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HYPERSPACES AND COMPLETE INVARIANCE PROPERTY

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Abstract. In this paper, the uniform flow over the hyperspaces 2^X of nonempty compact subsets of a noncompact metric space X with uniform flow, and $F_n(X)$ of nonempty subsets of a compact metric space X with uniform flow containing atmost n points is introduced and used to show that the hyperspace 2^X has the CIP and the hyperspace $F_n(X)$ has the CIPH.[1]

1. Introduction

A topological space X is said to possess the complete invariance property (CIP) if each of its nonempty closed subsets is the fixed point set, for some self continuous map f on X [14]. In case, f can be found to be a homeomorphism, we say that the space enjoys the complete invariance property with respect to homeomorphism ($CIPH$) [6].

A survey of results concerning the CIP may be found in [11] and a number of nonmetric results may be found in [9]. Some spaces known to have the $CIPH$ are even-dimensional Euclidean balls [13], compact surfaces and positive-dimensional spheres [12], the Hilbert cube and metrizable product spaces which have the real line or an odd-dimensional sphere as a factor [6]. In [9], it is shown that an uncountable self product of circles, real lines or two point spaces has the CIP and that connected subgroups of the plane and compact groups need not have the CIP.

In this paper, it is investigated as to when the hyperspace 2^X of nonempty compact subsets of a metric space X enjoys the notion of complete invariance property (CIP). It is also shown that the hyperspace $F_n(X)$ of nonempty subsets of a compact metric space X with uniform flow containing atmost n points has the complete invariance property with respect to homeomorphism ($CIPH$).

2. Pre-requisites

A. Hyperspaces

For a metric space $(X,\ d)$, the *hyperspace* 2^X of nonempty compact subsets and the *hyperspace* $C(X)$ of nonempty compact connected subsets are topologized by the *Hausdorff*

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metric, defined by

$$d_H(A,\ B) = max\{sup_{a\in A}d(a,\ B),\ sup_{b\in\ B}d(b,\ A)\}.$$

If $(X,\ d)$ is a compact metric space, then for each $n \in \mathbb{N}$ the *hyperspace* $F_n(X) = \{A \in 2^X : cardA \leq n\}$ is called the *n-symmetric* product of X. 1-*symmetric* product of X is *the hyperspace* $F_1(X)$ of singletons of X and $F_1(X) \approx X$. For a compact metric space $(X,\ d)$ *the hyperspace* 2^X is compact with respect to the *Hausdorff metric* and $F_n(X) \subset 2^X$ is closed for every n.

Definition 2.1.[10] A *continuum* is a nonempty, compact, connected metric space. A locally connected continuum is called *Peano continuum.* A continuum that contains more than one point is called *nondegenerate.*

Definition 2.2.[10] Any space homeomorphic to the closed interval $[0, 1]$ is called an *arc.* A *free arc* in the continuum X is an arc α with end points a and b such that $\alpha - \{a,\ b\}$ is open in X.

Result 2.3. *2^X is compact metric space if and only if X is a compact metric space. $C(X)$ is closed in 2^X, hence also compact.*

Result 2.4.[5] (Curtis - Shori theorem, a famous result concerning the topology of hyperspaces) *The hyperspace 2^X is homeomorphic to the Hilbert cube ($\mathcal{Q}$) if and only if X is a non-degenerate Peano continuum and $C(X)$ is homeomorphic to $\mathcal{Q}$ if and only if X is a non-degenerate Peano continuum with no free arcs.*

B. CIP and CIPH

A topological space X is said to possess *the complete invariance property* (CIP) [14] if every nonempty closed subset of X is the fixed point set of some continuous self map f on X. In case, f can be chosen to be a homeomorphism, the space is said to possess *the complete invariance property with respect to homeomorphism* (CIPH) [6].

Definition 2.5.[3] A continuous function $\varphi : X \times \mathbb{R} \to X$ on a metric space $(X,\ d)$ is called *uniform flow* if it satisfies the following conditions :

i. $\varphi(x,\ 0) = x$, for all $x \in X$.

ii. $\varphi(\varphi(x,\ s),\ t) = \varphi(x,\ s + t)$, for all $s,\ t \in \mathbb{R}$ and $x \in X$.

iii. $d(x,\ \varphi(x,\ t)) \leq C|t|$, for some positive C and for all $x \in X,\ t \in \mathbb{R}$.

iv. There is a real number $p \geq 0$ such that for all $t \in \mathbb{R}$ and $x \in X$, $\varphi(x,\ t) = x$ iff $t \in p\mathbb{Z}$.

The map $\varphi_t : X \to X$ defined by $\varphi_t(x) = \varphi(x,\ t)$ is a homeomorphism.

Definition 2.6.[8](1) A space X has *property* Q if for every nonempty closed subset K of X there is a point $p \in K$, a retract R of X containing K and a deformation $H : X \times I \to R$ such that $H(x,t) \neq x$ if $x \neq p$ and $t > 0$.

(2) If in (1) we omit p and stipulate that $H(x,t) \neq x$ if $x \notin K$ and $t > 0$, then we say that X has *property* $Q(weak)$.

(3) A space X has *property* W if for every point $p \in X$, there is a deformation $H : X \times I \to X$ such that $H(x,t) \neq x$ if $x \neq p$ and $t > 0$.

(4) If in (3) $H(x,t) \neq x$, whenever $t > 0$, we say that X has *property* $W(strong)$.

Remark 2.7. *$W(strong) \Rightarrow W \Rightarrow Q \Rightarrow Q(weak)$.*

Result 2.8.[8] *A metric space (X,d) with property W has the CIP.*

Result 2.9.[6] *The Hilbert cube $\mathcal{Q}$ has the CIPH.*

Result 2.10.[3] L*et (X,d) be a compact metric space with a uniform flow φ. Then every nonempty closed subset of X is the fixed point set of an orbit-preserving autohomeomorphism of X. In particular, X has the CIPH.*

3. CIP and CIPH over hyperspaces

In this section, the notion of uniform flow over the hyperspaces 2^X and $F_n(X)$ is introduced and using the property of uniform flow we study the CIP and the CIPH over hyperspaces with respect to the Hausdorff metric.

Theorem 3.1. *Let $(X,\ d)$ be a noncompact metric space with uniform flow φ satisfying $\varphi_t(A) = A$ if and only if $t = 0$, for all nonempty compact subsets A of X. Then the map $\Phi : 2^X \times \mathbb{R} \to 2^X$ on the hyperspace of all nonempty compact subsets of X, defined by $\Phi(A,\ t) = \{\varphi(a,\ t) : a \in A\}$ is a uniform flow.*

Proof. Let constants of uniform flow φ be p and C. Now we show that the map $\Phi : 2^X \times \mathbb{R} \to 2^X$ defined by $\Phi(A,\ t) = \{\varphi(a,\ t) : a \in A\}$ is a uniform flow.

i. $\Phi(A,\ 0) = \{\varphi(a,\ 0) : a \in A\} = A$ for all $A \in 2^X$.

ii. $\Phi(A,\ s+t) = \Phi(\Phi(A,\ s),\ t)$ for all $A \in 2^X$ and $s,t \in \mathbb{R}$.

iii. Consider

$$d_H(A,\ \Phi(A,\ t)) = \max\{\sup_{a\in A} d(a,\ \Phi(A,\ t)),\ \sup_{\varphi(a,\ t)\in\Phi(A,\ t)} d(A,\ \varphi(a,\ t))\}.$$

By noting that

$$d(a,\ \Phi(A,\ t)) = \inf_{\varphi(a',\ t)\in\Phi(A,\ t)} d(a,\ \varphi(a',\ t)) \leq C|t|,$$

we have

$$\sup_{a\in A} d(a,\ \Phi(A,\ t)) \leq C|t|,$$

and also

$$d(A,\ \varphi(a,\ t)) = \inf_{a' \in A} d(a',\ \varphi(a,\ t)) \le C|t|,$$

provides

$$\sup_{\varphi(a,\ t) \in \Phi(A,\ t)} d(A,\ \varphi(a,\ t)) \le C|t|.$$

Thus, we have

$$d_H(A,\ \Phi(A,\ t)) \le C|t|,$$

for all $A \in 2^X$.

iv. From the condition $\varphi_t(A) = A$ if and only if $t = 0$ we have
$\Phi(A,\ t) = \{\varphi(a,\ t) : a \in A\} = \varphi_t(A) = A$, if and only if $t = 0$.

Remark. If X is compact metric space, then $X \in 2^X$. In this case $\varphi_t(X) = X$ for all $t \in \mathbb{R}$.

Example 3.2. Let $(S^1,\ d_1)$ be a metric space where S^1 is the unit circle and d_1 is arc length metric. If $(X,\ d_2)$ is any metric space, then the product $X \times S^1$ is a metric space with the metric D defined by

$$D((x_1,\ y_1),\ (x_2,\ y_2)) = max\{d_2(x_1,\ x_2),\ d_1(y_1,\ y_2)\}.$$

The map $\varphi : (X \times S^1) \times \mathbb{R} \to X \times S^1$ defined by $\varphi((x,\ e^{i\alpha}),\ t) = (x,\ e^{i(\alpha+t)})$, $\alpha \in [0,\ 2\pi), t \in \mathbb{R}$ is a uniform flow with $p = 2\pi$ and $C = 1$.

If we take $A = \{x_0\} \times S^1$, then A is a nonempty compact subset of $X \times S^1$ such that $\varphi_t(A) = A$ for all $t \in \mathbb{R}$.

Example 3.3. The map $\varphi : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$ defined by $\varphi((x,\ y),\ t) = (x + t,\ y)$, $t \in \mathbb{R}$ is a uniform flow with $p = 0$ and $C = 1$.

If A is a nonempty compact subset of $\mathbb{R}^2$, then $\varphi_t(A) = A$ if and only if $t = 0$.

Example 3.4. Let $(S^1,\ d)$ be a metric space where S^1 is the unit circle and d is the arc length metric. If $\mathbb{T}^2 = S^1 \times S^1$, then $\mathbb{T}^2$ is a metric space with the metric D defined by

$$D((e^{2\pi i x_1},\ e^{2\pi i y_1}),\ (e^{2\pi i x_2},\ e^{2\pi i y_2})) = \sqrt{d^2(e^{2\pi i x_1},\ e^{2\pi i x_2}) + d^2(e^{2\pi i y_1},\ e^{2\pi i y_2})},$$

where $x_1, x_2, y_1, y_2 \in [0, 1]$.

Define a map φ on $\mathbb{T}^2$ by

$$\varphi((e^{2\pi i x},\ e^{2\pi i y}),\ t) = (e^{2\pi i(x+t)},\ e^{2\pi i(y+\sqrt{2}t)}).$$

Then
$D((e^{2\pi i x},\ e^{2\pi i y}),\ \varphi((e^{2\pi i x},\ e^{2\pi i y}),\ t)) = D((e^{2\pi i x},\ e^{2\pi i y}),\ (e^{2\pi i(x+t)},\ e^{2\pi i(y+\sqrt{2}t)})) \le C\,|t|$, where $C = 2\sqrt{3}\pi$ and $\varphi((e^{2\pi i x},\ e^{2\pi i y}),\ t) = (e^{2\pi i x},\ e^{2\pi i y})$ if and only if $t = 0$, shows that φ is a uniform flow.

Again the map $\Phi : (\mathbb{T}^2 \times \mathbb{R}) \times \mathbb{R} \longrightarrow \mathbb{T}^2 \times \mathbb{R}$ defined by

$$\Phi(((e^{2\pi ix},\ e^{2\pi iy}),\ a),\ t) = ((e^{2\pi i(x+t)},\ e^{2\pi i(y+\sqrt{2}t)}),\ a),$$

is a uniform flow with respect to the metric $\mathcal{D}$ given by

$$\mathcal{D}(((e^{2\pi ix_1},\ e^{2\pi iy_1}),\ a_1),\ ((e^{2\pi ix_2},\ e^{2\pi iy_2}),\ a_2)) = \max\{D((e^{2\pi ix_1},\ e^{2\pi iy_1}),\ (e^{2\pi ix_2},\ e^{2\pi iy_2})),\ |a_1 - a_2|\}.$$

Consider $A = \{e^{2\pi ix_0}\} \times S^1 \times \{a\}$, the nonempty compact set in $\mathbb{T}^2 \times \mathbb{R}$. We have $\{\Phi((((e^{2\pi ix_0},\ e^{2\pi iy}),\ a),\ 1)) : y \in [0,\ 1]\} = \{((e^{2\pi i(x_0+1)},\ e^{2\pi i(y+\sqrt{2})}),\ a) : y \in [0,\ 1]\} = \{((e^{2\pi ix_0},\ e^{2\pi i(y+\sqrt{2})}),\ a) : y \in [0,\ 1]\} = A$.
Thus, in this case, $\Phi_t(A) = A$ if and only if $t = 0$, is not true.

Theorem 3.5. *Let $(X,\ d)$ be a compact metric space with uniform flow φ satisfying $\varphi_t(A) = A$ if and only if $t \in p\mathbb{Z}$, for all subsets A of X containing atmost n points, then the map $\Phi : F_n(X) \times \mathbb{R} \to F_n(X)$ on the hyperspace of subsets of X containing atmost n points, defined by $\Phi(A,\ t) = \{\varphi(a,\ t) : a \in A\}$ is a uniform flow.*

Proof. Since (X, d) is a compact metric space and $A \in F_n(X)$ is closed in X, hence $A \in F_n(X)$ is a compact set in X. Since continuous image of a compact metric space is compact, we have the result from the proof of Theorem 3.1.

Theorem 3.6.[3] *Any metric space with uniform flow has property $W(strong)$.*

Proof. Let (X, d) be a metric space with uniform flow ϕ.
If $p = 0$, then define $H : X \times [0, 1] \to X$ by

$$H(x, t) = \phi(x, t), \qquad x \in X,\ t \in [0, 1].$$

Then H is continuous and $H(x, t) = x$ if and only if $t = 0$.

If $p > 0$, then $H : X \times [0, 1] \to X$ given by

$$H(x, t) = \phi(x, t/2p), \qquad x \in X,\ t \in [0, 1]$$

is a homotopy and $H(x, t) = x$ if and only if $t = 0$.

Theorem 3.7. *Any metric space with uniform flow has the CIP.*

Proof. It follows from the fact that a metric space having property $W(strong)$ has the CIP.

Theorem 3.8. *If $(X,\ d)$ is a noncompact metric space with uniform flow φ satisfying $\varphi_t(A) = A$ if and only if $t = 0$, for all nonempty compact subsets A of X, then the hyperspace 2^X of all nonempty compact subsets of X has the CIP.*

Proof. Since 2^X is a metric space with respect to the Hausdorff metric. The proof is immediate by Theorem 3.1 and Theorem 3.7.

Theorem 3.9. *If $(X,\ d)$ is a compact metric space with uniform flow φ satisfying $\varphi_t(A) = A$ if and only if $t \in p\mathbb{Z}$, for all subsets A of X containing atmost n points, then the hyperspace $F_n(X)$ of subsets of X containing atmost n points has the CIPH.*

Proof. Since for a compact metric space X, the hyperspace $F_n(X)$ is compact. From the Theorem 3.5 we get that $F_n(X)$ has uniform flow. Thus $F_n(X)$ is a compact metric space with uniform flow and hence the result follows from Theorem 2.10.

Theorem 3.10 *For a nondegenerate Peano continuum X, the hyperspace 2^X of nonempty compact subsets of X and the hyperspace $C(X)$ of nonempty compact connected subsets of X with no free arcs have the CIPH.*

Proof. On account of Result 2.3, 2^X and $C(X)$ are homeomorphic to the Hilbert cube $\mathcal{Q}$. Thus from 2.9 we get the result.

Example 3.11 The closed unit interval I, the Hilbert cube $\mathcal{Q}$ and closed unit sphere S^n are nondegenerate Peano continuum. Thus from Theorem 3.10 the hyperspaecs 2^I, $2^{\mathcal{Q}}$ and 2^{S^n} have the CIPH.

Example 3.12 For the closed unit interval I and the unit circle S^1, hyperspaecs $C(I)$ and $C(S^1)$ are homeomorphic to the product I^2. Hence $C(I)$ and $C(S^1)$ have the CIPH.

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