TOLERANCES ON MONO-UNARY ALGEBRAS WITH LONG PRE-PERIODS

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Received May 9, 2012 ; revised February 26, 2013

Abstract. Iterating a unary operation defined on a finite set A one obtains a chain

$$A \supseteq Imf \supseteq Imf^2 \supseteq \cdots \supseteq Imf^m = Imf^{m+1},$$

where Imf is the image of f, i.e. the set of all images and $f^i := \underbrace{f \circ \cdots \circ f}_{i-times}$ is the $i-th$ iteration of f, $f^0 := id_A$. The least integer $\lambda(f)$ with $Imf^{\lambda(f)} = Imf^{\lambda(f)+1}$ is called the pre-period of f. Let n be the cardinality of A. Then the pre-period of f is an integer between 0 and $n-1$. If $\lambda(f) = n-1$, then f is said to be a long-tailed operation (LT-operation) and if $\lambda(f) = n-2$, then f is said to be an LT_1-operation. In [3] the authors characterized LT- and LT_1-operations and their invariant equivalence relations. In [4] these were generalized to partial operations and in [5] (see also [7]) to n-ary operations ($n > 1$). In this paper we study invariant tolerance relations of LT- and LT_1-operations. An algebra $(A; f)$ where f is a unary operation on A is said to be a mono-unary algebra. For the theory of mono-unary algebras we refer the reader to the monograph [9]. Here we study the tolerance lattices of mono-unary algebras $(A; f)$, where f is an LT- or an LT_1-operation. Tolerances on mono-unary algebras were considered in [10] (see also [8]).

1 Preliminaries

To make this text independent we first repeat some results on LT- and LT_1-operations from [3].

Theorem 1.1 [3] *Let $f : A \to A$ be a unary operation and $|A| = n \geq 2$. Then the following propositions are satisfied:*

(i) $\lambda(f) = n-1$ *if and only if there exists an element $d \in A$ such that*

$$A = \{d, f(d), f^2(d), \dots, f^{n-1}(d) = f^n(d)\} (\text{see e.g. [2]}) .$$

(ii) *Assume now that $n \geq 3$. Then $\lambda(f) = n-2$ and $|Imf^{n-2}| = 1$ if and only if there are distinct elements $u, v \in A$ such that $A = \{u, v, f(v), \dots, f^{n-2}(v)\}$ and such that there is an exponent k with $0 \leq k \leq n-2$ with $f(u) = f^{k+1}(v)$ and there is an integer m with $m + k = n-2$ with $f^{m+1}(u) = f^m(u)$.*

(iii) *We have $\lambda(f) = n-2$ and $|Imf^{n-2}| = 2$ if and only if there are different elements $u, v \in A$ such that either*

a)$A = \{v, u, f(u), \dots, f^{n-2}(u)\}$ with $v = f(v)$ and $f^{n-1}(u) = f^{n-2}(u)$, or b) $A = \{u, f(u), f^2(u) \dots, v = f^{n-2}(u), f^{n-1}(u)\}$ where $v = f^n(u) = f^{n-2}(u)$.

2010 *Mathematics Subject Classification.* 03B50, 08A30, 08A55.
Key words and phrases. unary operation, pre-period, tolerance.

If we choose $A = \{0, 1, \cdots, n-2, n-1\}, d = n-1, f(k) = k-1$ for $k \neq 0, f(0) = 0$, then f can be pictured as a directed graph where the vertices are labeled by the elements and there is an directed edge from x to y if $f(x) = y$.

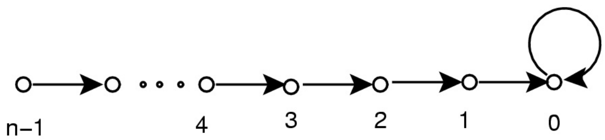

Figure 1: LT-operation

This graph is also called the graph of the mono-unary algebra $\mathcal{A} = (A; f)$. The following equations for an LT-operation f and $x, y \in A$ will be used many times.

$$f^y(x) = \begin{cases} x - y \text{ if } x \geq y \\ 0 \text{ if } x < y \end{cases} \qquad (*)$$

2 Tolerances on LT-algebras

A tolerance on a mono-unary algebra $(A; f)$ is a reflexive and symmetric binary relation T with the property that $(x, y) \in T$ implies $(f(x), f(y)) \in T$ for any $x, y \in A$. In a corresponding way, tolerances can be defined on arbitrary algebras. Congruences are transitive tolerances. For an element $a \in A$ let

$$[a]_T := \{x \in A \mid (a, x) \in T\}.$$

In [8] this set is called a class of T. Let $\leq$ be the usual order on $A = \{0, 1, \ldots, n-1\}$. The following facts are well-known and easy to prove (see [8], [10]).

Proposition 2.1 *Let $\mathcal{A} = (A; f)$ be an LT-algebra and let T be a tolerance on $\mathcal{A}$.*

(i) $[0]_T$ is convex with respect to $\leq$.

(ii) For any $x \neq y \in A$ if $T \neq \Delta_A$, there are distinct elements $u, v \in [0]_T$ such that $(u, v) \in T$.

(iii) Let $[0]_T = \{0, 1, \ldots, k-1\}$. If $T \neq \Delta_A$, then $|[0]_T| > 1$.

(iv) If T is a congruence, then $(x, y) \in T \Leftrightarrow \{x, y\} \subseteq [0]_T$ for all $x, y \in A$.

(v) For any $x, y \in A$, if $(x, y) \in T$ then $|x - y| < |[0]_T|$.

Proof. (i) We have to show: if $x \in [0]_T$ and $0 \leq y \leq x$, then $y \in [0]_T$. Because of $x \geq x - y$ by (*) we have $f^{x-y}(x) = x - (x - y) = y$. Now, $(0, x) \in T$ implies $(f^{x-y}(0), f^{x-y}(x)) = (0, y) \in T$.
(ii) We may assume that $\{x, y\} \not\subseteq [0]_T$, that $x < y$, and that $x \notin [0]_T$ since $y \notin [0]_T$ implies $x \notin [0]_T$ by (i). Let $[0]_T = \{0, 1, \ldots, k-1\}$. Then $f^{x-k+1}(x) = k-1$. If $y \in [0]_T$, then $(0, y) \in T$ and $(f^{x-k+1}(0), f^{x-k+1}(y)) = (0, f^{x-k+1}(y)) \in T$, i.e. $f^{x-k+1}(y) \in [0]_T$ and if $y \notin [0]_T$, then $x - k + 1 > y - k + 1$ implies that $f^{x-k+1}(y) \leq f^{y-k+1}(y) = k-1$, so $f^{x-k+1}(y) \in [0]_T$. Let $v := f^{x-k+1}(y)$. Now $(x, y) \in T$ implies $(k-1, v) = (f^{x-k+1}(x), f^{x-k+1}(y)) \in T$ and hence $u := k-1$ and v are the required elements.

(iii) Let $(x,y) \in T, x \neq y$. Then by (ii) there are elements $u, v \in \{0,1,\cdots,k-1\}, u \neq v$, such that $(u,v) \in T$. We may assume that $u \geq v$. Then $(f^v(u), f^v(v)) = (0, u-v) \in T$, where $0 < u - v < k$.
(iv) is clear.
(v) Suppose that there are elements $x, y \in A$ such that $(x,y) \in T$ and $|x-y| \geq |[0]_T|$. We may assume that $[0]_T = \{0,1,\ldots,k-1\}$. Then $|[0]_T| = k$ and $|x-y| = x-y$. Since $(x,y) \in T$, we have $(f^y(x), f^y(y)) = (x-y, 0) \in T$. By assumption and by (i) we get $(0,k) \in T$, hence $|[0]_T| \geq k+1$, a contradiction. ■

Let $Tol(\mathcal{A})$ denote the set of all tolerances of the mono-unary algebra $\mathcal{A}$. Clearly, the diagonal $\Delta_A = \{(a,a) \mid a \in A\}$ is contained in any $T \in Tol(\mathcal{A})$ and any $T \in Tol(\mathcal{A})$ is contained in the tolerance $A \times A$. The following properties of $Tol(\mathcal{A})$ for a mono-unary algebra $\mathcal{A}$ are well-known.

Proposition 2.2 ([10]) Let $(A; f)$ be an arbitrary mono-unary algebra. Then $Tol(\mathcal{A})$ forms an algebraic lattice with respect to set inclusion, which is a sublattice of the power set lattice $(\mathcal{P}(A \times A); \subseteq)$, and therefore, it is a distributive lattice.

Let $T(a,b)$ be the tolerance generated by the pair (a,b), i.e. the intersection of all tolerances on the given algebra $\mathcal{A}$ which contain the pair (a,b). Let $Con(\mathcal{A})$ be the congruence lattice of the algebra $\mathcal{A}$. Now we give two more properties of an LT-algebra.

Proposition 2.3 *Let $\mathcal{A}$ be an LT-algebra. Then*

(i) $\bigcap(Tol(\mathcal{A}) \setminus \{\Delta_A\}) = T(1,0)$.

(ii) *If $\alpha, \beta \in Con(\mathcal{A}) \setminus \{\Delta_A\}$, then $\alpha \cap \beta \neq \Delta_A$, i.e. $\mathcal{A}$ is subdirectly irreducible.*

Proof. (i) Let $T \in Tol(\mathcal{A}), T \neq \Delta_A$. Then there are elements $a, b \in A$ with $a \neq b$ and $(a,b) \in T$. Without loss of generality we may assume that $a < b$. There follows $(f^{b-1}(a), f^{b-1}(b)) = (0,1) \in T$, i.e. $T(0,1) \subseteq T$ for any $T \neq \Delta_A, T \in Tol(\mathcal{A})$. Then we have $T(0,1) \subseteq \bigcap(Tol(\mathcal{A}) \setminus \{\Delta_A\})$ and thus $\bigcap(Tol(\mathcal{A}) \setminus \{\Delta_A\}) = T(1,0)$.
(ii) In a similar way as in the proof of (i) we see that any congruence of $(A; f)$ contains the congruence generated by $(0,1)$ and therefore $\alpha, \beta \in Con(\mathcal{A}) \setminus \{\Delta_A\}$ implies $\alpha \cap \beta \neq \Delta_A$ and this means that $\mathcal{A}$ is subdirectly irreducible (see e.g. [1]). ■

By Proposition 2.1(iii) we have $|[0]_T| > 1$ for any $T \in Tol(\mathcal{A}), T \neq \Delta_A$ and by Proposition 2.1 (i) there is an element k with $1 < k \leq n$ such that $[0]_T = \{0,1,\ldots,k-1\}$. For each $1 \leq t \leq k-1$, there is the greatest integer $a_T^t \in A$ such that $(a_T^t, a_T^t + t) \in T$. We consider the set

$$B_T := \bigcup_{1 \leq t \leq k-1} \{(a_T^t - s, a_T^t + t - s) \mid 0 \leq s \leq a_T^t\}.$$

Since B_T is closed under application of f, the tolerance generated by B_T consists precisely of all pairs from B_T, all pairs from $B_T^d := \{(x,y) \in A \times A \mid (y,x) \in B_T\}$ and of all pairs from the diagonal Δ_A, i.e. $\langle B_T \rangle = B_T \cup B_T^d \cup \Delta_A$. By definition, $\langle B_T \rangle$ is the least tolerance containing B_T.

Lemma 2.4 (i) $\langle B_T \rangle = T$.

(ii) *If $T = T(m, m+1)$ is the tolerance generated by $(m, m+1)$ for $0 \leq m \leq n$,* *then* $B_T = \{(0,1),(1,2),(2,3),\cdots,(m,m+1)\}$.

(iii) *If* $T = T(a,b)$, *then there exists the greatest integer* m *such that*

$$B_T = \bigcup_{1\le t\le m} \{(a_T^t - s, a_T^t + t - s) \mid 0 \le s \le a_T^t\}.$$

(iv) $0 \le a_T^t \le n-t-1$ *for all* $1 \le t \le k-1$.

Proof. (i) We have $[0]_T = \{0,1,\ldots,k-1\}$. Clearly, $B_T \subseteq T$ and then $\langle B_T\rangle \subseteq T$. Assume that $(x,y) \in T$ with $x \neq y$. We may assume that $x > y$. Then there exists an element t with $0 < t < k$ such that $|x-y| = t$, i.e. $(x,y) \in \{(a_T^t - s, a_T^t + t - s) \mid 0 \le s \le a_T^t\} \subseteq B_T$ and then $T \subseteq \langle B_T\rangle$. Altogether, we have equality.
(ii) The set $\{(0,1),(1,2),\ldots,(m,m+1)\}$ can be written as $B_T = \bigcup_{1\le t\le 1} \{(a_T^1 - s, a_T^1 + 1 - s) \mid 0 \le s \le a_T^1\}$ with $a_T^1 = m, [0]_T = \{0,1\}$. We show that $T(m,m+1) = \langle B_T\rangle$. Indeed, $\{(0,1),(1,2),\ldots,(m,m+1)\} \subseteq T(m,m+1)$ implies $\langle B_T\rangle \subseteq T(m,m+1)$. Since $T(m,m+1)$ is the least tolerance containing $(m,m+1)$ and since $(m,m+1) \in \langle B_T\rangle$, we have $\langle B_T\rangle = T(m,m+1)$.
(iii) We may assume that $a > b$. Then $m = a - b$.
(iv) $(a_T^t, a_T^t + t) \in T$ implies $a_T^t + t \in A$, i.e. $a_T^t + t \le n-1$, i.e. $a_T^t \le n-t-1$. ■

To describe the tolerance lattice of $\mathcal{A}$ we classify all tolerances by the cardinality of $[0]_T = \{0,1,\ldots,k-1\}$.

Definition 2.5 Let $Tol_0(\mathcal{A}) = \{\Delta_A\}$ and for each $1 < k \le n$ let $Tol_{k-1}(\mathcal{A})$ be the set of all tolerances on $\mathcal{A}$ with $[0]_T = \{0,1,\ldots,k-1\}$.

By this definition, $Tol_1(\mathcal{A})$ is the set of all tolerances on $\mathcal{A}$ with $[0]_T = \{0,1\}$ and by Lemma 2.4, $Tol_1(\mathcal{A}) = \{T(m,m+1) \mid m = 0,1,\ldots,n-2\}$.

Let $\underline{n-1} := (\{0,\ldots,n-2\};\le)$ with the usual linear order $\le$ defined on A. The set $Tol_1(\mathcal{A})$ is partially ordered with respect to set inclusion.

Proposition 2.6 $(Tol_1(\mathcal{A});\subseteq) \cong \underline{n-1}$.

Proof. The mapping $g : \{0,1,\ldots,n-2\} \to Tol_1(\mathcal{A})$ defined by $g(m) = T(m,m+1)$ for all $0 \le m < n-1$ is order-preserving since $l \le m$ implies $T(l,l+1) \subseteq T(m,m+1)$. Conversely, from $T(l,l+1) \subseteq T(m,m+1)$ there follows $l \le m$. This shows that g is one-to-one, i.e. an order-isomorphism. Since by definition of $Tol_1(\mathcal{A})$ any tolerance from this set has the form $T(m,m+1)$ for $m \in \{0,1,\ldots,n-2\}$, the mapping g is onto and thus an order embedding. ■

Proposition 2.7 For $0 < k < n$, $Tol_k(\mathcal{A}) \cong \underline{n-k} \times Tol_{k-1}(\mathcal{A})$.

Proof. Let $T \in Tol_k(\mathcal{A})$. Then $[0]_T = \{0,1,\ldots,k-1\}$ and by Lemma 2.4 (i), $T = \langle B_T\rangle$ where $B_T = \bigcup_{1\le t\le k-1} \{(a_T^t - s, a_T^t + t - s) \mid 0 \le s \le a_T^t\}$. For $1 \le t \le k-1$, we consider the set $B_T^t := \{(a_T^t - s, a_T^t + t - s) \mid 0 \le s \le a_T^t\}$. We define $T_{k-1} := T \setminus (B_T^{k-1} \cup (B_T^{k-1})^d)$. Then $T_{k-1} \in Tol_{k-1}(\mathcal{A})$ and by Lemma 2.4(ii) we have $a_T^{k-1} \le n-(k-1)-1 = n-k$,

i.e. $a_T^{k-1} \in \underline{n-k}$ $(= \{0, 1, \ldots, n-k-1\}$. Therefore, the pair (a_T^{k-1}, T_{k-1}) belongs to the cartesian product $\underline{n-k} \times Tol_{k-1}(\mathcal{A})$. Let $c \in \underline{n-k}$ and $\alpha \in Tol_{k-1}(\mathcal{A})$. Define $\bar{\alpha} := \langle B_\alpha \rangle$ where $B_\alpha = \{(c-s, c+k-1-s) \mid 0 \leq s \leq c\}$. Then $[0]_{\bar{\alpha}} = \{0, 1, \ldots, k-1\}$ and thus $\bar{\alpha} \in Tol_k(\mathcal{A})$, where $c = a_{\bar{\alpha}}^{k-1}$ and $\bar{\alpha}_{k-1} = \alpha$.
Define $g : Tol_k(\mathcal{A}) \to \underline{n-k} \times Tol_{k-1}(\mathcal{A})$ by $g(T) = (a_T^{k-1}, T_{k-1})$. Since $[0]_\alpha = [0]_\beta$, we have $a_\alpha^{k-1} \leq a_\beta^{k-1}$, hence $B_\alpha^{k-1} \subseteq B_\beta^{k-1}$. Let $B_\alpha := B_\alpha^{k-1} \cup (B_\alpha^{k-1})^d$ and $B_\beta := B_\beta^{k-1} \cup (B_\beta^{k-1})^d$. Since a_α^{k-1} is the greatest element such that $(a_\alpha^{k-1}, a_\alpha^{k-1} + k - 1) \in \alpha$, we have $(B_\beta \setminus B_\alpha) \cap \alpha = \emptyset$ and then $\alpha_{k-1} = \alpha \setminus B_\alpha = \alpha \setminus B_\beta \subseteq \beta \setminus B_\beta = \beta_{k-1}$. Therefore, $g(\alpha) = (a_\alpha^{k-1}, \alpha_{k-1}) \leq (a_\beta^{k-1}, \beta_{k-1}) = g(\beta)$, where $\leq$ is defined component-wise, using the usual order on the natural numbers for the first component and set-inclusion for the second one. Now assume that $\alpha, \beta \in Tol_k(\mathcal{A})$ such that $g(\alpha) \leq g(\beta)$, i.e. $a_\alpha^{k-1} \leq a_\beta^{k-1}$ and $\alpha \subseteq \beta$. Then we have $B_\alpha \subseteq B_\beta$ and $\alpha = \alpha_{k-1} \cup B_\alpha \subseteq B_{k-1} \cup B_\beta = \beta$. Hence, g is an order-embedding and together with surjectivity we have an order isomorphism. ∎

Our construction has the following consequences:

Corollary 2.8 (i) *$Tol_k(\mathcal{A})$ is a product of chains: $Tol_k(\mathcal{A}) \cong \prod_{t=1}^{k} \underline{n-t}$ for all $0 \leq k \leq n$, especially, $Tol_{n-1}(\mathcal{A}) \cong \underline{1} \times \ldots, \underline{n-2} \times \underline{n-1}$.*

(ii) *$Tol_k(\mathcal{A}) \cap Tol_l(\mathcal{A}) = \emptyset$ for all $0 \leq k \neq l \leq n-1$ and $Tol(\mathcal{A}) = \bigcup_{0 \leq k \leq n-1} Tol_k(\mathcal{A})$.*

(iii) *$Tol_k(\mathcal{A})$ is isomorphic to a sublattice of $Tol(\mathcal{A})$ for all $0 \leq k \leq n-1$ and $Tol_{k-1}(\mathcal{A})$ is isomorphic to a sublattice of $Tol_k(\mathcal{A})$ for all $0 < k < n$.*

(iv) *$Tol_k(\mathcal{A}) \cong \pi_1(Tol_{k+1}(\mathcal{A}))$ where π_1 is the first projection.*

The ordered sum (see e.g. [6]) of a family of partially ordered sets will be denoted by the symbol $\sum$ and if the family consists of finitely many partially ordered sets we will also use the symbol $\overset{+}{\sqcup}$

Theorem 2.9 *The tolerance lattice of an LT-algebra is isomorphic to an ordered sum*

$$Tol(\mathcal{A}) \cong \sum_{0 \leq k \leq n-1} Tol_k(\mathcal{A}) \cong \sum_{0 \leq k \leq n-1} \prod_{t=1}^{k} \underline{n-t}.$$

Let $\alpha, \beta \in Tol(\mathcal{A})$. We define an equivalence relation on $Tol(\mathcal{A})$ as follows:

$$\alpha \sim \beta :\Leftrightarrow [0]_\alpha = [0]_\beta.$$

Then the quotient set $Tol(\mathcal{A})/\!\sim$ corresponds to the set $\{[0]_\alpha \mid \alpha \in Tol(\mathcal{A})\}$ which can be regarded as a linearly ordered set that is isomorphic to the congruence lattice of $(A; f)$. Altogether we have:

Corollary 2.10 *Let $\mathcal{A} = (A; f)$ be an LT-algebra. Then*

$$\underline{n-1} \;\cong\; Tol_1(\mathcal{A}) \;\cong\; Tol(\mathcal{A})/\!\sim \;\cong\; Con(\mathcal{A}).$$

Example 2.11 Let $A = \{0, 1, 2, 3\}$ and let $f : A \to A$ be given by the following table

x	$f(x)$	$f^2(x)$	$f^3(x)$
0	0	0	0
1	0	0	0
2	1	0	0
3	2	1	0.

Then
$Tol_0(\mathcal{A}) = \{\Delta_A\}$,
$Tol_1(\mathcal{A}) = \{T(0,1), T(1,2), T(2,3)\}$,
$Tol_2(\mathcal{A}) = \{T(0,2), T((0,2),(1,2)), T((0,2),(2,3)), T(1,3),$
$T((1,3),(1,2)), T((1,3),(2,3))\}$,
$Tol_3(\mathcal{A}) = \{T(0,3), T((0,3),(1,3)), T((0,3),(1,2)), T((0,3),(2,3)),$
$T((0,3),(1,3),(1,2)), T((0,3),(1,3),(2,3))\}$.

Then the tolerance lattice of $(A; f)$ is given by Figure 2.

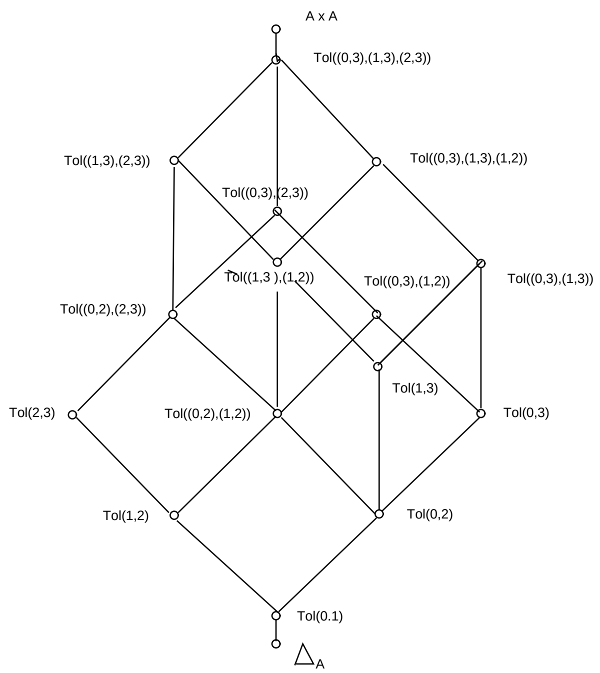

Figure 2: Tolerance Lattice of an LT-algebra

3 Tolerances on LT_1-algebras with $|Imf^{n-2}| = 2$

Corresponding to Theorem 1.1 (iii) we have to consider the cases that f has two fixed points and that f has no fixed point. For the first case, let $A = \{v, u, f(u), \ldots, f^{n-2}(u)\}$ where $v = f(v)$ and $f^{n-1}(u) = f^{n-2}(u), |A| = n \geq 3$ and $\lambda(f) = n - 2$. Without restriction of the generality we may assume that $A = \{0, 0', 1, 2, \ldots, n-2\}$ and $f(k) = k - 1$ if $k \notin \{0, 0'\}$ and $f(0) = 0, f(0') = 0'$. Let $B := \{0, 1, \ldots, n-2\}$ and let $f|B$ be the restriction of f onto B. Clearly, $f|B$ is an LT-operation on B. Then, Figure 3 shows

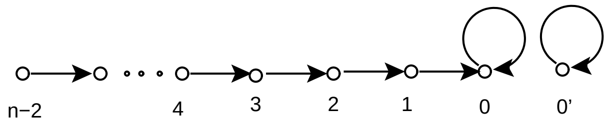

Figure 3: $LT1$-operation

the graph of the $LT1$-algebra $(A; f)$. Let $\leq$ be the usual order on $\{0, 1, \dots, n-2\}$. For an $LT1$-operation we have $f^y(x) = x - y$ if $x \geq y$, $f^y(0') = 0'$ for all y and $f^y(x) = 0$ if $0 \leq x < y$. For $T \in Tol(\mathcal{A})$ let T_B be the restriction of T onto B. If $|[0']_T| > 1$ let denote $[0']_T$ by $[0, 0']_T$ since in the next proposition we will prove that in this case $0 \in [0']_T$.

Proposition 3.1

(i) $[0]_T$ *is convex with respect to* $\leq$.

(ii) $[0']_T = \{0'\}$ *or* $[0']_T \setminus \{0'\}$ *is convex with respect to* $\leq$ *and* $0 \in [0']_T$.

(iii) $T_B \in Tol(\mathcal{B})$ *and* $T = T_B \cup T_{[0']_T}$.

(iv) *If* $0' \notin [0]_T$, *then* $[0]_T = [0]_{T_B}$.

(v) *There exists the greatest element* $k \in B$ *such that* $[0, 0']_T = \{0'\} \cup \{x \in B \mid x < k$ *and* $(x, 0') \in T\} = \{0', 0, 1, \dots, k-1\}$. *This* k *is denoted by* $(0_T)'$.

(vi) *For all* $\alpha, \beta \in Tol(\mathcal{A})$, *if* $\alpha \subseteq \beta$, *then* $[0]_\alpha \subseteq [0]_\beta, [0']_\alpha \subseteq [0']_\beta, [0, 0']_\alpha \subseteq [0, 0']_\beta$ *and* $\alpha_B \subseteq \beta_B$.

(vii) For all $\alpha, \beta \in Tol(\mathcal{A})$ we have $(0_\alpha)' \leq (0_\beta)' \Leftrightarrow [0, 0']_\alpha \subseteq [0, 0']_\beta$.

Proof. (i) can be proved in the same way as Proposition 2.1(i).
(ii) Assume that $[0']_T \neq \{0'\}$. Let $x \in [0']_T, x \neq 0'$ and $0 \leq y \leq x$. Then $(x, 0') \in T$ implies $(f^{x-y}(x), f^{x-y}(0')) = (y, 0') \in T$ and $(f^x(x), f^x(0')) = (0, 0') \in T$.
(iii) $T_B \in Tol(\mathcal{B})$ is clear. Since $T_B \subseteq T$ and $T_{[0']_T} \subseteq T$, we have $T_B \cup T_{[0']_T} \subseteq T$. Let $(x, y) \in T$. If $\{x, y\} \subseteq B$, then $(x, y) \in T_B$. If $\{x, y\} \not\subseteq B$, we may assume that $x = 0'$, so $x, y \in [0']_T$, hence $(x, y) \in T_{[0']_T}$.
(iv) is clear.
(v) Since $(0, 0') \in T$, there exists the greatest integer $k \in B$ such that $(0', k-1) \in T$. By (ii) we have $0 \leq x \leq k-1 \Leftrightarrow (0', x) \in T$. (vi) is clear.
(vii) We have $(0_\alpha)' \leq (0_\beta)' \Leftrightarrow [0, 0'] = \{0', 0, 1, \dots, (0_\alpha)' - 1\} \subseteq \{0', 0, 1, \dots, (0_\beta)' - 1\} = [0, 0']_\beta$ for all $\alpha, \beta \in Tol(\mathcal{A})$. ■

We define $\mathcal{C}_{0'} := \{[0']_T \mid T \in Tol(\mathcal{A})\}$. Then

Proposition 3.2 (i) $\mathcal{C}_{0'} = \{\{0'\}\} \cup \{[0, 0']_T \mid T \in Tol(\mathcal{A})\}$.

(ii) $(\mathcal{C}_{0'}; \subseteq) \cong \underline{n-1}$.

Proof. (i) The inclusion $\{\{0'\}\} \cup \{[0, 0']_T \mid T \in Tol(\mathcal{A})\} \subseteq \mathcal{C}_{0'}$ is obvious. Let $[0']_T \in \mathcal{C}_{0'}$. We may assume that $|[0']_T| > 1$. Then $0 \in [0']_T$ and $[0']_T = [0, 0']_T \in \mathcal{C}_{0'}$.
(ii) We consider the mapping $g : \mathcal{C}_{0'} \to \underline{n-1}$ defined by $\{0'\} \mapsto 0$ and $[0, 0']_T \mapsto (0_T)' + 1$. By Proposition 3.1 (vii) g is an order-embedding. We show that g is surjective. Let $k \in \underline{n-1}$. If $k = 0$, then $g(\{0'\}) = 0$ and if $k \in \{1, \dots, n-1\}$, then $k - 1 \in \{0, \dots, n-2\} = B$. Let denote $T_k := \{(x, 0') \mid x < k\}$ and define $T \subseteq A \times A$ by $T := \Delta_A \cup T_k \cup T_k^d$. Then

$T \in Tol(\mathcal{A})$ with $(0_T)' = k-1$. So $g(T) = (0_T)' + 1 = k - 1 + 1 = k$. Hence, g is an order-isomorphism. ■

Theorem 3.3 *For the tolerance lattice of an LT_1-algebra we have*

$$Tol(\mathcal{A}) \cong \underline{n-1} \times Tol(\mathcal{B}) \cong \underline{n-1} \times \sum_{0 \le k \le n-1} Tol_k(\mathcal{B}) \cong \underline{n-1} \times \sum_{0 \le k \le n-1} \prod_{t=1}^{k} \underline{n-t}.$$

Proof. We prove that $Tol(\mathcal{A}) \cong \mathcal{C}_{0'} \times Tol(\mathcal{B})$. Let $g : Tol(\mathcal{A}) \to \mathcal{C}_{0'} \times Tol(\mathcal{B})$ be defined by $T \mapsto ([0']_T, T_B)$ for all $T \in Tol(\mathcal{A})$. By Proposition 3.1 (vi), g is order-preserving. Now, let $\alpha, \beta \in Tol(\mathcal{A})$ such that $g(\alpha) \le g(\beta)$. Then $[0']_\alpha \subseteq [0']_\beta$ and $\alpha_B \subseteq \beta_B$. By Proposition 3.1 (iii) it remains to prove that $\alpha_{[0']_\alpha} \subseteq \beta_{[0']_\beta}$. Let $(x, y) \in \alpha_{[0']_\alpha}$. Then $\{x, y\} \subseteq [0']_\alpha$ and $(x, y) \in \alpha$. Therefore, $\{x, y\} \subseteq [0']_\beta$. We will prove that $(x, y) \in \beta$. If $x = y$ or $0' \in \{x, y\}$, then $(x, y) \in \beta$ since β is reflexive and $\{0', x\} \subseteq [0']_\beta$ or $\{0', y\} \subseteq [0']_\beta$. Thus we may assume that $x \neq y$ and $0' \notin \{x, y\}$. Then $\{x, y\} \subseteq B$. Hence $(x, y) \in \alpha$ and $\{x, y\} \subseteq B$ implies that $(x, y) \in \alpha_B \subseteq \beta_B$ and $(x, y) \in \beta$. The rest follows from Theorem 1.10, the fact that $(B; f|B)$ is an LT-algebra with $|B| = n - 2$ and from Proposition 2.3. ■

Example 3.4 Let $A = \{0', 0, 1, 2\}$ and let $f : A \to A$ be given by the table

x	$f(x)$	$f^2(x)$	$f^3(x)$
$0'$	$0'$	$0'$	$0'$
0	0	0	0
1	0	0	0
2	1	0	0.

Then the tolerance lattice can be pictured as in Figure 4.

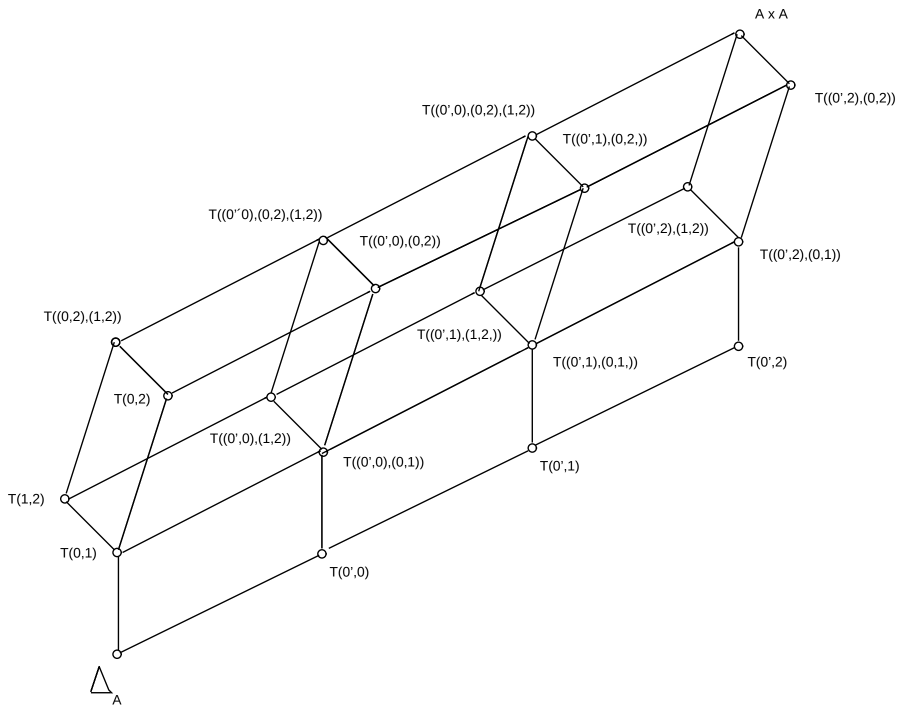

Figure 4: Tolerance Lattice of an LT_1-algebra

In the second case, $\lambda(f) = n-2, |Imf^{n-2}| = 2$ and f has no fixed points. Let $A = \{u, f(u), \ldots, f^{n-2}(u), f^{n-1}(u)\}, n \geq 3$ and $f^n(u) = f^{n-2}(u)$. Without restriction of the generality we may assume that $A = \{0, 1, 2, \ldots, n-2, n-1\}$ with $f(k) = k-1$ if $k \neq 0$ and $f(0) = 1$. Then Figure 5 shows the graph of the LT_1-algebra $(A; f)$.

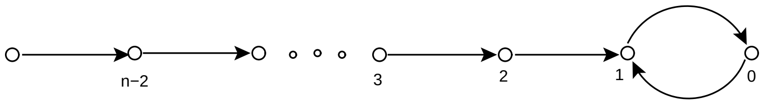

Figure 5: LT_1-operation

Let $\leq$ be the usual order on $\{0, 1, \ldots, n-1\}$. For an LT_1-operation of this kind we have $f^y(x) = x - y$ if $x \geq y$ and

$$f^y(x) \;=\; \begin{cases} 0 & \text{if} \quad x < y \quad \text{and} \quad y - x \quad \text{is even} \\ 1 & \text{if} \quad x < y \quad \text{and} \quad y - x \quad \text{is odd} \end{cases}$$

We consider the following set of tolerances on $\mathcal{A} : Tol_0(\mathcal{A}) = \{\Delta_A\}$ and let $Tol_k(\mathcal{A})$ be the set of all tolerance relations on $(A; f)$ such that k is the greatest integer in $A \setminus \{0\}$ with $(0, k) \in T$. Let $[\frac{n}{2}]$ be the greatest integer which is smaller than $\frac{n}{2}$. Then we have

Lemma 3.5 (i) *Let $0 < k < n$ and $T \in Tol_k(\mathcal{A})$. Then $|x - y| < k$ if $(x, y) \in T$ for all $x, y \in A$.*

(ii) *For all $T \in Tol_1(\mathcal{A})$ there is the greatest integer $m \in A$ such that $T = \Delta_A \cup \{(0,1), \ldots, (m, m+1)\} \cup \{(0,1), \ldots, (m, m+1)\}^d$ and $Tol_1(\mathcal{A}) \cong \underline{n-1}$.*

(iii) *For $0 \leq k \leq [\frac{n}{2}]$, if $T \in Tol_{2k+1}(\mathcal{A})$, then $(0,1) \in T$ and if $T \in Tol_{2k}(\mathcal{A})$, then $(0,2) \in T$.*

(iv) *If $T \in Tol(\mathcal{A})$, then there exists the greatest non-negative integer k such that $T \in Tol_k(\mathcal{A})$ and for each $0 < t \leq k$ there exists the greatest element $a_T^t \in A$ such that $(a_T^t, a_T^t + t) \in T$.*

(v) *$T = \langle B_T \rangle$ where $B_T = \bigcup\limits_{0 \leq t \leq k} \{(a_T^t - s, a_T^t + t + s) \mid 0 \leq s \leq a_T^t\}$.*

Proof. (i) Suppose that there are $x > y$ in A such that $(x, y) \in T$ and $x - y \geq k$. Then $(x, y) \in T$ implies $(f^y(x), f^y(y)) = (x - y, 0) \in T$. Therefore, there exists an integer t with $t \geq k$ and $(0, t) \in T$, which contradicts $T \in Tol_k(\mathcal{A})$.
(ii) This follows in a similar way as the corresponding proposition in section 1 and $Tol_1(\mathcal{A}) \cong \underline{n-1}$ can be proved by using the mapping g with $g(k) = T(k, k+1), k \in \{0, 1, \ldots, n-2\}$.
(iii) By definition of $T_{2k+1}(\mathcal{A})$ we have $(0, 2k+1) \in T$ and this implies

$$(f^{2k+1}(0), f^{2k+1}(2k+1)) = (1, 0).$$

If $T \in Tol_{2k}(\mathcal{A})$, then $(0, 2k) \in T$ implies

$$(f^{2k-2}(0), f^{2k-2}(2k)) = (0, 2k - (2k-2)) = (0, 2) \in T.$$

(iv) Let $T \in Tol(\mathcal{A}) \setminus \{\Delta_A\}$. Then there are elements $x \neq y$ in A such that $(x, y) \in T$. Without restriction of the generality we may assume that $y < x$ and $x = y + m$ for some

$m \geq 1$; hence $(x,y) \in T$ implies $(f^y(x), f^y(y)) = (x-y, 0) = (m, 0) \in T$, i.e. there exists an element $t \geq 1$ such that $(0,t) \in T$. Let $k \geq 1$ be the greatest integer in A such that $(0,k) \in T$. The second proposition is clear.

(v) This follows in the same way as Proposition 1.5. ■

Let

$\mathcal{T}^0 := \bigcup_{0 \leq k \leq [\frac{n}{2}]} \{T \in Tol_{2k}(\mathcal{A}) \mid (0,1) \notin T\}$,

$\mathcal{T}^1 := \bigcup_{0 \leq k \leq [\frac{n}{2}]} \{T \in Tol_{2k+1}(\mathcal{A}) \mid (0,2) \notin T\}$ and

$\mathcal{T}^2 := (\mathcal{T}^0 \wedge \mathcal{T}^1) \cup (\mathcal{T}^0 \vee \mathcal{T}^1)$, where

$\mathcal{T}^0 \wedge \mathcal{T}^1 := \{T \cap S \mid T \in \mathcal{T}^0 \text{ and } S \in \mathcal{T}^1\}$, $\mathcal{T}^0 \vee \mathcal{T}^1 := \{T \cup S \mid T \in \mathcal{T}^0, S \in \mathcal{T}^1\}$.

For each $0 \leq k \leq [\frac{n}{2}]$, let denote $\mathcal{T}^0_k := \mathcal{T}^0 \cap Tol_{2k}(\mathcal{A})$ and $\mathcal{T}^1_k := \mathcal{T}^1 \cap Tol_{2k+1}(\mathcal{A})$. Then we have

Proposition 3.6 (i) *For each* $0 < k \leq [\frac{n}{2}]$ *and for each* $T \in \mathcal{T}^0_k$, *if* $(x,y) \in T$, *then* $|x-y| = 2m$ *for some* $0 \leq m \leq k$.

(ii) *For each* $0 < k \leq [\frac{n}{2}]$ *and for each* $T \in \mathcal{T}^1_k$, *if* $(x,y) \in T$, *then* $|x-y| = 2m+1$ *for some* $0 \leq m \leq k$.

(iii) *For* $0 \leq k \leq [\frac{n}{2}]$, $\mathcal{T}^1_k \cong \underline{n-k-1} \times P_k$ *where* $P_0 := (\{0,1\}; \leq)$ *and* $P_k := \prod_{1 \leq t \leq k} \underline{n-2t-1}$ *and* $\mathcal{T}^0_k \cong \underline{n-k-1} \times Q_k$ *where* $Q_0 := (\{0,1\}; \leq)$ *and* $Q_k := \prod_{1 \leq k \leq k} \underline{n-2t-2}$.

Proof. (i) Assume that $(x,y) \in T$. If $x = y$, then we choose $m = 0$. Now we may assume that $x \neq y$. Since $\mathcal{T}^0_k \subseteq \mathcal{T}^0$, we have $(0,1) \notin T$. Suppose that $|x-y| = 2m+1$ for some $0 \leq m \leq k$. We may assume that $x = y+2m+1$. Then $(x,y) = (y+2m+1, y) \in T$ implies $(f^{y+2m+1}(y+2m+1), f^{y+2m+1}(y)) = (0,1) \in T$, a contradiction.

(ii) can be proved similar to (i).

(iii) $\mathcal{T}^1_1 = Tol_1(\mathcal{A}) \cong \underline{n-1} \cong \underline{n-1} \times \underline{1}$ is clear by Proposition 1.7. So, the proposition is true if $k = 0$. Assume that $\mathcal{T}^1_k \cong \underline{n-k-1} \times P_k$ for $k \geq 0$. We notice that $T \cup \{(0,2k+3),(2k+3,0)\} \in \mathcal{T}^1_{k+1}$ for all $T \in \mathcal{T}^1_k$ and define $\varphi : \mathcal{T}^1_k \to \mathcal{T}^1_{k+1}$ by $\varphi(T) = T \cup \{(0,2k+3),(2k+3,0)\}$ for all $T \in \mathcal{T}^1_k$. Clearly, φ is order-preserving. Since $(0,2k+3)$ implies $(f(0), f(2k+3)) = (1, 2k+2) \in T$ and this implies $(0,2k+1) \in T$ for all $T \in \mathcal{T}^1_{k+1}$, we have $\varphi(T(1,2k+2)) = \varphi(T(0,2k+1)) = T(0,2k+3)$ for $T(1,2k+2), T(0,2k+1)) \in \mathcal{T}^1_k$ and $T(0,2k+3) \in \mathcal{T}^1_{k+1}$ and hence $\varphi(\mathcal{T}^1_k)) \cong \mathcal{T}^1_k/ker\varphi \cong \underline{n-k-2} \times P_k$. Let $T \in \mathcal{T}^1_{k+1}$. Then $k+1$ is the greatest integer such that $k+1 \leq [\frac{n}{2}]$ and $(0,2k+3) \in T$. We have $T = \langle B^{k+1}_T \rangle$, where $B^{k+1}_T = \bigcup_{0 \leq m \leq k+1} \{(a^{2m+1}_T - s, a^{2m+1}_T + 2m+1-s) \mid 0 \leq s \leq a^{2m+1}_T\}$. Since $a^{2k+3}_T + 2(2k+1)+1 \leq n-1$ and $a^{2k+3}_T \leq n-1-2k-3 = n-2k-4$, we get $a^{2k+3}_T \in \underline{n-2k-3}$. Let $T' := T \setminus \{(a^{2k+3}_T - s, a^{2k+3}_T + 2k+3-s) \mid 0 \leq s \leq a^{2k+3}_t\}$. Then $T' \in \mathcal{T}^1_k$.

Let $g : \mathcal{T}^1_{k+1} \to \varphi(\mathcal{T}^1_k) \times \underline{n-2k-3}$ be defined by $g(T) := (\varphi(T'), a^{k+1}_T)$ for all $T \in \mathcal{T}^1_{k+1}$. If $T_1 \subseteq T_2$ in $\mathcal{T}^1_{k+1}$, then $T'_1 \subseteq T'_2$ and $a^{2k+3}_{T_1} \leq a^{2k+3}_{T_2}$. Conversely, if $g(T_1) \subseteq g(T_2)$ for $T_1, T_2 \in \mathcal{T}'_{k+1}$, then $\varphi(T'_1) \subseteq \varphi(T'_2)$ and $a^{2k+3}_{T_1} \leq a^{2k+3}_{T_2}$. Hence, $[T'_1]_{ker\varphi} \subseteq [T'_2]_{ker\varphi}$, which implies that $T'_1 \subseteq T'_2$ (using $\varphi(\mathcal{T}^1_k) \cong \mathcal{T}^1_k/ker\varphi$). Therefore, $T_1 = T'_1 \cup B^{k+1}_{T_1} \cup (B^{k+1}_{T_1})^d \subseteq T'_2 \cup B^{k+1}_{T_2} \cup (B^{k+1}_{T_2})^d = T_2$. Let $T \in \mathcal{T}^1_k$ and let $c \in \underline{n-2k-3}$ and let $T = \varphi(T) \cup B_c \cup B^d_c$ where $B_c := \{(c-s, c+2k+3-s) \mid 0 \leq s \leq c\}$. Then $\bar{T} \in \mathcal{T}^1_{k+1}$ where $a^{k+1}_T = c$ and $\varphi(T) = \bar{T}$. Then we have

$$\begin{array}{rcl} \mathcal{T}^1_{k+1} & \cong & \varphi(\mathcal{T}^1_k) \times \underline{n-2k-3} \\ & \cong & (\underline{n-k-2} \times \prod\limits_{1\le t\le k} \underline{n-2t-1}) \times \underline{n-2k-3} \\ & \cong & \underline{n-k-2} \times [(\prod\limits_{1\le t\le k} \underline{n-2t-1}) \times \underline{n-2k-3}] \\ & \cong & \underline{n-(k+1)-1} \times P_{k+1} \end{array}$$

(iv) can be proved in a similar way. ■

On $\mathcal{T}^1$, a partial order can be defined by:

$$a \le b :=\Leftrightarrow \begin{cases} a, b \in \mathcal{T}^1_k & \text{and } a \le_k b \text{ for some } 0 \le k \le [\frac{n}{2}] \text{ or} \\ a \in \mathcal{T}^1_k, b \in \mathcal{T}^1_{k+1} & \text{for some } 0 \le k \le [\frac{n}{2}] \end{cases} .$$

On $\mathcal{T}^0$ a partial order can be defined in a similar way. Finally, we have the following result:

Theorem 3.7 *For the tolerance lattice of an LT_1-algebra $(A; f)$ with $|Imf^{n-1}| = 2$ and f has no fixed point we have*

$$(Tol(\mathcal{A}); \subseteq) = \mathcal{T}^0 \cup \mathcal{T}^1 \cup \mathcal{T}^2$$

with $\mathcal{T}^0 \cong \sum\limits_{0\le k\le[\frac{n}{2}]} \mathcal{T}^0_k$ *and* $\mathcal{T}^1 \cong \sum\limits_{0\le k\le[\frac{n}{2}]} \mathcal{T}^1_k$.

Proof. It remains to prove that $Tol(\mathcal{A}) \subseteq \mathcal{T}^0 \cup \mathcal{T}^1 \cup \mathcal{T}^2$. Let $T \in Tol(\mathcal{A})$ and assume that there are elements $x, y, u, v \in A$ such that $(x, y) \in T$ and $(u, v) \in T$ with $|x - y| = 2t + 1, |u + v| = 2s$ for some $t \ge 0, s \ge 1$. Let k and m be the greatest integers with such properties, let $T_1 = \langle B_k\rangle, T_0 = \langle B_m\rangle$, where
$B_k = \bigcup\limits_{0\le t\le k} \{(a^{2t+1}_T - s, a^{2t+1}_t + 2t + 1 - s) \mid 0 \le s \le a^{2t+1}_T\}$
and $B_m = \bigcup\limits_{0\le t\le m} \{(a^{2t}_T - s, a^{2t}_T + 2t - s) \mid 0 \le s \le a^{2t}_T\}$.
Then $T_0 \in \mathcal{T}^0$ and $T_1 \in \mathcal{T}^1$ and $T = T_0 \vee T_1$. ■

Example 3.8 Let $A = \{0, 1, 2, 3\}$ and let $f : A \to A$ be given by the table

x	$f(x)$	$f^2(x)$	$f^3(x)$	$f^4(x)$
0	1	0	1	0
1	0	1	0	1
2	1	0	1	0
3	2	1	0	1.

Then the tolerance lattice can be pictured by Figure 6.

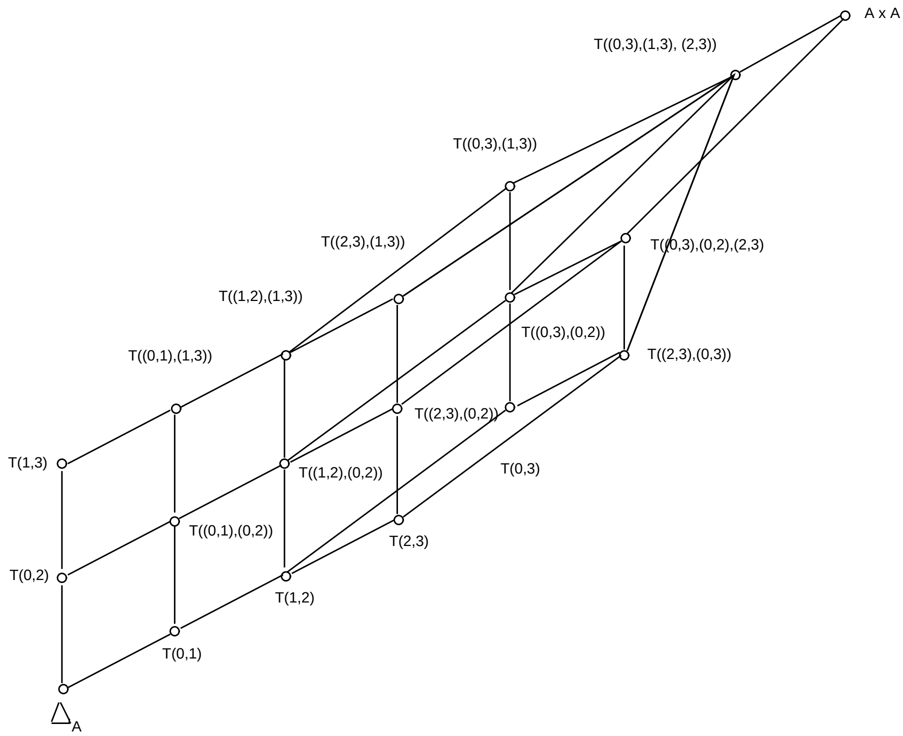

Figure 6: Tolerance Lattice of an LT_1-algebra

4 Tolerances on LT_1-algebras with $|Imf^{n-2}| = 1$

Let A be a set with $|A| = n \geq 3$ and $A = \{v, u, f(u), \ldots, f^{n-2}(u)\}$, where $f^{n-1}(u) = f^{n-2}(u)$ and $f(v) = f^m(u)$ for some $0 \leq m < n-2$. Without restriction of the generality we may assume that $A = \{0, 1, \ldots, m-1, m, m+1, \ldots, n-1\}$ and that $f(t) = t-1$ if $t \notin \{0, n-1\}, f(0) = 0$ and $f(n-1) = m$ for some $0 \leq m < n-2$.
Then Figure 7 shows the graph of the LT_1-algebra $(A; f)$.

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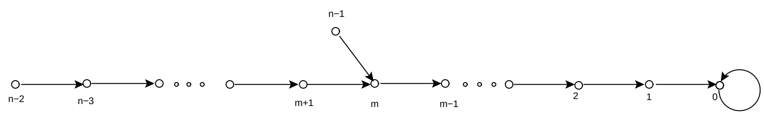

Figure 7: LT_1-algebra with $|Imf^{n-2}| = 1$

For an LT_1-operation with $|Imf^{n-2}| = 1$ we have $f^y(x) = x - y$ if $x \geq y, x < n-1, f^y(n-1) = m-(y-1)$ and $f^y(x) = 0$ if $x < y$.

We introduce the following notation:

$\mathcal{T} := \{T \in Tol(\mathcal{A}) \mid (m+1, n-1) \notin T\}$
and if $m > 0$, then we set $\mathcal{T}_0 := \{\Delta_A\}$, and for each $1 \leq k \leq n-2$ we define $\mathcal{T}_k := \mathcal{T} \cap Tol_k(\mathcal{A})$. Moreover, let $\mathcal{T}_k(\mathcal{A}) := \{T \in Tol(\mathcal{A}) \mid [0]_T := \{0, \ldots, k-1\}\}$. Clearly, $\mathcal{T}_k$ is a sublattice of $\mathcal{T}$. The following lemma turns out to be very useful for our next considerations.

Lemma 4.1 *Let $1 \leq m < n-2, k \neq m$, and assume that $T(m-k+1, n-1)$ or $T(n-1, m+k+1)$ are in $\mathcal{T}_k$. Let $\mathcal{X}$ and $\mathcal{Y}$ be disjoint isomorphic sublattices of $\mathcal{T}_k$. If $Y = \{T \cup T(m-k+1, n-1) \mid T \in X\}$ or $Y = \{T \cup T(n-1, m+k+1) \mid T \in X\}$, then $\mathcal{X} \dot\cup \mathcal{Y} \cong \underline{2} \times \mathcal{X}$.*

Proof. Define $g : \mathcal{X} \dot\cup \mathcal{Y} \to \underline{2} \times \mathcal{X}$ by

$$g(T) := \begin{cases} (0, T) & \text{if } T \in X \\ (0, T') & if T \in Y \text{ and } T = T' \cup T(m-k+1, n-1) \end{cases}$$

in the first case or

$$g(T) := \begin{cases} (0, T) & \text{if } T \in X \\ (0, T') & if T \in Y \text{ and } T = T' \cup T(n-1, m+k+1) \end{cases}$$

in the second one. It is clear that g is an isomorphism. ∎

For each $k \geq 1, i \geq 0$ and $T \in \mathcal{T}_k$, let $B_T^i := \{(i+t, i+t+k) \mid 0 \leq t \leq i\}$. As we have shown in section 1 we have $T(i, i+k) = \Delta_A \cup B_T^i \cup (B_T^i)^d$. Let $B := \{T(i, i+k) \mid 0 \leq i \leq n-k-2\}$. For $k \neq m$ let $C := \{T \cup T(m-k+1, n-1) \mid T \in B\}$ and $D := \{T \cup T(n-1, m+k+1) \mid T \in B \cup C\}$ and let $\bar{\mathcal{T}}_k := B \cup C \cup D$.

If $k = m$, one can see that $T(0, n-1) \supset T(0, m) \subset T(1, n-1)$. Moreover, we introduce the following notation:

$$\begin{array}{lcl} \bar{C} & := & \{T \cup T(0, n-1) \mid T \in B\}, \\ \bar{D} & := & \{T \cup T(1, n-1) \mid T \in B \cup \bar{C}\}, \\ \bar{E} & := & \{T \cup T(n-1, 2m+1) \mid T \in B \cup \bar{C} \cup \bar{D}\}, \end{array}$$

and let $\bar{\mathcal{T}}_m := B \cup \bar{C} \cup \bar{D} \cup \bar{E}$.

Now we consider the cases $1 < m < n-2$ and $m = 0$.

Proposition 4.2 *Let $1 < m < n-2$.*

1. *If $k \neq m$, then*

$$\bar{\mathcal{T}}_k \cong \begin{cases} \underline{m-k} \overset{+}{\sqcup} \underline{2} \overset{+}{\sqcup} \underline{2} \times \underline{k-1} \overset{+}{\sqcup} \underline{2}^2 \times \underline{n-m-k-1} \text{ if } 1 \leq k < m \\ \text{and } k \leq n-m-3, \\ \underline{m-k} \overset{+}{\sqcup} \underline{2} \overset{+}{\sqcup} \underline{2} \times \underline{n-m-2} \text{ if } 1 \leq k < m \text{ and } k > \underline{n-m-3}, \\ \underline{m} \overset{+}{\sqcup} \underline{2} \overset{+}{\sqcup} \underline{2} \times \underline{m-n-k-2} \text{ if } 1 < m < k \leq n-m-3, \\ \underline{n-k-1} \text{ if } 1 < m < k \\ \text{and } k > n-m-3. \end{cases}$$

2.

$$\bar{\mathcal{T}}_m \cong \begin{cases} \underline{2}^2 \times \underline{m} \overset{+}{\sqcup} \underline{2}^3 \times \underline{n-2m-1} & \text{if } 2m \leq n-3, \\ \underline{2}^2 \times \underline{n-m-1} & \text{if } 2m > n-3. \end{cases}$$

Proof. Let $1 < m < n-2$. We consider two cases.

1. $1 \leq k \neq m$. For each $0 \leq i \leq j \leq n-2$, let

$$\begin{array}{lll} B_{i,j} & := & \{T(t,t+k) \mid i \le t \le j\}, \\ C_{i,j} & := & \{T \cup T(m-k+1, n-1) \mid T \in B_{i,j}\}, \\ D_{i,j} & := & \{T \cup T(n-1, m+k+1) \mid T \in B_{i,j} \cup C_{i,j}\}. \end{array}$$

Then $B_{i,j} \subseteq B, C_{i,j} \subseteq C, D_{i,j} \subseteq D$ and $B_{i,j}$ is a sublattice of $\mathcal{T}_k$ which is isomorphic to $\underline{j-i+1}$.

<u>Case 1</u>: $1 \le k < m$. Then $1 < m < n-2$ and $1 \le k < m$ imply that $1 \le m-k < n-k-2$ which implies that $m-k+1 \le n-k-1 \le n-2$. Consequently, $T(m-k+1, n-1) \in \mathcal{T}_k$ and then $\mathcal{A} := B_{0,m-k-1} \dot\cup \{T(m-k,m) \subset T(m-k+1), n-1)\} \cong \underline{m-k}\overset{+}{\sqcup}\underline{2}$.
If $k > n-m-3$, then $n-2 < m+k+1$ and therefore $T(n-1, m+k+1) \notin \mathcal{T}_k$; hence $D = \emptyset$. Thus $\bar{\mathcal{T}}_k = B \cup C = \mathcal{A} \dot\cup B_{m-k+1,n-k-2} \dot\cup C_{m-k+1,n-k-2}$. Since $B_{m-k+1,n-k-2}$ is a sublattice of $\mathcal{T}_k$, Lemma 4.1 implies that $B_{m-k+1,n-k-2} \dot\cup C_{m-k+1,n-k-2} \cong \underline{2} \times B_{m-k+1,n-k-2} \cong \underline{2} \times \underline{n-m-2}$. Hence $\bar{\mathcal{T}}_k \cong \underline{m-k}\overset{+}{\sqcup}\underline{2}\overset{+}{\sqcup}\underline{2} \times \underline{n-m-2}$. But, if $k \le n-m-3$ then $T(n-1, m+k+1) \in \bar{\mathcal{T}}_k$. Therefore, $\bar{\mathcal{T}}_k = \mathcal{A} \dot\cup (B_{m-k+1,m-1} \dot\cup C_{m-k+1,m-1}) \dot\cup (B_{m,n-k-2} \dot\cup C_{m,n-k-2} \dot\cup D_{m,n-k-2})$. Since $B_{m-k+1,m-1}, B_{m,n-k-2}$ and $B_{m,n-k-2} \dot\cup C_{m,n-k-2}$ are sublattices of $\mathcal{T}_k$, Lemma 4.1 implies that

$$\begin{array}{l} B_{m-k+1,m-1} \dot\cup C_{m-k+1,m-1} \\ \quad \cong \quad \underline{2} \times B_{m-k+1,m-1} \\ \quad \cong \quad \underline{2} \times \underline{k-1}, \end{array}$$

$$\begin{array}{l} B_{m,n-k-2} \dot\cup C_{m,n-k-2} \\ \quad \cong \quad \underline{2} \times B_{m,n-k-2} \\ \quad \cong \quad \underline{2} \times \underline{n-m-k-1} \text{ and} \end{array}$$

$$\begin{array}{l} B_{m,n-k-2} \dot\cup C_{m,n-k-2} \dot\cup D_{m,n-k-2} \\ \quad \cong \quad \underline{2} \times (B_{m,n-k-2} \dot\cup C_{m,n-k-2}) \\ \quad \cong \quad \underline{2} \times \underline{2} \times \underline{n-m-k-1} \\ \quad \cong \quad \underline{2}^2 \times \underline{n-m-k-1}. \end{array}$$

Therefore, $\bar{\mathcal{T}}_k \cong \underline{m-k}\overset{+}{\sqcup}\underline{2}\overset{+}{\sqcup}\underline{2} \times \underline{k-1} \times \underline{2}^2 \times \underline{n-m-k-1}$.

<u>Case 2</u>: $m < k < n-2$. Then $m-k+1 \le 0$ and then $T(m-k+1, n-1) \notin \mathcal{T}_k$; hence, $C = \emptyset$. If $k > n-m-3$, then $n-2 < m+k+1$ and then $T(n-1, m+k+1) \notin \mathcal{T}_k$; hence $D = \emptyset$. Therefore, $\bar{\mathcal{T}}_k = B = B_{0,n-k-2} \cong \underline{n-k-1}$. If $k \le n-m-3$, then $T(n-1, m+k+1) \in \mathcal{T}_k$ and therefore, $D \ne \emptyset$. In this case,
$\bar{\mathcal{T}}_k = B_{0,m-1} \dot\cup \{T(m, m+k) \subset T(n-1, m+k+1)\} \dot\cup (B_{m+1,n-k-2} \dot\cup D_{m+1,n-k-2})$.
Since $B_{m+1,n-k-2}$ is a sublattice of $\mathcal{T}_k$, Lemma 4.1 implies that

$$B_{m+1,n-k-2} \dot\cup D_{m+1,n-k-2} \cong \underline{2} \times B_{m+1,n-k-2} \cong \underline{2} \times \underline{n-m-k-2}.$$

Hence, $\bar{\mathcal{T}}_k \cong \underline{m}\overset{+}{\sqcup}\underline{2}\overset{+}{\sqcup}\underline{2} \times \underline{n-m-k-2}$.

2. $1 \le k = m < n-2$. Then $T(0, n-1) \in \mathcal{T}_m$ and $T(1, n-1) \in \mathcal{T}_m$. If $2m > n-3$, then $2m+1 > n-2$; so, $T(n-1, 2m+1) \notin \mathcal{T}_m$, hence, $\bar{E} = \emptyset$. Therefore, $\bar{\mathcal{T}}_m = B \cup \bar{C} \cup \bar{D} = B_{0,n-m-2} \dot\cup C_{0,n-m-2} \dot\cup D_{0,n-m-2}$. Since $B_{0,n-m-2}$ and $B_{0,n-m-2} \dot\cup C_{o,n-m-2}$ are

sublattices of $\mathcal{T}_m$, Lemma 4.1 implies that

$$B_{0,n-m-2} \dot{\cup} C_{0,n-m-2} \cong \underline{2} \times B_{0,n-m-2} \cong \underline{2} \times \underline{n-m-1}$$

and

$$\begin{aligned}(B_{0,n-m-2} \dot{\cup} C_{0,n-m-2}) \dot{\cup} D_{0,n-m-2} &\\ \cong \quad & \underline{2} \times (B_{0,n-m-2} \dot{\cup} C_{0,n-m-2}) \\ \cong \quad & \underline{2} \times \underline{2} \times \underline{n-m-1}.\end{aligned}$$

Therefore, $\mathcal{T}_m \cong \underline{2^2} \times \underline{n-m-1}$.

If $2m \leq n-3$, then $T(n-1, 2m+1) \in \mathcal{T}_m$ and

$$\begin{aligned}\bar{\mathcal{T}}_m = B \cup \bar{C} \cup \bar{D} \cup \bar{E} &\\ = \quad & (B_{0,m-1} \dot{\cup} \bar{C}_{0,m-1} \dot{\cup} \bar{D}_{0,m-1}) \\ \dot{\cup} \quad & (B_{n,n-m-2} \dot{\cup} \bar{C}_{m,n-m-2} \dot{\cup} \bar{D}_{m,n-m-2} \dot{\cup} \bar{E}_{m,n-m-2}).\end{aligned}$$

Applying Lemma 4.1 in the same way as in the previous cases we obtain

$$\begin{aligned}B_{0,m-1} \dot{\cup} \bar{C}_{0,m-1} \dot{\cup} \bar{D}_{0,m-1} &\\ \cong \quad & \underline{2} \times (B_{0,m-1} \dot{\cup} \bar{C}_{0,m-1}) \\ \cong \quad & \underline{2} \times \underline{2} \times B_{0,m-1} \\ \cong \quad & \underline{2^2} \times \underline{m}, \text{ and}\end{aligned}$$

$$\begin{aligned}B_{m,n-m-2} \dot{\cup} \bar{C}_{m,n-m-2} \dot{\cup} \bar{D}_{m,n-m-2} \dot{\cup} \bar{E}_{m,n-m-2} &\\ \cong \quad & \underline{2} \times (B_{m,n-m-2} \dot{\cup} \bar{C}_{m,n-m-2} \dot{\cup} \bar{D}_{m,n-m-1}) \\ \cong \quad & \underline{2} \times \underline{2} \times (B_{m,n-m-2} \dot{\cup} C_{m,n-m-2}) \\ \cong \quad & \underline{2^2} \times \underline{2} \times B_{m,n-m-2} \\ \cong \quad & \underline{2^3} \times \underline{n-2m-1}.\end{aligned}$$

■

Now we consider the case $m = 0$.

Proposition 4.3 *If $m = 0$, then*

$$\mathcal{T}_k \cong \begin{cases} \underline{2} \times \underline{n-k-1} & \text{if } 1 \leq k \leq n-3 \\ \underline{1} & \text{if } k > n-3 \end{cases}.$$

Proof. Let $m = 0$ and $k \geq 1$. If $1 \leq k \leq n-3$, then $k+1 \leq n-2$ and therefore $T(n-1, k+1) \in \bar{\mathcal{T}}_k$. Let denote $G_{i,j} := \{T \cup T(n-1, k+1) \mid T \in B_{i,j}\}$. Then $\bar{\mathcal{T}}_k = B_{0,n-k-2} \dot{\cup} G_{0,n-k-2}$ and we have

$$\bar{\mathcal{T}}_k \cong \underline{2} \times B_{0,n-k-2} \cong \underline{2} \times \underline{n-k-1}$$

where the isomorphism is defined similar as in Lemma 4.1. If $k > n-3$, then $T(n-1, k+1) \notin \bar{\mathcal{T}}_k$. So, $\bar{\mathcal{T}}_k = B_{0,n-k-2} \cong \underline{n-k-1}$. From $k > n-3$ we obtain $n-k-1 < 2$, which shows that $\bar{\mathcal{T}}_k \cong \underline{1}$. ■

Moreover, we have

Corollary 4.4 *Let $0 \leq m < n-2$. Then*

1. *$\mathcal{T}_1 = \bar{\mathcal{T}}_1$ and*
2. *$\bar{\mathcal{T}}_k$ is a sublattice of $\mathcal{T}_k$ for all $k \geq 1$.*

Proposition 4.5

1. If $1 \leq m < n-2$, then $\mathcal{T}_k \cong \bar{\mathcal{T}}_k \times \mathcal{T}_{k-1}$ for all $k \geq 1$ and $\mathcal{T}_0 = \{\Delta_A\}$.

2. For $m = 0$, let $\mathcal{T}_0 := \{\Delta_A, T(0, n-1)\}$ and $\bar{\mathcal{T}} := \mathcal{T}_1 \cup \{T \cup T(0, n-1) \mid T \in \mathcal{T}_1\}$. Then $\bar{\mathcal{T}} \cong \underline{2} \times \mathcal{T}_1, \mathcal{T}_2 \cong \bar{\mathcal{T}}_2 \times \bar{\mathcal{T}}$ and $\mathcal{T}_k \cong \bar{\mathcal{T}}_k \times \mathcal{T}_{k-1}$ for all $k > 2$.

Proof. 1. Let $1 < m < n-2$ and $m \neq k > 1$ and let $T \in \mathcal{T}_k$. Since $A \setminus \{n-1\}$ is an LT-algebra with fundamental operation $f|A \setminus \{n-1\}$, we have $a_T^{k-1} \in \underline{n-k-1}$. Then $T \supseteq T(a_T^{k-1}, a_T^{k-1}+k) \in B \subseteq \bar{\mathcal{T}}_K$ or $T \supseteq T((a_T^{k-1}, a_T^{k-1}+k), (m-k+1, n-1)) \in C \subseteq \bar{\mathcal{T}}_k$ or $T \supseteq T((a_T^{k-1}, a_T^{k-1}+k), (m-k+1, n-1), (n-1, m+k+1)) \in D \subseteq \bar{\mathcal{T}}_k$ depending on $(m-k+1, n-1) \in T$ or $(n-1, m+k+1) \in T$ or neither. In each case one can see that there exists an element $S_T \in \bar{\mathcal{T}}_k$ such that $S_T \subseteq T$; and thus $T \setminus S_T \in \mathcal{T}_{k-1}$. For the cases $k = m > 1$ or $m = 1$ one concludes in a similar way. Now we define a mapping $g : \mathcal{T}_k \to \bar{\mathcal{T}}_k \times \mathcal{T}_{k-1}$ by $g(T) = (S_T, T \setminus S_T)$ for all $T \in \mathcal{T}_k$. Then g is an order-embedding and it is easy to prove that $T \cup S \in \mathcal{T}_k$ for all $S \in \bar{\mathcal{T}}_k$ and $T \in \bar{\mathcal{T}}_{k-1}$. Hence, g is an order-isomorphism.
2. Let $m = 0$. Proposition 4.3 and Corollary 4.4 imply that $\mathcal{T}_1 = \bar{\mathcal{T}}_1$ and this is isomorphic to either $\underline{2} \times \underline{n-k-1}$ or to $\underline{1}$. In the case $m = 0$ the set $Tol(\mathcal{A})$ of all tolerances on $\mathcal{A}$ contains $T(0, n-1)$. We set $H := \{T \cup T(0, n-1) \mid T \in \mathcal{T}_1\}$ for all $1 \leq i, j \leq n-2$ and $\bar{\mathcal{T}} := \mathcal{T}_1 \dot{\cup} H$. Then with an isomorphism defined similar as in Lemma 4.1 we get $\bar{\mathcal{T}} \cong \underline{2} \times \mathcal{T}_1$.] Hence, $\bar{\mathcal{T}} \cong \underline{2}$ or $\bar{\mathcal{T}} \cong \underline{2}^2 \times \underline{n-k-1}$.

Now we will show that $\mathcal{T}_2 \cong \bar{\mathcal{T}}_2 \times \bar{\mathcal{T}}$. Using an argumentation as in 1. One gets $T \supseteq T(a_T^1, a_T^1+2) \in B \subseteq \bar{\mathcal{T}}_2$ or $T \supseteq T((a_T^1, a_T^1+2), (n-1, 3)) \in C \subseteq \bar{\mathcal{T}}_2$ for all $T \in \mathcal{T}_2$. Hence for all $T \in \mathcal{T}_2$ there exists $S_T \in \bar{\mathcal{T}}_2$ such that $S_T \subseteq T$. Since $S_T \cup \bar{T} \in T$ for all $\bar{T} \in \bar{\mathcal{T}}$, we have $T = S_T \dot{\cup} \{\bar{T} \cup S_T \mid \bar{T} \in \bar{\mathcal{T}}\}$. Then the mapping defined by $T \mapsto (S_T, T \setminus S_T)$ for all $T \in \mathcal{T}_2$ is an isomorphism from $\mathcal{T}_2$ to $\bar{\mathcal{T}}_2 \times \bar{\mathcal{T}}$. A similar argumentation as in 1. shows $\mathcal{T}_k \cong \bar{\mathcal{T}}_k \times \mathcal{T}_{k-1}$ for all $k > 2$. ■

Our construction has the following consequences:

Corollary 4.6 1. *For $m \neq 0, \mathcal{T}_k \cap \mathcal{T}_l = \emptyset$ for all $0 \leq k \neq l \leq n-2$ and $\mathcal{T} = \bigcup_{0 \leq k \leq n-2} \mathcal{T}_k$.*

2. *For $m = 0, \mathcal{T}_k \cap \mathcal{T}_l = \emptyset$ for all $1 < k \neq l \leq n-2$, $\mathcal{T}_k \cap \bar{\mathcal{T}} = \emptyset$ and $\mathcal{T}_k \cap \mathcal{T}_0 = \emptyset$ for all $k \geq 2$ and $\bar{\mathcal{T}} \cap \mathcal{T}_0 = \emptyset$. Moreover, $\mathcal{T} = \mathcal{T}_0 \dot{\cup} \bar{\mathcal{T}} \dot{\cup} \bigcup_{2 \leq k \leq n-2} \mathcal{T}_k$.*
3. *For all $0 \leq k \leq n-2$, $\mathcal{T}_k$ and $\bar{\mathcal{T}}$ are isomorphic to a sublattice of $\mathcal{T}$.*
4. *For all $0 < k \leq n-2$, $\mathcal{T}_{k-1}$ is isomorphic to a sublattice of $\mathcal{T}_k$.*

Using the definition of an ordered sum we have:

Proposition 4.7 *1. For $m \neq 0$, $\mathcal{T} \cong \mathcal{T}_0 \overset{+}{\sqcup} \mathcal{T}_1 \overset{+}{\sqcup} \cdots \overset{+}{\sqcup} \mathcal{T}_{n-2}$ and $\mathcal{T}_0 = \{\Delta_A\}$.*

2. For $m = 0$, $\mathcal{T} \cong \underline{2} \overset{+}{\sqcup} \underline{2} \times \mathcal{T}_1 \overset{+}{\sqcup} \mathcal{T}_2 \cdots \overset{+}{\sqcup} \mathcal{T}_{n-2}$.

Finally we get our result:

Theorem 4.8 *$Tol(\mathcal{A}) \cong \underline{2} \times \mathcal{T}$.*

Proof. Let $\alpha : Tol(\mathcal{A}) \to \underline{2} \times \mathcal{T}$ be defined by

$$\alpha(T) = \begin{cases} (0, T) & \text{if } (n-1, m+1) \notin T \\ (1, T \setminus \{(n-1, m+1)\} & \text{if } (n-1, m+1) \in T \end{cases}.$$

Then, clearly, α is an order-isomorphism. ■

We consider the following example for $n = 4, m = 1$:

Example 4.9

x	$f(x)$	$f^2(x)$
0	0	0
1	0	0
2	1	0
3	1	0.

Then the tolerance lattice is pictured in Figure 8.

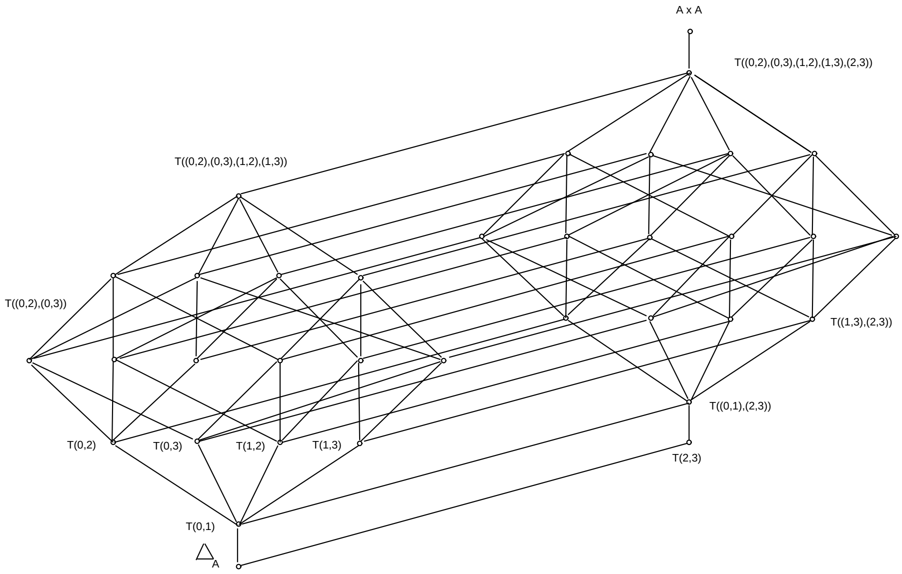

Figure 8: Tolerance Lattice of an LT_1-algebra

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