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A VISIT TO THE WEAK ELEMENTARY EUCLIDEAN PASCH FREE GEOMETRY

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Dedicated to the memory of Professor Kiyoshi Iséki

Abstract. In this note, we make a review of the weak elementary Euclidean Pasch free geometry.

1 The Weak Elementary Euclidean Pasch-free Geometry Professor Iséki had supervised the authors as master course students at Kobe university from 1981 to 1983. During the master course, the authors mainly studied topological vector spaces, but firstly Professor Iséki gave the authors copies of some papers [1, 2, 6, 7, 8, 9] of the weak elementary Euclidean Pasch free geometry and told them to study its contents. Because in those days Professor Iséki had been much interested in ways of new geometrics and one way of new geometrics is Pasch free geometry (see Iséki[4] and [5]). In the book[5], Professor Iséki expects that an analysis based on Pasch free geometry is quite different from the traditional analysis. But as far as the authors know, it seems that there isn't such a big progress of Pasch free geometry to construct a new analysis since the authors studied. Hence, the purpose of this note is, following our study in the master course, to make a review of the weak elementary Euclidean Pasch free geometry. The authors strongly hope that the note become a good sign to restudy the weak elementary Euclidean Pasch free geometry.

Before stating the definition of the weak elementary Euclidean Pasch free geometry, we prepare two primitive notions. One is the ternary relation $B(abc)$ of betweenness ($B(abc)$ means that point b is between points a and c). The other is the quaternary relation $D(abcd)$ of equidistance ($D(abcd)$ means that point a is as distant from point b as point c from point d). Furthermore, to make the expressions of the axioms (stated later) shorter, we define the ternary relation $L(abc)$ of collinearity ($L(abc)$ means that the points a, b, c are collinear.):

$$L(abc) \longleftrightarrow B(abc) \vee B(bca) \vee B(cab)$$

Then, the weak elementary Euclidean Pasch free geometry (*Euclidean Pasch free geometry* for short) is the elementary theory based on the following eleven axioms from A1 to A11 which are formulated in terms of two primitive notions B and D and one auxiliary relation L (see p. 659 in Szczerba and Szmielew[7]):

A1. $\forall(abc)\ [B(abc) \to B(cba)]$
A2. $\forall(abcd)\ [B(abd) \wedge B(bcd) \to B(abc)]$
A3. $\forall(ab)\ D(abba)$
A4. $\forall(abc)\ [D(abcc) \to a = b]$
A5. $\forall(abpqrs)\ [D(abpq) \wedge D(abrs) \to D(pqrs)]$
A6. $\forall(abcpa'b'c'p')\ [a \neq b \wedge B(abc) \wedge B(a'b'c') \wedge D(aba'b') \wedge D(bcb'c') \wedge D(pap'a')$
$\wedge D(pbp'b') \to D(pcp'c')]$

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A7. $\forall(acdp)\exists b\ [B(pab) \wedge D(abcd)]$
A8. $\forall(ac)\exists b\ [B(abc) \wedge D(abbc)]$
A9. $\exists(abc)\ [\neg L(abc)]$
A10. $\forall(abcpq)\ [p \neq q \wedge D(apaq) \wedge D(bpbq) \wedge D(cpcq) \rightarrow L(abc)]$
A11. $\forall(abc)\exists p\ [\neg L(abc) \rightarrow D(papb) \wedge D(papc)]$

By adding the following axiom A12 to the axioms A1 $\sim$ A11, we gets an axiom system of the weak elementary Euclidean geometry (*Euclidean geometry* for short):

A12. $\forall(abcpa')\exists b'\ [B(apa') \wedge B(ba'c) \rightarrow B(bpb') \wedge B(ab'c)]$

A12 is called the Pasch axiom. Before Szczerba and Szmielew[7] set this axiom system of the Euclidean geometry, Szczebra[6] gave another axiom system of the Euclidean geometry, which is a little bit different from Szczerba and Szmielew's version, and showed a model of a pseudo-Cartesian plane where all the axioms except the Pasch axiom hold but the Pasch axiom does not hold. After Szczebra[6], Szczerba and Szmielew[7] proved representations for any model of the Euclidean Pasch free geometry in terms of Cartesian plane over a semi-ordered field with some conditions.

In section 2, we introduce the results of representations by Szczerba and Szmielew[7]. Furthermore, we review results by Gupta and Prestel[2] on the triangle inequality and the Schwarz inequality in models of the Euclidean Pasch free geometry in section 3. Finally we give a slight consideration to convergence of sequences, another version of the Pasch axiom and convex sets in Szczebra's model[6] in section 4.

2 Representations for Models of the Euclidean Pasch Free Geometry We begin with explanations of a semi-ordered field and its properties.

A system $\mathcal{F} = (F, +, \cdot, 0, 1, P)$ is a *semi-ordered field* if and only if $(F, +, \cdot, 0, 1)$ is a commutative field and the distinguished subset P of F satisfies the following three conditions:

(i) $x \in P \vee -x \in P$
(ii) $x \in P \wedge -x \in P \rightarrow x = 0$
(iii) $x \in P \wedge y \in P \rightarrow x + y \in P$

Moreover if P satisfies the condition

(iv) $x \in P \wedge y \in P \longrightarrow xy \in P$,

together with (i), (ii), (iii), then $\mathcal{F}$ is called a *fully ordered field*. The distinguished subset P of a semi-ordered field $\mathcal{F}$ is called the *semi-positive cone* of $\mathcal{F}$. By the semi-positive cone P of a semi-ordered field $\mathcal{F}$, we can define the greater than or equal relation and the absolute value operation:

$$x \geq y \leftrightarrow x - y \in P,$$

$$|x| = \begin{cases} x & \text{if } x \in P, \\ -x & \text{if } -x \in P, \end{cases}$$

As is easily seen, if P is the semi-positive cone of a semi-ordered field $\mathcal{F}$, since $-P = \{-p : p \in P\}$ also satisfy the three conditions (i), (ii) and (iii), $-\mathcal{F} = (F, +, \cdot, 0, 1, -P)$ is isomorphic to $\mathcal{F} = (F, +, \cdot, 0, 1, P)$ as semi-ordered fields. Hence, without loss of generality, we assume that $1 \in P$.

Let $(F, +, \cdot, 0, 1)$ be a commutative field and $S = \{x^2 \mid x \in F\}$. Then $(F, +, \cdot, 0, 1)$ is said to be *formally real* if

(v) $x \in S$ and $-x \in S \rightarrow x = 0$.

is satisfied. And $(F, +, \cdot, 0, 1)$ is said to be *Pythagorean* if

(vi) $x \in S$ and $y \in S \to x + y \in S$

holds. In the rest of this note, we are concerned with formally real and Pythagorean semi-ordered fields.

In the usual way, any formally real and Pythagorean semi-ordered field $\mathcal{F} = (F, 0, 1, +, \cdot, P)$ generates the linear space $\mathcal{L}(\mathcal{F}) = (F^2, F, +, \cdot, \bullet)$ with the inner product $\bullet$ such that

$$a \bullet b = a_1 b_1 + a_2 b_2 \quad \text{for } a = (a_1, a_2), b = (b_1, b_2) \in F^2.$$

For any $a \in F^2$, we put $a^2 := a \bullet a$. Since $\mathcal{F}$ is formally real, we have

$$a^2 = 0 \longleftrightarrow a = (0, 0).$$

Noting that $\mathcal{F}$ is Pythagorean, we have the unique element $x \in F$ for any $a \in F^2$ satisfying that

$$(x^2 = a^2) \wedge (x \in P).$$

We denote the unique element x by $\|a\|$. As for $\|\cdot\|$, we easily obtain

$\|a\| = 0 \longleftrightarrow a = (0, 0)$
$\|a\| = \|-a\|$
$\|a\| + \|a\| = \|a + a\|$
$\|a\|^2 = a^2$
$\|a\| = \|b\| \longleftrightarrow a^2 = b^2$

Furthermore, introducing the betweenness relation $B_{\mathcal{F}}$ and the equidistance relation $D_{\mathcal{F}}$ such that

$$B_{\mathcal{F}}(abc) \longleftrightarrow \|c - a\| = \|c - b\| + \|b - a\|,$$

and

$$D_{\mathcal{F}}(abcd) \longleftrightarrow \|a - b\| = \|c - d\|$$

into the linear space $\mathcal{L}(\mathcal{F})$, we define the Cartesian plane $\mathcal{C}(\mathcal{F}) := (F^2, B_{\mathcal{F}}, D_{\mathcal{F}})$ over $\mathcal{F}$

Now we are position to state the representations of models of the axiom systems of A1 $\sim$ A11 or A1 $\sim$ A12.

Theorem 1 (Szczerba-Szmielew [7]) *The following statements hold.*
(1) *$\mathcal{U}$ is a model of the axiom system of A1 $\sim$ A11 if and only if $\mathcal{U}$ is isomorphic to a Cartesian plane over a formally real, Pythagorean, semi-ordered field.*
(2) *$\mathcal{U}$ is a model of the axiom system of A1 $\sim$ A12 if and only if $\mathcal{U}$ is isomorphic to a Cartesian plane over a formally real, Pythagorean, fully ordered field.*

Remark 1. In Szmielew [8], the circle axiom

$$\forall(abcp)\exists c' \; [B(abc) \to B(pbc') \wedge D(ac'ac)]$$

is considered and he proves that any model of the axiom system of A1 $\sim$ A11 and the circle axiom satisfies the Pasch axiom.

Here we show a concrete model of a Cartesian plane over the field of real numbers with a modified semi-positive cone by Szczerba[6], which does not satisfy A12 but A1 $\sim$ A11.

Szczerba's Example. Let $\mathcal{R} = (\mathbf{R}, 0, 1, +, \cdot)$ be the field of the real numbers and $\mathcal{Q} = (\mathbf{Q}, 0, 1, +, \cdot)$ the field of the rational numbers. Since $\mathcal{Q}$ is a subfield of $\mathcal{R}$, $\mathbf{R}$ can be regarded as a linear space over $\mathcal{Q}$. It is easily seen that the numbers (i.e., the vectors) $h_0 = 1, h_1 = \sqrt{2}, h_2 = \sqrt[4]{2}$ are linearly independent in the linear space $\mathbf{R}$. Let $H = \{h_\alpha\}$ be a Hamel's base of $\mathbf{R}$ which contains h_0, h_1, h_2. Since by the Hamel's base, any real number x is uniquely expressed as the finite sum $\sum w_\alpha h_\alpha$, $x_0 := w_0 h_0$ and $\bar{x} := x - x_0$ are the rational part and the non-rational part of x, respectively. We define the bijective linear transformation from $\mathbf{R}$ to $\mathbf{R}$

$$f(x) = x_0 - \bar{x}\ , \quad \text{for all } x \in \mathbf{R}$$

and put $P^* := \{f(x) \mid x \geqq 0\}$. Then, P^* is a semi-positive cone of $\mathcal{R}$. Since $\mathcal{R} = (\mathbf{R}, 0, 1, +, \cdot)$ satisfies (v) and (vi), $\mathcal{R} = (\mathbf{R}, 0, 1, +, \cdot, P^*)$ is a formally real, Pythagorean, semi-ordered field. By Representation Theorem, A1 $\sim$ A11 hold but A12 does not hold in the Cartesian plane $\mathcal{C}(\mathcal{R})$. Indeed, if we put $a = (0, -\sqrt[4]{2}), b = (-\sqrt[4]{2}, 0), c = (\sqrt{2}, 0), a' = (0, 0)$ and $p = (0, 1)$, since

$$\|c - a'\| + \|a' - b\| = \|c - b\|,$$

$$\|a - p\| + \|p - a'\| = \|a - a'\|,$$

$B_{\mathcal{R}}(ba'c)$ and $B_{\mathcal{R}}(apa')$ hold. Noting that three points x, y, z in $\mathbf{R}^2$ for which the relation $B_{\mathcal{R}}$ holds are collinear, we can't find a point b' for which $B_{\mathcal{R}}(bpb') \wedge B_{\mathcal{R}}(ab'c)$. Because the line in the usual sense on which the two points b and p lie is parallel to the line in the usual sense on which the two points a and c lie.

3 Some Properties of Models of Euclidean Pasch Free Geometry Let $\mathcal{F} = (F, 0, 1, +, \cdot, P)$ be a formally real and Pythagorean semi-ordered field and $\mathcal{L}(\mathcal{F})$ the linear space generated by $\mathcal{F}$. As for P, we consider the following properties:

(vii) $x^2 \in P$ for $x \in F$

(viii) $(x \in P) \wedge (y \in F) \longrightarrow xy^2 \in P$

Remark 2. Since we assume $1 \in P$, (vii) is weaker than (viii).

In $\mathcal{L}(\mathcal{F})$, we are concerned with the three inequalities:

(triangle inequality) $\|a - b\| \leq \|a\| + \|b\|$ for $a, b \in \mathcal{L}(\mathcal{F})$

(weak Schwartz inequlity) $(a \bullet b)^2 \leq a^2 b^2$ for $a, b \in \mathcal{L}(\mathcal{F})$

(Schwartz inequality) $|a \bullet b| \leq |\|a\|\|b\||$ for $a, b \in \mathcal{L}(\mathcal{F})$

Gupta and Prestel[2] obtained the following theorems of the relations between the properties (vii) and (viii) of P and three inequalities in $\mathcal{L}(\mathcal{F})$.

Theorem 2. *Let $\mathcal{F} = (F, 0, 1, +, \cdot, P)$ be a formally real and Pythagorean semi-ordered field and $\mathcal{L}(\mathcal{F})$ the linear space generated by $\mathcal{F}$. Then,* (vii) *holds if and only if weak Schwartz inequality holds in $\mathcal{L}(\mathcal{F})$.*

Theorem 3. *Let $\mathcal{F} = (F, 0, 1, +, \cdot, P)$ be a formally real and Pythagorean semi-ordered field and $\mathcal{L}(\mathcal{F})$ the linear space generated by $\mathcal{F}$. Then, the following statements are equivalent:*

(1) *The property* (viii) *holds.*
(2) *Triangle inequality holds in $\mathcal{L}(\mathcal{F})$.*
(3) *Schwartz inequality holds in $\mathcal{L}(\mathcal{F})$.*

Remark 3. Let $\mathcal{F} = (F, 0, 1, +, \cdot, P)$ be a formally real and Pythagorean semi-ordered field. As is easily seen, (iv) implies (viii). Indeed, Gupta and Prestel[1] showed an example of a formally real and Pythagorean semi-ordered field in which (iv) does not hold but (viii) holds.

Gupta and Prestel's Example (see p.21 in Gupta and Prestel[1]). Let $\mathbf{Q}$ be the field of rational numbers and x a positive transcendental real number. We consider $Q_1 = Q(\sqrt{2}, x)$ and extend Q_1 to the smallest Pythagorean real field K. We set $R = \{u \mid u = a^2 + b^2x + c^2\sqrt{2}, a, b, c \in K\}$. Since any element of R is non-negative numbers, R satisfies (ii). We easily see that (viii) holds for R. Furthermore, noting that K is Pythagorean, R satisfies (iii). To proceed to the next step, Lemma 1 and Lemma 2 in [1] are prepared.

Lemma 1. *Let K be a formally real, Pythagorean field and R a subset of K with the properties* (ii), (iii) *and* (viii). *If z belongs to $K - R$, then the subset*

$$R' = \{a \mid a = -b^2z + v \text{ for some } b \in K, v \in R\}$$

also satisfies (ii), (iii) *and* (viii), *and R' contains R.*

Lemma 2. *Let K be a formally real, Pythagorean field and R a subset of K with the properties* (ii), (iii) *and* (viii). *Then there exists a subset P of K such that $R \subseteq P$ and P satisfies* (i), (ii), (iii) *and* (viii).

Since $x\sqrt{2}$ belongs to $K - R$, $R_1 = \{a \mid a = -b^2x\sqrt{2} + v,\ b \in K, v \in R\}$ satisfies (ii), (iii) and (viii) by Lemma 1. Applying Lemma 2 to R_1, we obtain a subset P_1 such that $R_1 \subseteq P_1$ and P_1 satisfies (i), (ii), (iii) and (viii). $x, \sqrt{2}, -x\sqrt{2}$ belong to P_1, but $x\sqrt{2}$ does not belong to P_1 by (ii). Hence, P_1 does not have the property (iv). For the formally real and Pythagorean semi-ordered field $\mathcal{K} = (K, 0, 1, +, \cdot, P_1)$, if we consider the Cartesian plane $\mathcal{C}(\mathcal{K})$ over $\mathcal{K}$, then the Pasch axiom does not hold in $\mathcal{C}(\mathcal{K})$ by Theorem 1.

4 A Study on Szczerba's Example We give a slight consideration to convergence of sequences, another version of the Pasch axiom and convex sets in Szczebra's model[6]. Before doing this, we explain some notations. In the usual real line $\mathbf{R}$, $<$ denotes the order relation. And in Szczerba's real line, $<_*$ denotes the order relation and $|x|_*$ denotes the absolute value of the real number x.

(1) Convergence of sequences in Szczerba's real line First we consider the axiom of Archimedes.

Lemma 3. *For any $a, b \in P^* - \{0\}$, there exists a positive integer $n \in \mathbf{N}$ such that $a <_* nb$, i.e., $\dfrac{a}{n} <_* b$.*

Proof. We put $a = r_a + x_a, b = r_b + x_b$, where r_a, r_b denote the rational parts and x_a, x_b denote the non-rational parts. Since $a, b \in P^*$, we see that $r_a - x_a > 0$ and $r_b - x_b > 0$. By the axiom of Archimedes, we have a positive integer n such that $r_a - x_a < n(r_b - x_b)$. Since

$$nr_b - nx_b - r_a + x_a = (nr_b - r_a) + (-nx_b + x_a) > 0,$$

we obtain

$$(nr_b - r_a) - (-nx_b + x_a) = n(r_b + x_b) - (r_a + x_a) \ {}_*\!> 0,$$

which means that $a <_* nb$.

We make a definition of convergence of sequences in Szczerba's real line.

Definition 1. Let $\{a_n\}$ be a sequence of Szczerba's real line. Then $\{a_n\}$ converges to a number α if and only if for any $\varepsilon \in P^* - \{0\}$, there exists a positive integer N such that

$$|a_n - \alpha|_* <_* \varepsilon \quad \forall n \geq N.$$

By Lemma 3, for any $\varepsilon \in P^*$ and the positive element 1, there exists a positive integer n with $\frac{1}{n} <_* \varepsilon$. Hence, we have the following.

Theorem 4. *Let $\{a_n\}$ be a sequence of Szczerba's real line and α a number of Szczerba's real line. If for each positive integer $m \in \mathbf{N}$, there exists a positive integer $N \in \mathbf{N}$ such that*

$$|a_n - \alpha|_* <_* \frac{1}{m} \quad \forall n \geq N,$$

then $\{a_n\}$ converges to A.

(2) On Szczerba's Cartesian Plane In Szczerba's Cartesian Plane, $\|\cdot\|$ denotes the pseudo norm which is defined in section 2.

(a) The Pasch Axiom in Hilbert's Book The Pasch axiom in the book [3] is a little bit different from A 12. The Pasch axiom in the book [3] is expressed that in a triangle, a line passing through an inner point of a side of a triangle meets another side. By the following example, we can show that the Pasch axiom in the book [3] does not always hold in Szczerba's Cartesian plane.

Example. Let $a = (0,0), b = (\sqrt{2} + \sqrt[4]{2}, 0), c = (0, \sqrt[4]{2} + 1)$, $p = (\sqrt[4]{2}, 0), q = (0,1)$. Let us consider the line ℓ which passes through p and q. Since

$$\|a - p\| + \|p - b\| = \|a - b\|,$$

p is an inner point of the side ab. And since

$$\|q - a\| + \|a - c\| = \|q - c\|,$$

$B(qac)$ holds and ℓ does not pass through any point of side ac. Furthermore, since ℓ is parallel to the line m which passes through the points b and c, ℓ and m don't have a common point, which means that ℓ does not pass through any point of side bc. This shows that the Pasch axiom in the book [3] does not always hold in Szczerba's Cartesian plane.

(b) Convex Sets We begin with a definition of convex sets in Szczerba's Cartesian Plane.

Definition 2. A subset A of Szczerba's Cartesian plane is said to be *convex*, if for any $a, b \in A$ and for any number t in Szczerba's real line with $0 \leq_* t \leq_* 1$, $ta + (1-t)b$ belongs to A.

Let $B(\subset \mathbf{R}^2)$ be a bounded convex set containing at least two point in the usual sense. Suppose that $a, b \in B$ are distinct points both of which are boundary points or are very near to boundary points of B. Using the fact that $-1-\sqrt{2}$ satisfies $0 <_* -1-\sqrt{2} <_* 1$, we can observe that $(-1-\sqrt{2})a + (2+\sqrt{2})b$ does not belong to B.

Hence, we obtain the following.

Theorem 5. *Let B be a bounded convex set in $\mathbf{R}^2$ in the usual sense. Then B is not convex in Szczerba's Cartesian Plane.*

Here we give a slight consideration to convergence of sequences, another version of the Pasch axiom and convex sets in Szczebra's model. But we are sure that it is worthwhile to study some other properties in Szczerba's model or some concrete model where the Pasch axiom does not hold.

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References

[1] H. N. Gupta and A. Prestel, On a class of Pasch-free Euclidean Geometry, *Bull. Acad. Polon. Sci., Ser.Sci.Math.Astronom.Phys.* 20, No.1 (1972), 17-23.

[2] H. N. Gupta and A. Prestel, Triangle and Schwarz Inequality in Pasch-free Euclidean Geometry, *Bull. Acad. Polon. Sci., Ser.Sci.Math.Astronom.Phys.* 20, No.12 (1972), 999-1003.

[3] D. Hilbert, *Grundlagen der Geometrie*, 7th edition, Stuttgart, 1930.

[4] K. Iséki, *Non Greek Mathematics*(Lecture Notes in Mathematics), Fayetteville, The University of Arkansas , 1978.

[5] K. Iséki, *Sets and Logic*(in Japanese), Shinyou-sha Publishing Company, 1979.

[6] L.W. Szczebra, Independence of Pasch's Axiom, *Bull. Acad. Polon. Sci., Ser. Sci. Math. Astronom. Phys.* 18, No.8 (1970), 491-498.

[7] L. Szczebra and W. Szmielew, On the Euclidean Geometry without the Pasch Axiom, *Bull. Acad. Polon. Sci., Ser. Sci. Math. Astronom. Phys.* 18, No.14 (1970), 659-666.

[8] W. Szmielew, The Pasch axiom as a consequence of the circle axiom, *Bull. Acad. Polon. Sci., Ser. Sci. Math. Astronom. Phys.* 18, No.12 (1970), 751-758.

[9] A. Tarski, What is elementary Geometry? in : *The Axiomatic method* ..., Noth Holland Publ. Co., Amsterdam, 1959, 16-29.

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