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ON DEGREE OF NON-CONVEXITY OF FUZZY SETS

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Abstract. In the present paper, the convexity of fuzzy sets is generalized based on conjunctive aggregation functions, and the degree of the non-convexity of fuzzy sets is considered as an application of the generalized convexity. Then, the properties of the generalized convexity of fuzzy sets, and the properties of the degree of the non-convexity of fuzzy sets with respect to operations are investigated.

1. Introduction. The concept of fuzzy sets has been primarily introduced for representing sets containing uncertainty or vagueness by Zadeh [7] as fuzzy set theory. Then, fuzzy set theory has been applied in various areas of decision making theory including economics and optimization, etc., widely. We consider fuzzy sets on $\mathbb{R}^n$, and identify each fuzzy set on $\mathbb{R}^n$ with its membership function. The convexity of a fuzzy set is defined by the quasiconcavity of its membership function. Quasiconcavity of functions is defined using the minimum operation. Due to the importance in economics and optimization, etc., several generalizations of quasiconcavity of functions have been introduced and investigated; see [6] and the references therein. In [3], the quasiconcavity of membership functions is generalized by allowing arbitrary conjunctive aggregation functions instead of the minimum operation. In [5], the degree of the non-quasiconcavity of membership functions is proposed as an application of the generalized quasiconcavity. Since the convexity of a fuzzy set is defined by the quasiconcavity of its membership function, the generalized quasiconcavity of membership functions can be regarded as the generalized convexity of fuzzy sets, and the degree of the non-quasiconcavity of membership functions can be regarded as the non-convexity of fuzzy sets.

In the present paper, the properties of the generalized convexity of fuzzy sets, and the properties of the degree of the non-convexity of fuzzy sets with respect to operations are investigated.

The remainder of the present paper is organized as follows. In Section 2, some properties of continuous conjunctive aggregation functions are presented. In Section 3, some properties of fuzzy sets with respect to operations are presented. In Section 4, some properties of the generalized convexity of fuzzy sets, and some properties of the degree of the non-convexity of fuzzy sets with respect to operations are presented. Finally, conclusions are presented in Section 5.

2. Aggregation functions In this section, the properties of continuous conjunctive aggregation functions are investigated. Conjunctive aggregation functions are used in order to generalize the convexity of fuzzy sets. For details of aggregation functions, see [1,6].

For $a, b \in \mathbb{R} \cup \{-\infty, \infty\}$, we set $[a, b] = \{x \in \mathbb{R} : a \le x \le b\}, [a, b[= \{x \in \mathbb{R} : a \le x < b\},]a, b] = \{x \in \mathbb{R} : a < x \le b\}$, and $]a, b[= \{x \in \mathbb{R} : a < x < b\}$.

First, aggregation functions are defined.

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Definition 1. (See [1].) Let $G : [0,1]^2 \to [0,1]$. The function G is called an aggregation function if the following two axioms are satisfied: (G1) if $x_i, y_i \in [0,1], x_i \le y_i, i = 1,2$, then $G(x_1,x_2) \le G(y_1,y_2)$ (monotonicity), and (G2) $G(0,0) = 0$ and $G(1,1) = 1$ (boundary condition).

Next, the definition of a property of aggregation functions is given.

Definition 2. (See [1].) Let $G : [0,1]^2 \to [0,1]$ be an aggregation function. The aggregation function G is said to be conjunctive if $G(x,y) \le \min\{x,y\}$ for any $x,y \in [0,1]$.

Next, the definition of a relationship between two aggregation functions is given.

Definition 3. (See [4].) Let $G, G' : [0,1]^2 \to [0,1]$ be aggregation functions. G is said to dominate G' ($G \gg G'$) if $G(G'(x_1,y_1),\ G'(x_2,y_2)) \ge G'(G(x_1,\ x_2),\ G(y_1,\ y_2))$ for any $x_i, y_i \in [0,1], i = 1,2$.

The concept of the domination for aggregation functions is closely related to the preservation of T-transitivity in aggregating fuzzy relations, where T is a triangular norm; see [4].

In order to measure the non-convexity of fuzzy sets in Section 4, we consider continuous conjunctive aggregation functions $G^{(p)} : [0,1]^2 \to [0,1], p \in [1,\infty[$ defined as

$$G^{(p)}(x,y) = [\min\{x,y\}]^p \quad \text{for } x,y \in [0,1] \tag{1}$$

for each $p \in [1,\infty[$. The larger p is, the larger the difference between $G^{(p)}$ and $\min = G^{(1)}$ is.

The following proposition shows a property of continuous conjunctive aggregation function.

Proposition 1. Let $G : [0,1]^2 \to [0,1]$ be a continuous conjunctive aggregation function, and let $A, B \subset [0,1]$. Then,

$$\sup_{x\in A, y\in B} G(x,y) = G(\sup A, \sup B),$$

where $\sup \emptyset = 0$ for $\emptyset \subset [0,1]$.

Proof. If $A = \emptyset$ or $B = \emptyset$, then

$$\sup_{x\in A, y\in B} G(x,y) = 0 = G(\sup A, \sup B).$$

Assume that $A \ne \emptyset$ and $B \ne \emptyset$. Since $G(x,y) \le G(\sup A, \sup B)$ for any $x \in A$ and any $y \in B$ by the monotonicity of G, we have

$$\sup_{x\in A, y\in B} G(x,y) \le G(\sup A, \sup B).$$

Suppose that

$$\sup_{x\in A, y\in B} G(x,y) < G(\sup A, \sup B).$$

We set $\alpha = \sup A$ and $\beta = \sup B$. Note that $\alpha > 0$ and $\beta > 0$. We set

$$\varepsilon = \frac{G(\alpha,\beta) - \sup\limits_{x\in A, y\in B} G(x,y)}{2} > 0.$$

By the monotonicity of G and the continuity of G at (α, β), there exists δ such that $0 < \delta < \min\{\alpha, \beta\}$ and $G(\alpha, \beta) - \varepsilon < G(x, y) \leq G(\alpha, \beta)$ for any $(x, y) \in]\alpha - \delta, \alpha] \times]\beta - \delta, \beta]$. By the definitions of α and β, there exists $(x_0, y_0) \in (]\alpha - \delta, \alpha] \times]\beta - \delta, \beta]) \cap (A \times B)$. Then, it follows that

$$G(x_0, y_0) \leq \sup_{x \in A, y \in B} G(x, y) < G(\alpha, \beta) - \varepsilon < G(x_0, y_0),$$

which is a contradiction. □

The following proposition shows a relationship between $\min = G^{(1)}$ and $G^{(p)}, p \in [1, \infty[$, where $G^{(p)}, p \in [1, \infty[$ are the continuous conjunctive aggregation functions defined by (1).

Proposition 2. $\min = G^{(1)} \gg G^{(p)}$ for any $p \in [1, \infty[$.

Proof. Let $x_i, y_i \in [0, 1], i = 1, 2$, and let z be the minimum among $x_i, y_i, i = 1, 2$. Then, we have

$$\min\{G^{(p)}(x_1, y_1), G^{(p)}(x_2, y_2)\} = z^p = G^{(p)}(\min\{x_1, x_2\}, \min\{y_1, y_2\}).$$

□

3. Fundamental properties of fuzzy sets In this section, the properties of fuzzy sets with respect to operations are investigated.

We consider fuzzy sets on $\mathbb{R}^n$, and identify a fuzzy set $\widetilde{a}$ on $\mathbb{R}^n$ with its membership function $\widetilde{a} : \mathbb{R}^n \to [0, 1]$. Let $\mathcal{F}(\mathbb{R}^n)$ be the set of all fuzzy sets on $\mathbb{R}^n$.

For $\widetilde{a} \in \mathcal{F}(\mathbb{R}^n)$ and $\alpha \in]0, 1]$, the set

$$[\widetilde{a}]_\alpha = \{\boldsymbol{x} \in \mathbb{R}^n : \widetilde{a}(\boldsymbol{x}) \geq \alpha\}$$

is called the α-level set of $\widetilde{a}$.

For a crisp set $S \subset \mathbb{R}^n$, the function $c_S : \mathbb{R}^n \to \{0, 1\}$ defined as

$$c_S(\boldsymbol{x}) = \begin{cases} 1 & \text{if } \boldsymbol{x} \in S, \\ 0 & \text{if } \boldsymbol{x} \notin S \end{cases}$$

for each $\boldsymbol{x} \in \mathbb{R}^n$ is called the indicator function of S. Whenever we consider c_S as a fuzzy set, $c_S : \mathbb{R}^n \to \{0, 1\}$ is interpreted as $c_S : \mathbb{R}^n \to [0, 1]$.

A fuzzy set $\widetilde{a} \in \mathcal{F}(\mathbb{R}^n)$ can be represented as

$$\widetilde{a} = \sup_{\alpha \in]0,1]} \alpha c_{[\widetilde{a}]_\alpha}, \tag{2}$$

which is well-known as the decomposition theorem; see [2].

A fuzzy set $\widetilde{a} \in \mathcal{F}(\mathbb{R}^n)$ is said to be convex if $\widetilde{a}(\lambda \boldsymbol{x} + (1 - \lambda)\boldsymbol{y}) \geq \min\{\widetilde{a}(\boldsymbol{x}), \widetilde{a}(\boldsymbol{y})\}$ for any $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ and any $\lambda \in]0, 1[$. That is, $\widetilde{a} \in \mathcal{F}(\mathbb{R}^n)$ is said to be convex if $\widetilde{a}$ is a quasiconcave function.

We set

$$\mathcal{S}(\mathbb{R}^n) = \{\{S_\alpha\}_{\alpha \in]0,1]} : S_\alpha \subset \mathbb{R}^n, \alpha \in]0, 1], \text{ and } S_\beta \supset S_\gamma \text{ for } \beta, \gamma \in]0, 1] \text{ with } \beta < \gamma\},$$

and define $M : \mathcal{S}(\mathbb{R}^n) \to \mathcal{F}(\mathbb{R}^n)$ as

$$M(\{S_\alpha\}_{\alpha \in]0,1]}) = \sup_{\alpha \in]0,1]} \alpha c_{S_\alpha}$$

for each $\{S_\alpha\}_{\alpha\in]0,1]} \in \mathcal{S}(\mathbb{R}^n)$. For $\{S_\alpha\}_{\alpha\in]0,1]} \in \mathcal{S}(\mathbb{R}^n)$ and $\boldsymbol{x} \in \mathbb{R}^n$, it follows that

$$M(\{S_\alpha\}_{\alpha\in]0,1]})(\boldsymbol{x}) = \sup_{\alpha\in]0,1]} \alpha c_{S_\alpha}(\boldsymbol{x}) = \sup\{\alpha \in]0,1] : \boldsymbol{x} \in S_\alpha\},$$

where $\sup\emptyset = 0$ for $\emptyset \subset]0,1]$. The decomposition theorem (2) can be represented as

$$\widetilde{a} = M(\{[\widetilde{a}]_\alpha\}_{\alpha\in]0,1]})$$

for $\widetilde{a} \in \mathcal{F}(\mathbb{R}^n)$.

When $\widetilde{a} = M(\{S_\alpha\}_{\alpha\in]0,1]})$ for $\widetilde{a} \in \mathcal{F}(\mathbb{R}^n)$ and $\{S_\alpha\}_{\alpha\in]0,1]} \in \mathcal{S}(\mathbb{R}^n)$, $\widetilde{a}$ is called the fuzzy set generated by $\{S_\alpha\}_{\alpha\in]0,1]}$, and $\{S_\alpha\}_{\alpha\in]0,1]}$ is called the generator of $\widetilde{a}$.

The following proposition shows a relationship between the inclusion relation of two generators of two fuzzy sets and the inclusion relation of the two fuzzy sets.

Proposition 3. Let $\{S_\alpha\}_{\alpha\in]0,1]}, \{T_\alpha\}_{\alpha\in]0,1]} \in \mathcal{S}(\mathbb{R}^n)$. If $S_\alpha \subset T_\alpha$ for any $\alpha \in]0,1]$, then $M(\{S_\alpha\}_{\alpha\in]0,1]}) \le M(\{T_\alpha\}_{\alpha\in]0,1]})$.

Proof. For any $\boldsymbol{x} \in \mathbb{R}^n$, it follows that

$$\{\alpha \in]0,1] : \boldsymbol{x} \in S_\alpha\} \subset \{\alpha \in]0,1] : \boldsymbol{x} \in T_\alpha\},$$

and that

$$\begin{aligned} M(\{S_\alpha\}_{\alpha\in]0,1]})(\boldsymbol{x}) &= \sup\{\alpha \in]0,1] : \boldsymbol{x} \in S_\alpha\} \\ &\le \sup\{\alpha \in]0,1] : \boldsymbol{x} \in T_\alpha\} = M(\{T_\alpha\}_{\alpha\in]0,1]})(\boldsymbol{x}). \end{aligned}$$

□

The following proposition shows a relationship between a generator of a fuzzy set and level sets of the fuzzy set.

Proposition 4. Let $\{S_\alpha\}_{\alpha\in]0,1]} \in \mathcal{S}(\mathbb{R}^n)$, and let $\widetilde{a} = M(\{S_\alpha\}_{\alpha\in]0,1]})$. In addition, let $\alpha \in]0,1]$. Then,

$$[\widetilde{a}]_\alpha = \bigcap_{\beta\in]0,\alpha[} S_\beta.$$

Proof. It follows that

$$\begin{aligned} \boldsymbol{x} \in [\widetilde{a}]_\alpha &\Leftrightarrow \widetilde{a}(\boldsymbol{x}) = \sup\{\beta \in]0,1] : \boldsymbol{x} \in S_\beta\} \ge \alpha \\ &\Leftrightarrow \boldsymbol{x} \in S_\beta, \beta \in]0,\alpha[\\ &\Leftrightarrow \boldsymbol{x} \in \bigcap_{\beta\in]0,\alpha[} S_\beta. \end{aligned}$$

□

The following definitions are addition and scalar multiplication on $\mathcal{F}(\mathbb{R}^n)$ by Zadeh's extension principle. See [2] for Zadeh's extension principle.

Definition 4. For $\widetilde{a}, \widetilde{b} \in \mathcal{F}(\mathbb{R}^n)$ and $\lambda \in \mathbb{R}$, we define $\widetilde{a} + \widetilde{b}, \lambda\widetilde{a} \in \mathcal{F}(\mathbb{R}^n)$ as

$$(\widetilde{a} + \widetilde{b})(\boldsymbol{x}) = \sup_{\boldsymbol{x}=\boldsymbol{y}+\boldsymbol{z}} \min\left\{\widetilde{a}(\boldsymbol{y}), \widetilde{b}(\boldsymbol{z})\right\}, \quad (\lambda\widetilde{a})(\boldsymbol{x}) = \sup_{\boldsymbol{x}=\lambda\boldsymbol{y}} \widetilde{a}(\boldsymbol{y})$$

for each $\boldsymbol{x} \in \mathbb{R}^n$, respectively.

The following proposition shows a property of level sets of fuzzy sets with respect to scalar multiplication.

Proposition 5. Let $\widetilde{a} \in \mathcal{F}(\mathbb{R}^n)$, and let $\lambda \in \mathbb{R}$. In addition, let $\alpha \in]0,1]$. Then, $[\lambda\widetilde{a}]_\alpha \supset \lambda[\widetilde{a}]_\alpha$.

Proof. Let $\boldsymbol{x} \in \lambda[\widetilde{a}]_\alpha$. Then, there exists $\boldsymbol{y}_0 \in [\widetilde{a}]_\alpha$ such that $\boldsymbol{x} = \lambda\boldsymbol{y}_0$. Since $\widetilde{a}(\boldsymbol{y}_0) \geq \alpha$, it follows that $(\lambda\widetilde{a})(\boldsymbol{x}) \geq \widetilde{a}(\boldsymbol{y}_0) \geq \alpha$. Therefore, we have $\boldsymbol{x} \in [\lambda\widetilde{a}]_\alpha$. □

The following proposition shows a relationship between scalar multiplication of fuzzy sets and generators of the fuzzy sets.

Proposition 6. Let $\{S_\alpha\}_{\alpha\in]0,1]} \in \mathcal{S}(\mathbb{R}^n)$, and let $\widetilde{a} = M(\{S_\alpha\}_{\alpha\in]0,1]})$. In addition, let $\lambda \in \mathbb{R}$. Then,

$$\lambda\widetilde{a} = M(\{\lambda S_\alpha\}_{\alpha\in]0,1]}) = \sup_{\alpha\in]0,1]} \alpha c_{\lambda S_\alpha}.$$

Proof. For each $\alpha \in]0,1]$, it follows that $[\widetilde{a}]_\alpha = \bigcap_{\beta\in]0,\alpha[} S_\beta \supset S_\alpha$ from Proposition 4, and that $[\lambda\widetilde{a}]_\alpha \supset \lambda[\widetilde{a}]_\alpha \supset \lambda S_\alpha$ from Proposition 5. Thus, it follows that $\lambda\widetilde{a} \geq M(\{\lambda S_\alpha\}_{\alpha\in]0,1]})$ from Proposition 3 and the decomposition theorem (2). Suppose that there exists $\boldsymbol{x}_0 \in \mathbb{R}^n$ such that $(\lambda\widetilde{a})(\boldsymbol{x}_0) > M(\{\lambda S_\alpha\}_{\alpha\in]0,1]})(\boldsymbol{x}_0)$. We set $\gamma = M(\{\lambda S_\alpha\}_{\alpha\in]0,1]})(\boldsymbol{x}_0)$. Then, since $(\lambda\widetilde{a})(\boldsymbol{x}_0) = \sup_{\boldsymbol{x}_0=\lambda\boldsymbol{y}} \widetilde{a}(\boldsymbol{y}) > \gamma$, there exists $\boldsymbol{y}_0 \in \mathbb{R}^n$ such that $\boldsymbol{x}_0 = \lambda\boldsymbol{y}_0$ and $\widetilde{a}(\boldsymbol{y}_0) > \gamma$. We set $\eta = \widetilde{a}(\boldsymbol{y}_0) > \gamma$. It follows that $\boldsymbol{y}_0 \in [\widetilde{a}]_\eta = \bigcap_{\beta\in]0,\eta[} S_\beta$ from Proposition 4, and that $\boldsymbol{x}_0 = \lambda\boldsymbol{y}_0 \in \lambda S_\beta$ for any $\beta \in]0,\eta[$. Therefore, we have $\gamma = M(\{\lambda S_\alpha\}_{\alpha\in]0,1]})(\boldsymbol{x}_0) = \sup\{\alpha \in]0,1] : \boldsymbol{x}_0 \in \lambda S_\alpha\} \geq \eta > \gamma$, which is a contradiction. □

The following proposition shows the properties of scalar multiplication of fuzzy sets.

Proposition 7. Let $\widetilde{a} \in \mathcal{F}(\mathbb{R}^n)$, and let $\lambda, \mu \in \mathbb{R}$.

(i) $(\lambda\mu)\widetilde{a} = \lambda(\mu\widetilde{a})$.

(ii) $1\widetilde{a} = \widetilde{a}$.

Proof.
(i) From the decomposition theorem (2) and Proposition 6, we have

$$\begin{aligned} (\lambda\mu)\widetilde{a} &= M\left(\{(\lambda\mu)[\widetilde{a}]_\alpha\}_{\alpha\in]0,1]}\right) \\ &= M\left(\{\lambda(\mu[\widetilde{a}]_\alpha)\}_{\alpha\in]0,1]}\right) \\ &= \lambda M\left(\{\mu[\widetilde{a}]_\alpha\}_{\alpha\in]0,1]}\right) \\ &= \lambda(\mu\widetilde{a}). \end{aligned}$$

(ii) From the decomposition theorem (2) and Proposition 6, we have

$$\begin{aligned} 1\widetilde{a} &= M\left(\{1[\widetilde{a}]_\alpha\}_{\alpha\in]0,1]}\right) \\ &= M\left(\{[\widetilde{a}]_\alpha\}_{\alpha\in]0,1]}\right) \\ &= \widetilde{a}. \end{aligned}$$

□

4. Generalized convexity and degree of non-convexity In this section, the properties of the generalized convexity of fuzzy sets, and the properties of the degree of the non-convexity of fuzzy sets with respect to operations are investigated.

The following definition is a generalization of the convexity of fuzzy sets by allowing arbitrary conjunctive aggregation functions instead of the minimum operation, and is first proposed in [3] as the generalized quasiconcavity of membership functions. We consider the generalized quasiconcavity of membership functions as the generalized convexity of fuzzy sets.

Definition 5. (See [3].) Let $G : [0,1]^2 \to [0,1]$ be a conjunctive aggregation function, and let $\widetilde{a} \in \mathcal{F}(\mathbb{R}^n)$. The fuzzy set $\widetilde{a}$ is said to be G-convex if

$$\widetilde{a}(\lambda \boldsymbol{x} + (1-\lambda)\boldsymbol{y}) \geq G(\widetilde{a}(\boldsymbol{x}), \widetilde{a}(\boldsymbol{y}))$$

for any $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ and any $\lambda \in]0,1[$.

For a conjunctive aggregation function $G : [0,1]^2 \to [0,1]$ and $\widetilde{a} \in \mathcal{F}(\mathbb{R}^n)$, if $\widetilde{a}$ is convex, then $\widetilde{a}$ is G-convex from Definition 5.

The following proposition shows the properties of the G-convexity of fuzzy sets with respect to operations.

Proposition 8. Let $G : [0,1]^2 \to [0,1]$ be a continuous conjunctive aggregation function, and let $\widetilde{a}, \widetilde{b} \in \mathcal{F}(\mathbb{R}^n)$. In addition, let $\lambda \in \mathbb{R}$.

(i) Assume that $\min = G^{(1)} \gg G$. If $\widetilde{a}$ and $\widetilde{b}$ are G-convex, then $\widetilde{a} + \widetilde{b}$ is G-convex.

(ii) If $\widetilde{a}$ is G-convex, then $\lambda\widetilde{a}$ is G-convex. When $\lambda \neq 0$, if $\lambda\widetilde{a}$ is G-convex, then $\widetilde{a}$ is G-convex.

Proof.
(i) Let $\boldsymbol{x}, \boldsymbol{x}' \in \mathbb{R}^n$, and let $\mu \in]0,1[$. Fix any $\boldsymbol{y}, \boldsymbol{z}, \boldsymbol{y}', \boldsymbol{z}' \in \mathbb{R}^n$ such that $\boldsymbol{x} = \boldsymbol{y} + \boldsymbol{z}$ and $\boldsymbol{x}' = \boldsymbol{y}' + \boldsymbol{z}'$. Since $\mu\boldsymbol{x} + (1-\mu)\boldsymbol{x}' = (\mu\boldsymbol{y} + (1-\mu)\boldsymbol{y}') + (\mu\boldsymbol{z} + (1-\mu)\boldsymbol{z}')$, it follows that

$$\begin{aligned}
&(\widetilde{a} + \widetilde{b})(\mu\boldsymbol{x} + (1-\mu)\boldsymbol{x}') \\
&\geq \min\{\widetilde{a}(\mu\boldsymbol{y} + (1-\mu)\boldsymbol{y}'), \widetilde{b}(\mu\boldsymbol{z} + (1-\mu)\boldsymbol{z}')\} \\
&\geq \min\{G(\widetilde{a}(\boldsymbol{y}), \widetilde{a}(\boldsymbol{y}')), G(\widetilde{b}(\boldsymbol{z}), \widetilde{b}(\boldsymbol{z}'))\} \quad (\text{from the } G\text{-convexity of } \widetilde{a} \text{ and } \widetilde{b}) \\
&\geq G(\min\{\widetilde{a}(\boldsymbol{y}), \widetilde{b}(\boldsymbol{z})\}, \min\{\widetilde{a}(\boldsymbol{y}'), \widetilde{b}(\boldsymbol{z}')\}) \quad (\text{from } \min = G^{(1)} \gg G).
\end{aligned}$$

By the arbitrariness of $\boldsymbol{y}, \boldsymbol{z}, \boldsymbol{y}', \boldsymbol{z}'$, we have

$$\begin{aligned}
&(\widetilde{a} + \widetilde{b})(\mu\boldsymbol{x} + (1-\mu)\boldsymbol{x}') \\
&\geq \sup_{\substack{\boldsymbol{x} = \boldsymbol{y} + \boldsymbol{z} \\ \boldsymbol{x}' = \boldsymbol{y}' + \boldsymbol{z}'}} G(\min\{\widetilde{a}(\boldsymbol{y}), \widetilde{b}(\boldsymbol{z})\}, \min\{\widetilde{a}(\boldsymbol{y}'), \widetilde{b}(\boldsymbol{z}')\}) \\
&= G\left(\sup_{\boldsymbol{x}=\boldsymbol{y}+\boldsymbol{z}} \min\{\widetilde{a}(\boldsymbol{y}), \widetilde{b}(\boldsymbol{z})\}, \sup_{\boldsymbol{x}'=\boldsymbol{y}'+\boldsymbol{z}'} \min\{\widetilde{a}(\boldsymbol{y}'), \widetilde{b}(\boldsymbol{z}')\} \right) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\text{from Proposition 1}) \\
&= G((\widetilde{a} + \widetilde{b})(\boldsymbol{x}), (\widetilde{a} + \widetilde{b})(\boldsymbol{x}')).
\end{aligned}$$

(ii) Assume that $\widetilde{a}$ is G-convex. Let $\boldsymbol{x}, \boldsymbol{x}' \in \mathbb{R}^n$, and let $\mu \in]0,1[$. Fix any $\boldsymbol{y}, \boldsymbol{y}' \in \mathbb{R}^n$ such that $\boldsymbol{x} = \lambda \boldsymbol{y}$ and $\boldsymbol{x}' = \lambda \boldsymbol{y}'$. Since $\mu \boldsymbol{x} + (1-\mu)\boldsymbol{x}' = \lambda(\mu \boldsymbol{y} + (1-\mu)\boldsymbol{y}')$, it follows that

$$(\lambda \widetilde{a})(\mu \boldsymbol{x} + (1-\mu)\boldsymbol{x}') \geq \widetilde{a}(\mu \boldsymbol{y} + (1-\mu)\boldsymbol{y}') \geq G(\widetilde{a}(\boldsymbol{y}), \widetilde{a}(\boldsymbol{y}'))$$

from the G-convexity of $\widetilde{a}$. By the arbitrariness of $\boldsymbol{y}, \boldsymbol{y}'$, we have

$$\begin{aligned}
&(\lambda \widetilde{a})(\mu \boldsymbol{x} + (1-\mu)\boldsymbol{x}') \\
&\geq \sup_{\substack{\boldsymbol{x} = \lambda \boldsymbol{y} \\ \boldsymbol{x}' = \lambda \boldsymbol{y}'}} G(\widetilde{a}(\boldsymbol{y}), \widetilde{a}(\boldsymbol{y}')) \\
&= G\left(\sup_{\boldsymbol{x} = \lambda \boldsymbol{y}} \widetilde{a}(\boldsymbol{y}), \sup_{\boldsymbol{x}' = \lambda \boldsymbol{y}'} \widetilde{a}(\boldsymbol{y}') \right) \quad \text{(from Proposition 1)} \\
&= G((\lambda \widetilde{a})(\boldsymbol{x}), (\lambda \widetilde{a})(\boldsymbol{x}')).
\end{aligned}$$

The latter assertion follows from the former assertion and Proposition 7. □

Let $G : [0,1]^2 \to [0,1]$ be a conjunctive aggregation function. Assume that $\widetilde{a} \in \mathcal{F}(\mathbb{R}^n)$ is G-convex. Then, the larger the difference between G and $\min = G^{(1)}$ is, the larger the allowable non-convexity of $\widetilde{a}$ is.

Now, the degree of the non-convexity of fuzzy sets is defined. The degree of the non-convexity of fuzzy sets is first proposed in [5] as the degree of the non-quasiconcavity of membership functions. We consider the degree of the non-quasiconcavity of membership functions as the degree of the non-convexity of fuzzy sets.

Definition 6. (See [5].) For $\widetilde{a} \in \mathcal{F}(\mathbb{R}^n)$, the value

$$D(\widetilde{a}) = \min\{p \in [1, \infty[: \widetilde{a} \text{ is } G^{(p)}\text{-convex. }\} - 1 \tag{3}$$

is called the degree of the non-convexity of $\widetilde{a}$, where $\min \emptyset = \infty$ for $\emptyset \subset [1, \infty[$, and $G^{(p)}, p \in [1, \infty[$ are the continuous conjunctive aggregation functions defined by (1).

In (3), if $\{p \in [1, \infty[: \widetilde{a} \text{ is } G^{(p)}\text{-convex. }\} \neq \emptyset$, then the minimum is attained; see [5]. The degree of the non-convexity of $\widetilde{a}$ defined by (3) means that $\widetilde{a}$ is convex when $D(\widetilde{a}) = 0$, and that the larger $D(\widetilde{a})$ is, the larger the non-convexity of $\widetilde{a}$ is.

The following proposition shows the properties of the degree of the non-convexity of fuzzy sets.

Proposition 9. (See [5].) Let $\widetilde{a} \in \mathcal{F}(\mathbb{R}^n)$.

(i) $\widetilde{a}$ is convex if and only if $D(\widetilde{a}) = 0$.

(ii) $\widetilde{a}$ is not $G^{(p)}$-convex for any $p \in [1, D(\widetilde{a}) + 1[$.

(iii) $\widetilde{a}$ is $G^{(p)}$-convex for any $p \in [D(\widetilde{a}) + 1, \infty[$.

The following example illustrates the degree of the non-convexity of fuzzy sets.

Example 1. (See [5].) For each $\alpha \in \left[0, \frac{1}{2}\right[$, we define $\widetilde{a}_\alpha \in \mathcal{F}(\mathbb{R})$ as

$$\widetilde{a}_\alpha(x) = \begin{cases} 0 & \text{if } x \in]-\infty, 0] \cup [6, \infty[, \\ \frac{1}{2}x & \text{if } x \in [0,1], \\ \alpha \sin 4x\pi + \frac{1}{2} & \text{if } x \in [1,2], \\ \frac{1}{2}x - \frac{1}{2} & \text{if } x \in [2,3], \\ -\frac{1}{2}x + \frac{5}{2} & \text{if } x \in [3,4], \\ \alpha \sin 4x\pi + \frac{1}{2} & \text{if } x \in [4,5], \\ -\frac{1}{2}x + 3 & \text{if } x \in [5,6] \end{cases}$$

(Figure 1). Then, $D(\widetilde{a}_\alpha) = \frac{\log\left(\frac{1}{2}-\alpha\right)}{\log\left(\frac{1}{2}+\alpha\right)} - 1$ (Figure 2).

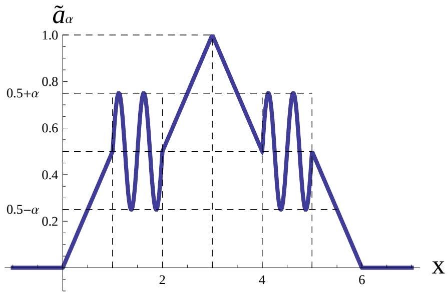

Figure 1. $\widetilde{a}_\alpha(x)$ $(\alpha = 0.25)$.

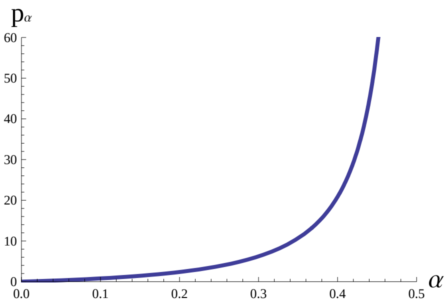

Figure 2. $p_\alpha = \frac{\log\left(\frac{1}{2}-\alpha\right)}{\log\left(\frac{1}{2}+\alpha\right)} - 1,\ \alpha \in \left[0, \frac{1}{2}\right[$.

The following proposition shows the properties of the degree of the non-convexity of fuzzy sets with respect to operations.

Proposition 10. Let $\widetilde{a}, \widetilde{b} \in \mathcal{F}(\mathbb{R}^n)$, and let $\lambda \in \mathbb{R}$.

(i) $D(\widetilde{a} + \widetilde{b}) \leq \max\{D(\widetilde{a}), D(\widetilde{b})\}$.

(ii) $D(\lambda\widetilde{a}) \leq D(\widetilde{a})$. When $\lambda \neq 0$, $D(\lambda\widetilde{a}) = D(\widetilde{a})$.

Proof.
(i) From Proposition 9 (iii), $\widetilde{a}$ is $G^{(p)}$-convex for $p \in [D(\widetilde{a}) + 1, \infty[$, and $\widetilde{b}$ is $G^{(p)}$-convex for $p \in [D(\widetilde{b}) + 1, \infty[$. Thus, $\widetilde{a}$ and $\widetilde{b}$ are $G^{(p)}$-convex for $p \in [\max\{D(\widetilde{a}) + 1, D(\widetilde{b}) + 1\}, \infty[$. From Propositions 2 and 8 (i), $\widetilde{a} + \widetilde{b}$ is $G^{(p)}$-convex for $p \in [\max\{D(\widetilde{a}) + 1, D(\widetilde{b}) + 1\}, \infty[$. Therefore, we have

$$D(\widetilde{a} + \widetilde{b}) \leq \max\{D(\widetilde{a}), D(\widetilde{b})\}.$$

(ii) From the former assertion of Proposition 8 (ii), we have

$$D(\lambda\widetilde{a}) \leq D(\widetilde{a}).$$

From the latter assertion of Propositions 8 (ii), we have

$$D(\lambda\widetilde{a}) = D(\widetilde{a}).$$

when $\lambda \neq 0$. □

The following example illustrates the degree of the non-convexity of fuzzy sets with respect to operations.

Example 2. Consider $\widetilde{a}_\alpha \in \mathcal{F}(\mathbb{R})$ for $\alpha \in \left]0, \frac{1}{2}\right[$ defined in Example 1. We set $p_\alpha = \frac{\log\left(\frac{1}{2}-\alpha\right)}{\log\left(\frac{1}{2}+\alpha\right)} - 1$. Then, $p_\alpha > 0$ and $D(\widetilde{a}_\alpha) = p_\alpha$. We set $\widetilde{\mathbb{R}} = c_{\mathbb{R}} \in \mathcal{F}(\mathbb{R})$ and $\widetilde{0} = c_{\{0\}} \in \mathcal{F}(\mathbb{R})$.

(i) Let $\widetilde{a} = \widetilde{a}_\alpha$, and let $\widetilde{b} = \widetilde{\mathbb{R}}$. Since $\widetilde{a} + \widetilde{b} = \widetilde{\mathbb{R}}$, $D(\widetilde{a}) = p_\alpha$, and $D(\widetilde{b}) = 0$, we have

$$D(\widetilde{a} + \widetilde{b}) = 0 < p_\alpha = \max\{D(\widetilde{a}), D(\widetilde{b})\}.$$

(ii) Let $\widetilde{a} = \widetilde{a}_\alpha$, and let $\widetilde{b} = \widetilde{0}$. Since $\widetilde{a} + \widetilde{b} = \widetilde{a}$, $D(\widetilde{a}) = p_\alpha$, and $D(\widetilde{b}) = 0$, we have

$$D(\widetilde{a}+\widetilde{b}) = p_\alpha = \max\{D(\widetilde{a}), D(\widetilde{b})\}.$$

(iii) Let $\widetilde{a} = \widetilde{a}_\alpha$, and let $\lambda = 0$. Since $\lambda\widetilde{a} = \widetilde{0}$, $D(\widetilde{a}) = p_\alpha$, and $D(\lambda\widetilde{a}) = 0$, we have

$$D(\lambda\widetilde{a}) = 0 < p_\alpha = D(\widetilde{a}).$$

5. Conclusions We dealt with the G-convexity of fuzzy sets, which was a generalization

of the convexity by allowing arbitrary conjunctive aggregation functions instead of the minimum operation. Then, the properties of the G-convexity of fuzzy sets with respect to operations were investigated. The degree of the non-convexity of fuzzy sets was considered as an application of the G-convexity. Then, the properties of the degree of the non-convexity of fuzzy sets with respect to operations were investigated.

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