33.11 ([154, Lemma 7.7]). *Keep to the same assumption as Lemma* 33.10. *Assume in addition $p(\cdot)$ satisfies* (14.1) *and* (14.2). *For $g \in \mathcal{L}_{q',\phi_3,d}(\mathbb{R}^n)$, for $f \in L^{q,d}_{\mathrm{comp}}(\mathbb{R}^n)$, and for any decomposition $f = \sum_j \kappa_j a_j$, where the convergence takes place in $H^{p(\cdot),q}_{\mathrm{atom}}(\mathbb{R}^n) \cap H^{p_\infty}(\mathbb{R}^n) \cap H^1(\mathbb{R}^n)$, the following equality holds:*

$$\int_{\mathbb{R}^n} f(x) g(x) \, dx = \sum_j \kappa_j \int_{\mathbb{R}^n} a_j(x) g(x) \, dx. \tag{33.16}$$

Theorem 33.12 ([154, Theorem 7.5]). *Let $p \in LH$ be a variable exponent and $0 < p_- \le p_+ \le 1$. Define*

$$\phi(Q) := \frac{\|\chi_Q\|_{L^{p(\cdot)}}}{|Q|} \quad (Q \in \mathcal{Q}).$$

Then,

$$(H^{p(\cdot)}(\mathbb{R}^n))^* \simeq \mathcal{L}_{1,\phi,d}(\mathbb{R}^n).$$

Remark 33.5. About the dual of $H^{p(\cdot)}(\mathbb{R}^n)$, the cases $0 < p_- \le p_+ \le 1$ and $p_- > 1$ are known and the remaining case $p_- < 1 < p_+$ is still an open problem.

See [155] for the case of Orlicz-Hardy spaces and their duals. See also [12, 15, 109, 111, 228] for Musielak-Orlicz-Hardy spaces.

34 Besov and Triebel-Lizorkin spaces with variable exponents In 2009 and 2010, Diening, Hästö and Roudenko [42] and Almeida and Hästö [3] investigated inhomogeneous Besov and Triebel-Lizorkin spaces with three variable exponents. Noi and Sawano [162] (2012) considered the complex interpolation. Dong and Xu characterized function spaces by using the local means [47]. Shi and Xu considered Besov spaces and Triebel-Lizorkin spaces based on the Herz spaces $K^{\alpha(\cdot)}_{p(\cdot),q}$ with variable exponents in [45, 210], where the necessary vector-valued inequality is investigated in [45, Theorem 2.8]. In [52], Besov spaces and Triebel-Lizorkin spaces based upon the variable Morrey space $M^{p(\cdot)}_{q(\cdot)}$ is investigated and Fu and Xu obtained a discrete characterization in [52, Theorems 3.1 and 3.2]. See [68, 209, 218, 219, 220, 221, 222, 223] for more related results.

We now define Triebel-Lizorkin spaces of variable integrability. We use $\mathcal{F}$ and $\mathcal{F}^{-1}$ to denote the Fourier transform and its inverse respectively. Let $\Phi \in \mathcal{S}(\mathbb{R}^n)$ be a function satisfying $\chi_{B(0,1/2)} \le \mathcal{F}\Phi \le \chi_{B(0,1)}$, where $B(0,r) = \{x \in \mathbb{R}^n : |x| < r\}$ as usual. Set $\Phi_j(x) := 2^{nj}\Phi(2^j x)$ for $j \in \mathbb{Z}_0 = \mathbb{Z} \cup \{0\}$. If we define $\theta_j := \Phi_j - \Phi_{j-1}$ for $j \in \mathbb{N}$ and $\theta_0 := \Phi_0$, then we have $\sum_{j=0}^{\infty} \theta_j \equiv 1$. To describe the vector-valued norm, for $p, q \in \mathcal{B}_0(\mathbb{R}^n) \cap C^{\log}(\mathbb{R}^n)$, we write $L^{p(\cdot)}(\ell^{q(\cdot)})$ to denote the space consisting of all sequences $\{g_j\}_0^\infty$ of measurable functions on $\mathbb{R}^n$ such that

$$||\{g_j\}_0^\infty||_{L^{p(\cdot)}(\ell^{q(\cdot)})} = ||\, ||\{g_j\}_0^\infty||_{\ell^{q(\cdot)}}||_{L^{p(\cdot)}} = \left\| \left(\sum_{j=0}^{\infty} |g_j(\cdot)|^{q(\cdot)} \right)^{\frac{1}{q(\cdot)}} \right\|_{L^{p(\cdot)}} < \infty.$$

Definition 34.1. Let $p(\cdot), q(\cdot) \in C^{\log}(\mathbb{R}^n) \cap \mathcal{B}_0(\mathbb{R}^n)$ and $s(\cdot) \in C^{\log}$. Then define the variable exponent Triebel-Lizorkin space $F^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ as the collection of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that $||f||_{F^{s(\cdot)}_{p(\cdot),q(\cdot)}} = \left\| \left\{ 2^{s(\cdot)j}\theta_j * f \right\}_0^\infty \right\|_{L^{p(\cdot)}(\ell^{q(\cdot)})} < \infty$.

As is established in [42], the definition of $F^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ does not depend on the starting function Φ.

The definition and formulation of our result for Besov spaces is analogous. Let $p(\cdot)$, $q(\cdot) \in \mathcal{P}_0(\mathbb{R}^n)$. The space $\ell^{q(\cdot)}(L^{p(\cdot)})$ is the collection of all sequences $\{g_j\}_0^\infty$ of measurable functions on $\mathbb{R}^n$ such that

$$||\{g_j\}_{j=0}^\infty||_{\ell^{q(\cdot)}(L^{p(\cdot)})} = \inf \left\{ \mu > 0 \,:\, \varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})} \left(\left\{ \frac{f_j}{\mu} \right\}_{j=0}^\infty \right) \le 1 \right\} < \infty,$$

where

$$\varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})} \left(\{f_j\}_{j=0}^\infty \right) = \sum_{j=0}^{\infty} \inf \left\{ \lambda_j > 0 \,:\, \int_{\mathbb{R}^n} \left(\frac{|f_j(x)|}{\lambda_j^{\frac{1}{q(x)}}} \right)^{p(x)} \mathrm{d}x \le 1 \right\}.$$

Since we assume that $q_+ < \infty$,

$$\varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})} \left(\{f_j\}_{j=0}^\infty \right) = \sum_{j=0}^{\infty} \left\| |f_j|^{q(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}} \tag{34.1}$$

holds.

Definition 34.2. Let $p(\cdot), q(\cdot) \in C^{\log}(\mathbb{R}^n) \cap \mathcal{P}_0(\mathbb{R}^n)$ and $s(\cdot) \in C^{\log}(\mathbb{R}^n)$. The Besov space with variable exponents $B^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ is the collection of $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$||f||_{B^{s(\cdot)}_{p(\cdot),q(\cdot)}} = \left\| \left\{ 2^{js(\cdot)}\theta_j * f \right\}_0^\infty \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} < \infty.$$

Here we content ourselves with stating the definition of these spaces. See [3, 42] for more information. See also [229].

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