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EXAMPLES ON IRRESOLVABILITY

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Abstract. We construct several examples of Hausdorff (resp. regular) open-hereditarily irresolvable not hereditarily irresolvable or hereditarily irresolvable not submaximal spaces. Also, examples of separable or countable (connected or not) irresolvable spaces are constructed.

1 Introduction

The concepts of maximal, submaximal and irresolvable spaces were introduced by E. Hewitt in [12], while the concept of open-hereditarily irresolvable space was introduced by E. K. van Douwen in [27], and the concept of maximal connected space was introduced by J. P. Thomas in [25]. These properties have been widely studied in the last sixty years.

In the sequel all spaces are considered to be crowded (without isolated points).

Definition 1.1. A space X is called:

1. Resolvable ([12]) if X has two disjoint dense subsets, and it is called irresolvable ([12]) if it is not resolvable.
2. Open-hereditarily irresolvable ([27]), if every open subspace of X is irresolvable.
3. Hereditarily irresolvable ([12]), if every subspace of X is irresolvable.

Irresolvable spaces have been also studied by K. Kunen, A. Szymański and F. Tall in [17], by J. Dontchev, M. Ganster and D. Rose in [8], by O. T. Alas, M. Sanchis, M. G. Tkačenko, V. V. Tkachuk and R. G. Wilson in [1] and by W.W. Comfort and S. Garcia-Ferreira in [7] where a number of relevant references is provided, as well as a number of interesting open problems is listed.

Definition 1.2. A space (X, τ) is called:

1. Submaximal ([12]), if every dense subset of X is open.
2. Maximal connected ([25]), if every finer topology than τ is not connected.
3. Maximal Hausdorff ([12]), if τ is maximal in the set of Hausdorff crowded topologies on X.
4. Maximal regular ([12]), if τ is maximal in the set of regular crowded topologies on X.
5. Extremally disconnected, if the closure of every open subset is open.

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Submaximal spaces as well as maximal topologies are studied in detail by N. Bourbaki [4], D. E. Cameron [5], E. K. van Douwen [27], A. V. Arhangel'skii and P. J. Collins [2], R. Levy and J. R. Porter [19] .

It is known that all maximal Hausdorff spaces are submaximal ([13], [15]) and that all maximal connected spaces are also submaximal ([6]). Since every subspace of a submaximal is submaximal ([4]) and every connected subspace of a maximal connected is maximal connected ([10]) it follows that every submaximal (connected or not) is hereditarily irresolvable, hence open-hereditarily irresolvable and hence irresolvable.

The examples constructed by D. Rose, K. Sizemore and B. Thurston in [23], by G. Bezhanishvili, R. Mines and P. J. Morandi in [3] and by E. K. van Douwen in [27] prove that none of the previous implications is reversible. We note that the Example 1.12 in [27] is a regular disconnected (or totally disconnected) space and the Example 1.9. in [27] is a regular extremally disconnected space.

In this paper we prove that every Hausdorff (resp. regular) space S can be embedded as a closed nowhere dense subset in a open-hereditarily irresolvable Hausdorff (resp. regular) space T. The space T is obtained by attaching to S an auxiliary space Z which is the cone constructed from a space X. Since the properties of the final space T depend on the properties of S and Z, it follows that the attachment of Z to S leads to several examples of all kinds of irresolvability.

We note that using spaces with appropriate properties, either for S or for Z, the attachment presented in this paper can expand the known examples of different kinds of irresolvability so that the final spaces become in addition connected. In Remarks 3.2 and 3.3 we present several relevant examples. Moreover, by weakening the topology of the space Z, the space Z itself leads to several examples of connected spaces on irresolvability.

2 The auxiliary space Z.

In the sequel we will use the following Lemma 2.1 and Lemma 2.2 whose statements are well known.

Lemma 2.1.

(1) *Let (X, τ) be a Hausdorff space. The set of all topologies finer than τ having the same regular-open sets as τ, has a regular-open maximal topology which is Hausdorff submaximal.*

(2) *If (X, τ) is Hausdorff connected (resp. countable connected), then the regular-open maximal topology is Hausdorff connected (resp. countable connected) submaximal.*

Proof. (1) This is proved in [20].
(2) It follows from the fact that the two topologies have the same regular-open sets. □

Lemma 2.2.

(1) *Every Hausdorff and maximal connected is submaximal.*
(2) *Every maximal Hausdorff is submaximal.*
(3) *Every Hausdorff submaximal is hereditarily irresolvable.*
(4) *Every maximal regular is hereditarily irresolvable.*

Proof. (1) By [6] a maximal connected space is submaximal.
(2) By [4] (Exercise 21 of §11) a Hausdorff space X is maximal Hausdorff if and only if X is submaximal and extremally disconnected.
(3) Obviously every submaximal is hereditarily irresolvable. Since by [4] (Exercise 22 of §8) every subspace of a submaximal space is submaximal, it follows that it is hereditarily irresolvable.
(4) Let X be maximal regular. If D is dense in X, then by [4] (Exercise 21 of §11) the subset $IntD$ is open-dense. Hence X is irresolvable. If a subspace of X contains isolated

points, then obviously it is irresolvable. If a subspace is crowded then by [27] it is maximal regular and hence irresolvable. □

We now consider the cone Z constructed from an arbitrary topological space X. The space Z will be used in the sequel as auxiliary space attached to a space S with specific properties. The final space T gives several examples of irresolvable spaces. For the cone and its applications see J. K. Kohli [16], S. Watson [28] and J. R. Porter [22].

Let X be a topological space and let $X_i, i \in I$, be pairwise disjoint homeomorphic copies of X. We fix a point $x \in X$ and let x_i be the copy of x in X_i for every $i \in I$. We set $Y = X \setminus \{x\}$ and $Y_i = X_i \setminus \{x_i\}, i \in I$. We identify the points x_i, for every $i \in I$ and we denote this common point by z.

The cone constructed from X is the set $Z = \{z\} \cup (\cup_{i\in I} Y_i)$ with the following topology: Each copy Y_i keeps the subspace topology of the space X_i, that is, for every $y_i \in Y_i, i \in I$ a basis of open neighborhoods of y_i in Z is the (homeomorphic) copy of a basis of open neighborhoods of y in Y whose copy in Y_i is y_i. For the point z, a basis of open neighborhoods in Z consists of the subsets $O_z = \{z\} \cup W$, where for every $i \in I$ the set $W \cap Y_i$ is an open deleted neighborhood of x_i in X_i, that is the set $W \cap Y_i$ is the (homeomorphic) copy in Y_i of a deleted open neighborhood of x in X.

Lemma 2.3.

(1) *If X is Hausdorff (resp. regular), then Z is Hausdorff (resp. regular).*

(2) *If X is submaximal, then Z is submaximal.*

(3) *If X is countable submaximal and the index set I is countable, then Z is countable submaximal.*

(4) *If X is maximal Hausdorff, then Z is Hausdorff submaximal not extremally disconnected.*

(5) *If X is countable maximal Hausdorff and the index set I is countable, then Z is countable Hausdorff submaximal not extremally disconnected.*

(6) *If X is maximal regular, then Z is regular hereditarily irresolvable not extremally disconnected.*

(7) *If X is countable maximal regular and the index set I is countable, then Z is countable regular hereditarily irresolvable not extremally disconnected.*

(8) *If X is separable submaximal and the index set I is countable, then Z is separable submaximal.*

(9) *If X is connected submaximal, then Z is connected submaximal.*

(10) *If X is separable connected submaximal and the index set I is countable, then Z is separable connected submaximal.*

(11) *If X is countable connected submaximal and the index set I is countable, then Z is countable connected submaximal.*

Proof. (1) Let X be Hausdorff and $a, b \in Z \setminus \{z\}$. If both a, b belong to the same copy Y_i for some $i \in I$, then since Y_i is Hausdorff there exist in Y_i disjoint open neighborhoods U_a, U_b of a, b respectively. If $a \in Y_i, b \in Y_j, i \neq j$ then the subspaces Y_i, Y_j are disjoint open in Z containing a, b respectively. Let $a \in Z \setminus \{z\}$ and $b = z$. Then $a \in Y_i$ for some $i \in I$. Since X_i is Hausdorff it follows that for the points a, x_i there exist in X_i open neighborhoods U_a, U_{x_i} of a, x_i respectively such that $U_a \cap U_{x_i} = \emptyset$. Therefore the sets U_a and $\{z\} \cup W$ where $W \cap Y_i = W_{x_i} \setminus \{x_i\}$ are disjoint open sets in Z containing a, z respectively. Thus, Z is Hausdorff.

Let X be regular. Obviously, the space Z is regular at every point $y_i \in Y_i$, for every $i \in I$. For the point z, let $O_z = \{z\} \cup W$ be an open neighborhood of $z \in Z$. By the definition of topology in Z the set $W_{x_i} = W \cap Y_i$ is an open neighborhood of $x_i \in X_i$.

Since for every $i \in I$ the space X_i is regular, it follows that for every W_{x_i} there exist an open neighborhood O_{x_i} of x_i in X_i such that $Cl_{X_i}(O_i \setminus \{x_i\}) \subseteq W_{x_i}$. Therefore the set $O_z = \{z\} \cup O$ such that $O \cap Y_i = O_{x_i}$ is an open neighborhood of the point $z \in Z$ such that $Cl_Z(\{z\} \cup O) \subseteq \{z\} \cup W$ that is, Z is regular at z.
(2) Let D be dense in Z, and $z \in D$. Since for every $i \in I$ the subset $D \cap Y_i$ is open-dense in Y_i, it follows that for the point z in the subspace $\{z\} \cup Y_i$, there exists an open set $U_{i(D)}$ (depended on $D \cap Y_i$) containing z and such that $U_{i(D)} \setminus \{z\} \subseteq D \cap Y_i$. Therefore the set $\{z\} \cup W$ for which $W \cap Y_i = U_{i(D)} \setminus \{z\}$, $\forall i \in I$, is an open set in Z containing z and included in D. That is, the point z is an interior point of D. Therefore D is open.
(4) By Lemma 2.2 (2), X is submaximal. Hence Z is submaximal. Since for the open subset Y_i of Z it holds that $Cl_Z Y_i = \{z\} \cup Y_i$, it follows that Z is not extremally disconnected.
(6) Let A be a subspace of Z. By Lemma 2.2 (4), X is hereditarily irresolvable. Since for every $i \in I$ the subspace $\{z\} \cup Y_i$ is homeomorphic to X it follows that A is a disjoint union of hereditarily irresolvable subspaces. Hence, A is irresolvable. That Z is not extremally disconnected is proved as previously.

The remaining statements (3), (5), (7), (8), (9), (10), (11) are obvious. □

3 The space T. In [27] E. K. van Douwen constructs two maximal regular spaces. The first (Example 1.9) is not maximal Hausdorff while the second (Example 3.3) is countable and maximal Hausdorff. In [19] R. Levy and J. R. Porter construct uncountable Hausdorff (and Tychonoff) submaximal separable spaces. The first example of a connected submaximal Hausdorff space is constructed by K. Padmavally [21]. Maximal connected Hausdorff spaces are constructed by A. G. El'kin [9], J. A. Guthrie, H. E. Stone and M. L. Wage [11], and G. J. Kennedy and S. D. McCartan [14]. For countable connected Hausdorff spaces see the list of references in [26].

Since a submaximal space is hereditarily irresolvable (Lemma 2.2 (3)), it follows that in all cases the initial space X is a hereditarily irresolvable space, implying that the space Z is also hereditarily irresolvable. Therefore, as initial space X it can be used any submaximal space of Lemma 2.1 or any hereditarily irresolvable space of Lemma 2.2, as well as any of the previous specific spaces.

Proposition 3.1. *Every Hausdorff (resp. regular) space S can be embedded as a closed nowhere dense subset in a open-hereditarily irresolvable Hausdorff (resp. regular) space T. If in addition S is separable, then T is separable.*

Proof. Let S be a Hausdorff (resp. regular) space. We consider a hereditarily irresolvable Hausdorff (resp. regular) space X and we construct the space Z, the index set I having the same cardinality as the set S. In the space X we fix a point $a \neq x$ and let a_i be the copy of a in Y_i. Hence, for every $i \in I$, $a_i \neq x_i$ and therefore $a_i \neq z$, because by the construction of the space Z the point z is defined by identifying the points x_i. We attach the space Z to the space S identifying every point of S with a point a_i of Z.

On the set $T = S \cup (Z \setminus \{a_i : i \in I\})$ we define the following topology: The subset $Z \setminus \{a_i : i \in I\}$ keeps the subspace topology of Z. For every open subset U of S the subset O_U of T is open in T if and only if $O_U = U \cup W(U)$ where $W(U) = \bigcup_{a_i \in U} U_{a_i}$, and U_{a_i} is an open deleted neighborhood of a_i in Y_i. It can be easily verified that this is a topology observing that if U, V are open sets in S, then for the sets O_U, O_V it holds that $O_U \cap O_V = (U \cap V) \cup W(U \cap V) = O_{U \cap V}$. Also, if $U_i, i \in I$ are open sets in S then for the sets O_{U_i} it holds that $\cup O_{U_i} = (\cup U_i) \cup W(\cup U_i) = O_{\cup U_i}$ because $\cup W(U_i) = \cup(\bigcup_{a_i \in U_i} U_{a_i}) = W(\cup U_i)$.

We prove that T is Hausdorff. Let $x, y \in T$. If $x, y \in Z \setminus S$ then the points x, y either belong to a common $Y_i \setminus \{a_i\}$ for some $i \in I$ or $x \in Y_i \setminus \{a_i\}$ and $y \in Y_j \setminus \{a_j\}$, $i \neq j$, or $x \in Y_i \setminus \{a_i\}$ and $y = z$. The proof of these cases is the same as in Lemma 2.3 (1). Let

$x, y \in S$. Since S is Hausdorff, there exist open subsets U, V in S containing the points x, y respectively and such that $U \cap V = \emptyset$. Since for every $i \in I$ all copies Y_i are pairwise disjoint it is obvious that we can choose $W(U)$ and $W(V)$ such that $W(U) \cap W(V) = \emptyset$. Hence the corresponding sets O_U, O_V are the required open subsets.

Let $x \in T \setminus S$, $x \neq z$ and $s \in S$. Then $x \in Y_i$, for some $i \in I$. If Y_i is attached to s' and $s' \neq s$ then the proof is as in the previous case. If Y_i is attached to s, that is $a_i = s$, then, since Y_i is Hausdorff, it follows that there exist open sets W_{a_i} and V_x in Y_i containing s and x respectively, and such that $W_{a_i} \cap V_x = \emptyset$. Hence if U is an open set in S containing s, then the corresponding set $O_U = U \cup W(U)$ for which $W(U) \cap Y_i = W_{a_i}$ and the set V_x are the required open subsets.

It remains the case for the point z and a point $s \in S$. Let U be an open set in S containing the point s. Let Y_j, $j \in I' \subset I$ be the copies whose points a_j are attached to U. For the point z and for every a_j there exist open sets W_j, W_{a_j} in the subspace $\{z\} \cup Y_j$ containing z and a_j respectively, and such that $W_j \cap W_{a_j} = \emptyset$. Hence the subset $O_z = \{z\} \cup W$ for which $W \cap Y_j = W_j$ and the subset $O_U = U \cup W(U)$ for which $W(U) \cap Y_j = U_{a_j}$ are the required open subsets.

We now prove that T is regular. By the definition of the topology on T, it follows that T is regular at every point of $T \setminus S$. Since S is regular, for every $s \in S$ there exist open sets U, V in S such that

$$s \in V \subseteq Cl_S V \subseteq U.$$

We consider $W(Cl_S V)$, that is the subset of $W(U)$ for which $W(U) \cap Y_i$ is an open set in Y_i containing those a_i which are attached to the points of $Cl_S V$. Since each Y_i is regular, then for every such open set there exists an open set W_{a_i} in Y_i containing a_i and such that

$$W_{a_i} \subseteq Cl_{Y_i} W_{a_i} \subseteq W \cap Y_i.$$

Hence,

$$\bigcup_{a_i \in V} W_{a_i} \subseteq \bigcup_{a_i \in Cl_T V} W_{a_i} \subseteq \bigcup_{a_i \in Cl_T V} Cl_{Y_i} W_{a_i} \subseteq W,$$

and therefore

$$s \in O_V \subseteq Cl_T O_V \subseteq O_U,$$

that is, T is regular. Obviously, the subset $T \setminus S$ is open. Since for every open set U in S it holds that $O_U \cap (T \setminus S) \neq \emptyset$, it follows that $T \setminus S$ is also dense. Hence S is closed nowhere dense in T.

It remains to prove that T is open-hereditarily irresolvable. Let U be an open subspace of T. If U is a subset of $T \setminus S$ then, since Z is hereditarily irresolvable, is follows that U is an irresolvable subspace of T. If the open set is of the form $O_U = U \cup W(U)$ then, by the definition of O_U the subset U is open in S and nowhere dense in T and the subset $W(U)$ is an open subset of T. Hence $W(U)$ is irresolvable and therefore O_U is irresolvable. Hence T is open-hereditarily irresolvable.

Finally, let S be separable. Let D be a countable dense subset of S, and Z be as in (8) of Lemma 2.3. We attach Z to S, identifying every point of D with a point a_i, $i = 1, 2, \dots$ of Z. The topology on the set $T = S \cup (Z \setminus \{a_i : i \in \mathbb{N}\})$ is defined in exactly the same manner as above. Obviously T is separable. That T is open-hereditarily irresolvable is proved as previously. □

The following remarks are consequences of Lemma 2.3 and the previous Proposition, indicating that none of the following implications
" submaximal ⇒ hereditarily irresolvable ⇒ open-hereditarily irresolvable ⇒ irresolvable "

is reversible. We must mention that the two examples constructed by D. Rose, K. Sizemore and B. Thurston in [23] (Examples 2.5), the Example 3.2 constructed by G. Bezhanishvili, R. Mines and P. J. Morandi in [3] and the Examples 1.12 and 1.9 constructed by E. K. van Douwen in [27] give an answer to this. Specifically, the first example in [23] is a crowded T_1 hereditarily irresolvable not submaximal and the second is a crowded T_1 open-hereditarily irresolvable not hereditarily irresolvable. The example in [3] is a connected crowded T_1 irresolvable not open-hereditarily irresolvable. Obviously, the space X in this example is Hausdorff (resp. regular) if both spaces Y, Z used for the construction of X are Hausdorff (resp. regular). We observe that in order to be connected it is needed both Y, Z to be connected. The Example 1.12 in [27] is a regular disconnected (or totally disconnected) open-hereditarely irresolvable but not hereditarely irresolvable space. The Example 1.9. in [27] is maximal regular but not maximal Hausdorff. Hence by [4] (Exercise 21 of §11) it is extremally disconnected and by Lemma 2.2 (4) it is hereditarily irresolvable. Since a Hausdorff space is maximal Hausdorff if and only if it is extremally disconnected and submaximal, it follows that this space is not submaximal.

Remark 3.2 below is referred to open-hereditarily irresolvable not hereditarily irresolvable spaces. Specifically, the cases (3), (4), and part of (5) deal with connected spaces. Remark 3.3 is referred to hereditarily irresolvable not submaximal spaces, some of which are also connected. The final space T is, in all cases, Hausdorff (resp. regular) if both S, Z are Hausdorff (resp. regular). Using Lemma 2.3 and Proposition 3.1 we can expand the different kinds of irresolvability so that the spaces become in addition connected. In Remark 3.4 we examine whether the set of all dense subsets in these spaces, is a filter. In what follows $\mathcal{D}(X)$ denotes the set of all dense subsets of X.

Remark 3.2. (1) If S is resolvable and Z is as in (2), (4) or (6) of Lemma 2.3, then T is open-hereditarily irresolvable but not hereditarily irresolvable.

(2) If S is separable resolvable and Z is as in (8) of Lemma 2.3 , then T is in addition separable .

(3) If S is a resolvable not necessarily connected space and Z is as in (9) of Lemma 2.3 , then T is in addition connected. If S is separable (resp. countable) resolvable not necessarily connected space and Z is as in (10) (resp. (11)) of Lemma 2.3, then T is in addition separable (resp. countable) connected. The space T is connected, either if Z is attached to the whole of S or to a countable dense subset D of S, because the subset $D \cup (Z \setminus \{a_i : i \in \mathbb{N}\})$ is dense connected and therefore $Cl_T(D \cup (Z \setminus \{a_i : i \in \mathbb{N}\})) = T$ is connected.

(4) If S is countable resolvable (not necessarily connected) and Z is as in (11) of Lemma 2.3 , then T is countable connected open-hereditarily irresolvable not hereditarily irresolvable.

(5) Consider [18] (Chapter I, §9) the set of rational numbers of the interval $[0, 1]$, written as irreducible fractions $\frac{p}{q}$. We set $D = \{(\frac{p}{q}, \frac{1}{q}) : p, q \in \mathbb{N}\}$. The subspace $S = D \cup [0, 1]$ (resp. $S = D \cup (\mathbb{Q} \cap [0, 1])$) of the plane is regular and the subset D of isolated points is countable and dense. Obviously the subspace $[0, 1]$ (resp. $\mathbb{Q} \cap [0, 1]$) is resolvable (resp. countable resolvable). Hence, by Proposition 3.1 , the attachment of any hereditarily irresolvable space Z of Lemma 2.3 to the subspace D of S leads to a space T being in all cases open-hereditarily irresolvable but not hereditarily irresolvable.

Specifically, if Z is as in (9), (10) or (11) of Lemma 2.3 , then T is in addition connected, separable connected or countable connected (if $S = D \cup (\mathbb{Q} \cap [0, 1])$), respectively.

We observe that in all the previous cases the space T is not submaximal since the closed nowhere dense subset S of T is not discrete.

We note that in all cases $\mathcal{D}(T)$ is a filter (as it was expected, see Remark 3.4) because if

L, M are dense subsets of T then since the subset $T \setminus S$ is open-dense it follows that both subsets $L \cap (T \setminus S)$ and $M \cap (T \setminus S)$ are dense in T. Since $T \setminus S$ is hereditarily irresolvable it follows that both subsets $Int_T(L \cap (T \setminus S))$ and $Int_T(M \cap (T \setminus S))$ are open-dense. Therefore the set $L \cap M$ is dense.

This construction is actually based on the construction of Example 1.12 in [27], with the following modification: instead of attaching to the set D disjoint copies of spaces, we attach to D the space Z of Lemma 2.3. We note that attaching disjoint copies of spaces to D the final space is not connected even if all copies are connected.

Remark 3.3. Let S, Z be any hereditarily irresolvable spaces. We attach the space Z to S as in Proposition 3.1. The space T is always hereditarily irresolvable not submaximal (even if both spaces S, Z are submaximal). In order to prove this, we consider an open-dense subset D of Z, not containing anyone of the points a_i which are attached to S. Since S is closed nowhere dense in T, it follows that D is open-dense in T. Hence, if $s \in S$ then the set $D \cup \{s\}$ is dense in T but not open. It is obvious that if Z is as in (9), (10) or (11) of Lemma 2.3, then T is in addition connected. Specifically, if S is separable and Z as in (10) then T is separable connected. If S is countable and Z as in (11) then T is countable connected.

Hereditarily irresolvable not submaximal spaces can also be constructed as follows: Let (Z, τ) be any submaximal space of Lemma 2.3. We weaken the topology on (Z, τ) changing the topology only at the point z as follows: The subset O_z is open in Z containing z if and only if $O_z = \{z\} \cup W$, where for every finite subset $I' \subset I$ and for every $i \in I'$, the subset $W \cap Y_i$ is an open deleted neighborhood of x_i in X_i while $W \cap Y_i = Y_i$, $\forall i \in I \setminus I'$. We denote this topology by τ^*. It can be easily proved that (Z, τ^*) remains Hausdorff (resp. regular) if (Z, τ) is Hausdorff (resp. regular). The proof that in all cases (Z, τ^*) is hereditarily irresolvable is the same as in (6) of Lemma 2.3.

We prove that (Z, τ^*) is not submaximal. Let D be a proper dense subset of X. We set $D_i = D \cap Y_i$. Since each D_i is open-dense in Y_i it follows that $\bigcup_{i \in I} D_i$ is open-dense in Z. But $\{z\} \cup (\bigcup_{i \in I} D_i)$ is not open because every open neighborhood of z contains all but finite copies of Y_i. We observe that again the set $\mathcal{D}(Z)$ is a filter on (Z, τ^*).

Obviously, if (Z, τ) is as in (9), (10) or (11) of Lemma 2.3, hence connected, then since $\tau^* \subset \tau$ it follows that (Z, τ^*) is in addition connected, separable connected or countable connected, respectively.

Remark 3.4. The Example 3.2 in [3] has the additional property that on the space X the set $\mathcal{D}(X)$ is not a filter. This can occur only to irresolvable not open-hereditarily irresolvable spaces. For if X is submaximal then every dense subset is open and hence $\mathcal{D}(X)$ is a filter. If X is hereditarily irresolvable then $\mathcal{D}(X)$ is a filter ([3] , Theorem 2.4). Finally, if X is open-hereditarily irresolvable then $\mathcal{D}(X)$ is again a filter. In order to prove this it suffices to prove that for every dense subset D of X it holds that the set $IntD$ is open-dense. Obviously $IntD \neq \emptyset$. If $IntD$ is not dense then there exists an open set U such that $U \cap IntD = \emptyset$. Obviously the sets $D \cap U$ and $U \setminus D$ are non-empty, dense in U, that is the open set U is resolvable, which is a contradiction. Examples of irresolvable Hausdorff (resp. regular) spaces with additional properties and on which the set of all dense subsets is not a filter can be also constructed (using the space Z of Lemma 2.2 and imitating the wedge construction of Example 3.2 in [3]).

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