NOTES ON SOME GENERALIZED CONVEX SETS

ANNA B.ROMANOWSKA

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ABSTRACT. This paper discusses some properties of convex subsets of affine spaces over the principal ideal domains $\mathbb{Z}[1/p]$, where p is a prime number. In particular, it is shown that such convex sets are equivalent to certain algebras with finitely many operations, and a minimal number of generators for the segments of the line $\mathbb{Z}[1/p]$ is provided. In the case of p = 3, relations between convex sets and a certain groupoid variety are discussed.

1 Introduction. It is well known that affine spaces over a subring R of the field \mathbb{R} of real numbers (or *affine R-spaces*) may be described as abstract algebras (A, P, \underline{R}) with infinitely many binary operations

$$xy\underline{r} = x(1-r) + yr$$

indexed by the elements of R, and the ternary Mal'cev operation

$$xyzP = x - y + z.$$

For a subfield F of \mathbb{R} , the restriction of the set of basic operations to the operations indexed by the open unit interval $I^o(F) = \{x \in F \mid 0 < x < 1\}$ of F provides an algebraic description of convex subsets of affine spaces over F (or F-convex sets) as algebras $(C, \underline{I}^o(F))$. (See e.g.[13].) In this case the algebraic and geometric definitions of convex sets coincide.

The class of F-convex sets forms a quasivariety Cv(F) (a subquasivariety of the variety of so-called *barycentric algebras* over F), and is defined by the identities

of *idempotence*, the identities

of skew-commutativity, the identities

(1.3)
$$xy \underline{p} \ z \underline{q} = x \ yz q/(p+q-pq) \ \underline{p+q-pq}$$

of *skew-associativity* and the quasi-identities

$$(1.4) \qquad (xyp = xzp) \longrightarrow (y = z)$$

of cancellation for all p, q in $I^{o}(F)$. (See [6,9,10,13].)

Let us note that both affine R-spaces and F-convex sets may be considered as subreducts of affine \mathbb{R} -spaces, i.e. subalgebras of appropriate types of reducts of real affine spaces. Recall as well that algebras describing affine spaces and convex sets belong to the family

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of algebras called *modes*, algebras that are idempotent and entropic. Indeed, any two operations p and q of affine spaces satisfy the entropic law:

Both the geometric and algebraic concepts of F-convex sets were generalized in [2] to the case of convex subsets of faithful affine spaces over principal ideal subdomains R of the ring \mathbb{R} . Algebraically they are described as algebras $(C, \underline{I}^o(R))$, where $\underline{I}^o(R)$ is the set of binary operations $xy\underline{r}$ for $r \in I^o(R) = \{x \in R \mid 0 < x < 1\}$. Such $\underline{I}^o(R)$ -algebras were called R-convex sets. Geometric R-convex sets in finite dimensional affine spaces R^n are described as the intersections of \mathbb{R} -convex subsets of \mathbb{R}^n with its subspace R^n . If R is not a field, then the algebraic and geometric definitions of R-convex sets do not coincide. An essential role is played by the segments (closed intervals) of R-lines (or simply lines), one-dimensional affine R-spaces (R, P, \underline{R}) . For a < b in R, the segment or interval with the endpoints a and b is defined to be $[a, b]_R = \{x \in R \mid a \le x \le b\}$. Such segments are geometric R-convex sets. Note however that there are algebraic convex subsets of R-lines that are not segments. (For more detail, see [2].)

In this paper we are interested in convex sets over the principal ideal domains $R_p = \mathbb{Z}[1/p]$, where p is a prime number. Such R_p -convex sets are considered here as $\underline{I}^o(R_p)$ -subreducts $(C, \underline{I}^o(R_p))$ of faithful affine R_p -spaces (A, P, \underline{R}_p) .

The paper contains three results. In Section 2, we show that R_p -convex sets (and even more generally $\underline{I}^o(R_p)$ -subreducts of affine R_p -spaces) may be described as certain modes with finitely many binary operations. Section 3 deals with the segments of R_p -lines. It was shown in [2] that, unlike the real case, such segments are not necessarily pairwise isomorphic, and are not necessarily generated by their endpoints. They are however finitely generated. In Section 4, we provide an exact number of minimal sets of generators for any one of them. In the final section, we show that R_3 -convex sets belong to a certain variety of groupoid modes, such that all its members embeddable into affine spaces in a non-trivial way, actually embed into affine R_3 -spaces.

We refer the readers to the monographs [9,11,13] for additional information about the algebraic concepts used in the paper, especially those concerning convex sets, barycentric algebras, and affine spaces; to [1] for basic geometric properties of convex subsets of \mathbb{R}^n , and to [2] for more information about generalizations of convex sets. Our notation generally follows the conventions established in the first three monographs mentioned above.

2 Convex sets over the rings $R_p = \mathbb{Z}[1/p]$. In this section, we show that R_p -convex sets may be described as certain modes with finitely many binary operations. First recall that for a positive integer k, the elements of the interval $[0, k]_{R_p}$ of R_p may be described as follows:

(2.1)
$$[0,k]_{R_n} = \{m/p^n \mid m, n \in \mathbb{Z}, 0 \le m \le kp^n\}.$$

Moreover, all the intervals $[0, p^n]_{R_p}$, where *n* is any integer, are isomorphic, and each is generated as an $\underline{I}^o(R_p)$ -algebra by its endpoints [2].

Let us consider affine R_p -spaces (A, P, \underline{R}_p) and their reducts

$$(A, \Omega_p) = \left(A, \underline{1/p}, \underline{2/p}, \dots, \underline{(p-1)/p}\right).$$

We call such reducts *p*-reducts and their subalgebras *p*-subreducts of (A, P, \underline{R}_p) , or simply *p*-modes. Note that *p*-modes obviously satisfy the identities (1.1) and (1.2) defining real barycentric algebras. However they do not necessarily satisfy (1.3). For example, in the case

of 2-modes, if p = q = 1/2, then q/(p + q - pq) = 2/3 does not belong to R_2 . All *p*-modes form a quasivariety. (This follows from a more general result saying that the subreducts of members of a given quasivariety again form a quasivariety [3, §11.1].)

Evidently, for any $\underline{I}^{o}(R_{p})$ -subreduct of an affine R_{p} -space A, the set $\underline{I}^{o}(R_{p})$ of operations contains all the operations of Ω_{p} . We will show that Ω_{p} generates all the operations of $\underline{I}^{o}(R_{p})$.

Lemma 2.1. Let $(C, \underline{I}^o(R_p))$ be an $\underline{I}^o(R_p)$ -subreduct of an affine R_p -space. Then the set Ω_p of operations of C generates all the operations of the set $\underline{I}^o(R_p)$.

Proof. First recall, that the interval

$$[0,1]_{R_n} = \{m/p^n \mid m, n \in \mathbb{Z}, 0 \le m \le p^n\}$$

of the line $R_p = \mathbb{Z}[1/p]$ is the free algebra on two free generators 0 and 1 in the quasivariety $Q(R_p)$ of the $\underline{I}^o(R_p)$ -subreducts of affine R_p -spaces. (See [8] and [13, Ch. 5].) Its elements represent the binary term operations of $\underline{I}^o(R_p)$ -subreducts of affine R_p -spaces. To show that the operations of Ω_p generate all the other operations of $\underline{I}^o(R_p)$, it is enough to show that the elements of the interval $[0, 1]_{R_p}$ can be obtained by applying the operations of Ω_p to the elements 0 and 1. For each natural number n, let us define

$$I_n := \{m/p^n \mid 0 \le m \le p^n\}$$

and observe that $[0,1]_{R_p}$ is a directed union of the sets I_n . Note also that any two elements of $[0,1]_{R_p}$ can always be written as quotients with common denominators. It will be shown, by induction on n, that a single application of the operations of Ω_p to elements of I_n generates the elements of I_{n+1} . Evidently, single applications of Ω_p to I_0 produces the elements $0, 1/p, \ldots, (p-1)/p, 1$ of I_1 . Now assume that the elements of I_n have already been obtained, and apply Ω_p to I_n . Then for $i = 1, \ldots, p-1$, and $0 \le m < k \le p^n$, one has

$$\frac{m}{p^n}\frac{k}{p^n}\frac{i}{p} = \frac{m}{p^n}\cdot\frac{p-i}{p} + \frac{k}{p^n}\cdot\frac{i}{p} = \frac{m(p-i)+ki}{p^{n+1}}$$

Moreover, this element lies between $m/(p^n)$ and $k/(p^n)$. Hence it belongs to I_{n+1} . In particular, for $m < p^n$,

$$\frac{m}{p^n} \frac{m+1}{p^n} \frac{i}{p} = \frac{m}{p^n} \cdot \frac{p-i}{p} + \frac{m+1}{p^n} \cdot \frac{i}{p} = \frac{m(p-i) + (m+1)i}{p^{n+1}} = \frac{mp+i}{p^{n+1}}.$$

Hence one does indeed obtain all the elements of I_{n+1} . It follows that this procedure will generate precisely the elements of $[0,1]_{R_p}$.

Corollary 2.2. Let A be an affine R_p -space. Then each $\underline{I}^o(R_p)$ -subreduct $(B, \underline{I}^o(R_p))$ of A is equivalent to the p-subreduct (B, Ω_p) of A. Similarly, each p-subreduct (C, Ω_p) of A is equivalent to the $\underline{I}^o(R_p)$ -subreduct $(C, \underline{I}^o(R_p))$ of A.

Proposition 2.3. For a fixed prime number p, all p-modes satisfy the following identities:

$$xy \frac{1/p}{p} = yx (p-1)/p,$$

$$xy \frac{2/p}{p} = yx (p-2)/p,$$

$$\dots$$

$$xy \lfloor p/2 \rfloor/p = yx (p - \lfloor p/2 \rfloor)/p.$$

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Proof. This follows directly by the skew commutativity (1.2). Indeed, for any $i = 1, \ldots, |p/2|$,

$$\begin{aligned} xy \, \underline{i/p} &= x(1 - i/p) + yi/p \\ &= y(1 - (1 - i/p)) + x(1 - i/p) \\ &= yx \, \underline{(p - i)/p}. \end{aligned}$$

Note that for p = 2, the identities of Proposition 2.3 reduce to the commutativity of the operation $x \cdot y = (x + y)/2$. In fact, each \mathbb{D} -convex set $(C, \underline{I}^o(\mathbb{D}))$, where $\mathbb{D} = R_2 = \mathbb{Z}[1/2]$ is the set of rational dyadic numbers, is equivalent to the groupoid (C, \cdot) . Such groupoids are called *commutative groupoid* (or binary) modes. The variety $\mathcal{V}_2 = \mathcal{CB}$ of commutative binary modes is known to be generated by 2-modes [4]. This suggests the following question.

Question 2.4. Let p > 2 be a fixed prime number. Let \mathcal{V}_p be the variety of modes of the same type as *p*-modes, defined by the identities of Proposition 2.3. Does the class of all *p*-modes generate the variety \mathcal{V}_p ?

We will return to this question in Section 5.

3 Intervals of the lines $R_p = \mathbb{Z}[1/p]$. In what follows intervals (segments) of the line $R_p = \mathbb{Z}[1/p]$ will be denoted simply by [a, b] instead of $[a, b]_{R_p}$, and will be considered as $\underline{I}^o(R_p)$ -subreducts $([a, b], \underline{I}^o(R_p))$ of the affine R_p -space $(R_p, P, \underline{R}_p)$.

All such intervals were classified (up to isomorphism) in [2]. They are geometric R_p convex sets, and form an obvious generalization of the segments of the dyadic line $(\mathbb{D}, P, \mathbb{D})$ [5]. The intervals of the lines R_p are not necessarily pairwise isomorphic. In fact, each one is isomorphic to some interval [0, k], where k is a positive integer not divisible by p, and two such intervals are isomorphic precisely when their right hand endpoints coincide [2,5].

4 Generating intervals of R_p . In what follows, we fix a prime number p > 2. It was shown in [2] that the intervals of the line R_p are not necessarily generated by their endpoints, though they are finitely generated. Note that each positive integer k has a unique representation

(4.1)
$$k = k_1 p^{n_1} + k_2 p^{n_2} + \dots + k_j p^{n_j},$$

where the n_i , for i = 1, ..., j, are pairwise distinct non-negative integers such that $n_1 > n_2 > \cdots > n_j$, and $k_i \in \{0, 1, ..., p-1\}$. It was shown in [2] that for k > 1, the set

$$G = \{0, p^{n_1}, 2p^{n_1}, \dots, k_1 p^{n_1}, k_1 p^{n_1} + p^{n_2}, \dots, k_1 p^{n_1} + k_2 p^{n_2}, \dots, \sum_{i=1}^j k_i p^{n_i} = k\}$$

forms a convenient set of generators of the interval [0, k]. However, it is not necessarily a minimal set of generators. In [5], it was shown that the intervals of the dyadic line \mathbb{D} are minimally generated by two or three elements. We will find a minimal number of generators for intervals of R_p by proving the following theorem. For brevity, we will say that an interval is *m*-generated if it may be generated by m, but no fewer elements.

Theorem 4.1. Each interval of the line R_p is m-generated for some integer $1 < m \le p+1$.

First recall that each interval of R_p is isomorphic to an interval [0, k] for some positive integer k not divisible by p. Since all such intervals are finitely generated, we may further assume that each of them is isomorphic to an interval generated by some integers contained within it. It suffices to observe that the interval [0, k] is isomorphic to each interval $[0, kp^n]$, and take n sufficiently large. So in what follows we will drop the requirement that k is not divisible by p and consider the intervals [0, k] for any positive integer k and with integral generators. Recall that each set of generators of [0, k] must contain the endpoints 0 and k.

Proof. The proof will be based on a case analysis, and will be divided into lemmas labeled by capital letters. First recall the following two properties (A) and (B) (see [2]):

- (A) The interval $[0, p^n]$ is generated by its endpoints 0 and p^n , and thus is 2-generated.
- (B) Each interval $[0, ip^n]$, for $1 \le i < p$, is isomorphic to the interval [0, i].

Next we show the following property.

(C) If k > 1 is not divisible by p, then the interval [0, k] is not 2-generated.

Proof. Note that in the interval [0, k], the elements 0 and k generate the set

(4.2)
$$\{km/p^r \mid r \in \mathbb{N}, m = 0, 1, \dots, p^r\}.$$

Since $k \neq p^r$ for any $0 < r \in \mathbb{N}$, none of the numbers $1, 2, \ldots, k-1$ belongs to [0, k]. \Box

(D) Each interval [0, i], where $1 \le i < p$, is (i + 1)-generated.

Proof. By (C), the interval [0, i] must contain at least three generators. It is clear that [0, 2] is generated by 0, 1, and 2. Now if 2 < i < p, then [0, i] cannot have less than i + 1 integer generators, since otherwise these generators would not generate all the integers between 0 and i.

As a corollary of (B) and (D), we get the following.

- (E) Each interval $[0, ip^n]$, where $1 \le i < p$, is (i + 1)-generated.
- (F) Each interval $[0, ip^n + j]$, where $1 \le i < p$ and $1 \le j < p^n$, is (i + 2)-generated.

Proof. By (E), the interval $[0, ip^n]$ is (i + 1)-generated. The interval $[(i - 1)p^n + j, ip^n + j]$ is generated by its endpoints, the first of which is already contained in $[0, ip^n]$.

(G) Let $k = ip^n + (p-1)p^{n-1} + \dots + (p-1)p + (p-1)$, where $1 \le i < p$. Then [0, k] is (i+2)-generated.

Proof. First note that

(4.3)
$$j := (p-1)p^{n-1} + \dots + (p-1)p + (p-1) < p^n.$$

Indeed,

$$j = (p-1)p^{n-1} + \dots + (p-1)p + (p-1)$$

= $(p-1)(p^{n-1} + \dots + p + 1)$
= $(p-1)[(p^{n-1} - 1)/(p-1)]$
= $p^{n-1} - 1 < p^{n-1} < p^n$.

Then, by (F), it follows that the interval $[0, ip^n + j]$ is (i + 1)-generated.

(H) Let k have the form of (4.1), so $k = k_1 p^{n_1} + k_2 p^{n_2} + \dots + k_j p^{n_j}$, with $n_1 > n_2 > \dots > n_j$. Then [0, k] is (i + 2)-generated.

Proof. Let $j := k_2 p^{n_2} + \dots + k_j p^{n_j}$. Then by (4.3),

$$j < (p-1)p^{n_1-1} + (p-1)p^{n_1-2} \dots (p-1)p + (p-1) < p^{n_1}.$$

Hence, by (F), the interval [0, k] is (i + 2)-generated.

The theorem follows by (A), (E) and (H).

5 Barycentric algebras over $R_3 = \mathbb{Z}[1/3]$. In this section we consider $\underline{I}^o(R_3)$ -subreducts of affine R_3 -spaces. As demonstrated below, the case p = 3 is rather special.

Proposition 5.1. Each $\underline{I}^{o}(R_{3})$ -subreduct $(C, \underline{I}^{o}(R_{3}))$ of an affine R_{3} -space is equivalent to the groupoid (C, \cdot) , where $a \cdot b = ab 1/3$, satisfying the identity

Proof. By Corollary 2.2, each such $\underline{I}^o(R_3)$ -subreduct C is equivalent to the 3-mode (C, Ω_3) . By Proposition 2.3, (C, Ω_3) is equivalent to the groupoid (C, \cdot) , where the multiplication is defined by $a \cdot b = ab 1/3$. Note also that ab 2/3 = ba 1/3.

Then C satisfies $\overline{\text{the}}$ identity (5.1), since

$$\begin{aligned} xy \cdot yx &= (x(2/3) + y(1/3))(2/3) + (y(2/3) + x(1/3))(1/3) \\ &= x(5/9) + y(4/9) \\ &= (y(2/3) + x(1/3))(2/3) + x(1/3) = yx \cdot x. \end{aligned}$$

Let \mathcal{T} be the variety of groupoid modes defined by the identity (5.1). Let $Q(R_3)$ be the quasivariety of $\underline{I}^o(R_3)$ -subreducts of affine R_3 -spaces, and let $Q(R_3^{\bullet})$ be the quasivariety of groupoids (as in Proposition 5.1) equivalent to such subreducts. Observe the following obvious fact.

Lemma 5.2. The variety \mathcal{T} contains $Q(R_3^{\bullet})$ as a subquasivariety. It also contains the variety \mathcal{RZ} of right-zero semigroups and the variety \mathcal{SL} of semilattices.

Proof. The first statement follows by Proposition 5.1. The varieties \mathcal{RZ} and \mathcal{SL} are known to be varieties of modes. Moreover, since the identities satisfied by \mathcal{RZ} are precisely the identities with the same last variable on both sides, and the identities satisfied by \mathcal{SL} are precisely the identities having the same sets of variables on both sides, it follows that they both satisfy the identity (5.1) [13, §5.2].

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Note that non-trivial semilattices do not embed into affine spaces. On the other hand, right-zero semigroups may be considered as <u>1</u>-subreducts of any affine space, so they trivially embed into any affine space. If a groupoid of \mathcal{T} has no right semi-group as a subgroupoid, and embeds into an affine space, we will say that it embeds *non-trivially*. We will show that groupoids of \mathcal{T} that embed non-trivially as subreducts into affine spaces are in fact subreducts of affine R_3 -spaces. This can be done by looking for the so-called affinization of the variety \mathcal{T} . The affinization of a variety \mathcal{V} of modes is the variety $\frac{R(\mathcal{V})}{R(\mathcal{V})}$ of affine spaces over a certain ring $R(\mathcal{V})$, with the property that all members of \mathcal{V} that embed as subreducts into some affine spaces are in fact subreducts of affine $R(\mathcal{V})$ -spaces. (See [13, §7.1], and also [7,12].) The variety $\frac{R(\mathcal{V})}{R(\mathcal{V})}$ is determined by the ring $R(\mathcal{V})$, which may be constructed using a method described in [13, §7.1]. The affinization ring $R = R(\mathcal{T})$ of the variety \mathcal{T} is computed as follows. The ring R is a quotient of the ring $\mathbb{Z}[X]$, and the corresponding binary operation on an affine R-space is $x \cdot y = x(1-X) + yX$. The identity (5.1) holds for $x \cdot y$ in $R(\mathcal{T})$. It follows that

$$xy \cdot yx = [x(1-X) + yX](1-X) + [y(1-X) + xX]X$$
$$= x(1-2X+2X^2) + y(2X-2X^2)$$

is equal to

$$yx \cdot x = [y(1-X) + xX](1-X) + xX = x(2X - X^{2}) + y(1 - 2X + X^{2}).$$

Equating coefficients of x and y in the above expressions shows that

$$3X^2 - 4X + 1 = 0,$$

whence X = 1/3 or X = 1. If X = 1, then $x \cdot y = y$, and the groupoid operation is the operation of a right-zero semi-group. If X = 1/3, then R is a quotient of $\mathbb{Z}[1/3]$. Conversely, affine spaces over $\mathbb{Z}[1/3]$ are \mathcal{T} -groupoids under $x \cdot y = x(2/3) + y(1/3) = xy \underline{1/3}$. Thus $R = \mathbb{Z}[1/3] = R_3$. This is summarized as follows.

Proposition 5.3. The groupoids in \mathcal{T} that embed non-trivially into affine spaces as subreducts all belong to the quasivariety $Q(R_3^{\bullet})$.

It is clear that the variety \mathcal{V}_3 of Question 2.4 is the variety \mathcal{GM} of all groupoid modes, and the quasivariety of 3-modes coincides with $Q(R_3^{\bullet})$ and is (properly) contained in the variety \mathcal{T} . It follows that it cannot generate the variety \mathcal{V}_3 , and we have a negative answer to Question 2.4.

We are left with the following

Question 5.4. Does the quasivariety $Q(R_3^{\bullet})$ generate the variety \mathcal{T} ?

References

- [1] A. Brøndsted, An Introduction to Convex Polytopes, Springer-Verlag, New York, 1983.
- [2] G. Czédli and A. Romanowska, Generalized convexity and closure conditions, International Journal of Algebra and Computation, to appear.
- [3] A.I. Mal'cev, Algebraic Systems, Springer-Verlag, Berlin, 1973.
- [4] K. Matczak and A. Romanowska, Quasivarieties of cancellative commutative binary modes, Studia Logica 78 (2004), 321–335.
- K. Matczak, A.B. Romanowska and J.D.H. Smith, *Dyadic polygons*, International Journal of Algebra and Computation 21 (2011), 387–408. DOI:10.1142/80218196711006248

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- [6] W.D. Neumann, On the quasivariety of convex subsets of affine spaces, Arch. Math. 21 (1970), 11–16.
- [7] A. Pilitowska and A. Romanowska, Embedding modes into semimodules, Part I, Demonstratio Math. 44 (2011), 523–534.
- [8] K. Pszczoła, A. Romanowska and J.D.H. Smith, Duality for some free modes, Discuss. Math. General Algebra and Appl. 23 (2003), 45–62.
- [9] A.B. Romanowska and J.D.H. Smith, Modal Theory, Heldermann, Berlin, 1985.
- [10] A.B. Romanowska and J.D.H. Smith, On the structure of barycentric algebras, Houston J.Math. 16 (1990), 431–448.
- [11] A.B. Romanowska and J.D.H. Smith, Post-Modern Algebra, Wiley, New York, NY, 1999.
- [12] A.B. Romanowska and J.D.H. Smith, Embedding sums of cancellative modes into functorial sums of affine spaces, in Unsolved Problems on Mathematics for the 21st Century, a Tribute to Kiyoshi Iséki's 80th Birthday (J. M. Abe and S. Tanaka, eds.), IOS Press, Amsterdam, 2001, pp. 127–139.
- [13] A.B. Romanowska and J.D.H. Smith, Modes, World Scientific, Singapore, 2002.

Anna B. Romanowska Faculty of Mathematics and Information Science Warsaw University of Technology 00-661 Warsaw, Poland http://www.mini.pw.edu.pl/~aroman/ E-mail: aroman@mini.pw.edu.pl