A SINGLE AXIOM FOR BOOLEAN ALGEBRA

Shotaro Tanaka

Received January 10, 2012 ; revised May 16, 2013

Dedicate to the memory of the late emeritus professor Kiyoshi ISÉKI

Dedication. Prof.K.Iséki and I chose Symbolic Logic in Foundation of Mathematics as the object of study. In 1956 we started to read $<$ Elements of Mathematical Logic $>$ (J. Łukasiewicz, 1929). Our knowledge of logic was changed greatly. The book was written in Polish notations (or Łukasiewicz's notation). We thought that by using the method we could obtain a great many results. Indeed, we wrote papers and published them as raging billows surge. In this paper, I return to those times.

Abstract. We shall show that the thesis $C\delta pC\delta Cp0\delta q$ is a single axiom for Boolean algebra. That is, we deduce Huntington's first axioms system of Boolean algebra(BA) from that single thesis.

Introduction. The genesis of BA is founded in the works of G. Boole in 1847,1848. E.V. Huntington condensed formulas of BA to ten axioms, and published his first axioms system for BA in 1904.

G. Frege improved logic, and J.Łukasiewicz introduced Polish notation into logic in 1929. C.A.Meredith showed that $C\delta pC\delta Np\delta q$ is a single axiom for propositional calculus in 1953. Our single axiom has $Cp0$ instead of Np.

1. Single Axiom System : $C\delta pC\delta Cp0\delta q$

Inference rules

1. Detachment rule (modus ponens):

If a thesis $C\alpha\beta \in$ M and a thesis $\alpha \in$ M, then thesis $\beta \in$ M. For details, see Łukasiewicz [5], and see proof lines between Thesis 1,Thesis 2 and Thesis 3.

2. Substitution rule:

(1) If a thesis $\alpha(p) \in$ M and $\beta \in$ M, then a thesis $\alpha(\beta) \in$ M.

(2) If let α, $\delta\alpha \in$ M, δ/f, then $f\alpha \in$ M, δ: functorial-variable, f:any function. For details of δ, see Prior [12], Meredith [9] or Łukasiewicz [6].

Replacement rule is not used, see the following definitions.

3. Definitions.

If $p, q \in$ M, then the following definitions hold in M:

2010 *Mathematics Subject Classification.* 06E05.

Key words and phrases. Boolean algebra, Propositional calculus, Single axiom, Huntington's first system, Łukasiewicz's notation.

01.1	$C\delta Np\delta Cp0$	01.2	$C\delta Cp0\delta Np$
02.1	$C\delta Apq\delta CCp0q$	02.2	$C\delta CCp0q\delta Apq$
	$02'.1$ $C\delta Apq\delta CNpq$		$02'.2$ $C\delta CNpq\delta Apq$
03.1	$C\delta Kpq\delta CCpCq00$	03.2	$C\delta CCpCq00\delta Kpq$
	$03'.1$ $C\delta Kpq\delta NCpNq$		$03'.2$ $C\delta NCpNq\delta Kpq$
04.1	$C\delta Epq\delta NCCpqNCqp$	04.2	$C\delta NCCpqNCqp\delta Epq$
	$04'.1$ $C\delta Epq\delta KCpqCqp$		$04'.2$ $C\delta KCpqCqp\delta Epq$
05.1	$C\delta 1\delta C00$	05.2	$C\delta C00\delta 1$
06.1	$CIpq[\delta](KE\delta p\delta qE\delta q\delta p)$	06.2	$C[\delta](KE\delta p\delta qE\delta q\delta p)Ipq$
	$06'.1$ $CNIpq[\exists\delta](ANE\delta p\delta qNE\delta q\delta p)$		$06'.2$ $C[\exists\delta](ANE\delta p\delta qNE\delta q\delta p)NIpq$

2. Deduction of theses from $C\delta pC\delta Cp0\delta q$.

1 $C\delta pC\delta Cp0\delta q$.
Substituting 1 $\delta/CpCCp0'$, $q/Cp0$, we obtain

$$C\boldsymbol{CpCCp0p}CCpCCp0Cp0CpCCp0Cp0,$$

the antecedent of this implication is equiform with substitution 1 $\delta/'$, q/p, then by Detachment rule we obtain the consequent i.e.Thesis 2 , this process is expressed :

$$1\ \ \delta/CpCCp0',\ q/Cp0\ *\ C1\delta/',\ q/p-2,$$

2 $CCpCCp0Cp0CpCCp0Cp0$.
We prove the next Thesis:
1 $\delta/'$, $p/CpCCp0Cp0$, $q/CCpCCp0Cp00$ $^*C1$ $\delta/'$, $q/Cp0-3$

$$C\boldsymbol{CpCCp0Cp0}CCCpCCp0Cp00CCpCCp0Cp00,$$

the antecedent of this is substitution 1 $\delta/'$, $q/Cp0$, then by Detachment rule we obtain the consequent i.e.Thesis 3,

3 $CCCpCCp0Cp00CCpCCp0Cp00$.

$$1\ \ \delta/C'',\ p/CpCCp0Cp0,\ q/p\ ^*\ C2{-}C3{-}4,$$

4 Cpp.

$$1\ \ \delta/C''\ ^*\ C4{-}5,$$

5 $CCCp0Cp0Cqq$.

$$1\ \ \delta/CC''Cqq,\ p/Cp0,\ q/r\ ^*\ C5{-}C5\ p/Cp0{-}6,$$

6 $CCrrCqq$.

$$1\ \ \delta/',\ p/Cpp\ ^*\ C4{-}7,$$

7 $CCCpp0q$.

$$1\ \ \delta/C'Cqq,\ p/Cpp,\ q/r\ ^*\ C6\ r/p{-}C7\ q/Cqq{-}8,$$

8 $CrCqq$.

$$1\ \ \delta/CrC'r,\ p/r\ ^*\ C8\ q/r{-}C1\delta/',p/r,q/r{-}9,$$

9 $CrCqr$.

1 $\delta/CCp0'$, $p/Cp0$, q/p $*$ $C4$ $r/Cp0$—10,

10 $CCCp0CCp00CCp0p$.

1 $\delta/CCCp0C'0C'p$ $*$ $C6$ $r/Cp0$, q/p—$C10$—11,

11 $CCCp0Cq0Cqp$.

1 $\delta/CCpCCp0rCC'CprCpC'r$ $*$ $C8$ $r/CpCCp0r$, $q/CpCpr$ — $C9$ $r/CpCCp0r, q/CCp0Cpr$ — $C1$ $\delta/'$, q/r—12,

12 $CCqCprCpCqr$.

1 δ/Cp', q/r $*$ $C4$—13,

13 $CCpCp0Cpr$.

9 $r/CCpCp0Cpr$, $q/CCp0r * C13$—14,

14 $CCCp0rCCpCp0Cpr$.

1 $\delta/CC'rCCp'Cpr$ $*$ $C9$ r/Cpr, q/Cpp—$C14$—15,

15 $CCqrCCpqCpr$.

12 q/Cqr, p/Cpq, r/Cpr $*$ $C15$—16,

16 $CCpqCCqrCpr$.

Following Meredith, We use Condensed Detachment Dxy. For the details, see appendix II in Prior [12].

$DD12 \cdot 4 \cdot 7 = 17$,

17 $CCCCCpp0qCCpp0CCpp0$.

9 r/Cpp, $q/CCCppqCpp$ $*$ $C4$—18,

18 $CCCCppqCppCpp$.

1 $\delta/CCC'q''$, p/Cpp, q/p $*$ $C18$—$C17$—19,

19 $CCCpqpp$.

$DDDD16 \cdot 15 \cdot 15 \cdot 12 \cdot 15 = 20$,

20 $CCsCqrCCpsCqCpr$.

20 $p/s, q/Cpq$, r/Cpr, s/Cqr $*$ $C15$—21,

21 $CCsCqrCCpqCsCpr$.

20 p/s, q/Cqr, $r/Cpr, s/Cpq$ $*$ $C16$—22,

22 $CCsCpqCCqrCsCpr$.

$DD16 \cdot 16 \cdot 19 = 23$,

23 $CCpCpqCpq$.

22 p/Cpq, $q/CpCpr$, r/Cpr, $s/CpCqr$ $*$ $C21$ s/p—$C23$ q/r—24,

24 $CCpCqrCCpqCpr$.

12 q/Cpq, r/q $*$ $C4$ p/Cpq—25,

25 $CpCCpqq$.

$1\ \delta/'\ \ —26,$

26 $CpCCp0q.$

$12\ p/Cp0,\ q/p,\ r/q\ *\ C26—27,$

27 $CCp0Cpq.$

$24\ p/Cq0,\ r/p\ *\ C27\ p/q,\ q/p—28,$

28 $CCCp0qCCq0p.$

$15\ q/Cpq,\ r/CCqrCpr,\ p/CCq0Cp0\ *\ C16—C11\ p/q, q/p—29,$

29 $CCCq0Cp0CCqrCpr.$

$15\ q/CCq0Cp0,\ r/CCqrCpr,\ p/CCq0q\ *\ C29—C28\ p/Cp0—30,$

30 $CCCq0qCCqrCpr.$

$12\ q/CCq0q,\ p/Cqq,\ r/CCCq0qq\ *\ C30r/q,\ p/CCq0q—C4\ p/q—31,$

31 $CCCq0qCCCq0qq.$

$23\ p/CCp0p,\ q/p\ *\ C31\ q/p—32,$

32 $CCCp0pp.$

$16\ p/Cp0,\ q/Cpq\ *\ C27—33,$

33 $CCCpqrCCp0r.$

$33\ p/Cp0,\ q/p,\ r/p\ *\ C32—34,$

34 $CCCp00p.$

$11\ p/CCp00,\ q/p\ *\ C34\ p/Cp0—35,$

35 $CpCCp00.$

$16\ p/q,\ q/CCq00,\ r/Cp0\ *\ C35\ p/q—36,$

36 $CCCCq00Cp0CqCp0.$

$16\ p/CCp00,\ q/p,\ r/q\ *\ C34—37,$

37 $CCpqCCCp00q.$

$15\ p/CCp00,\ r/CCq00\ *\ C35\ p/q—38,$

38 $CCCCp00qCCCp00CCq00.$

$16\ p/Cpq,\ q/CCCp00q,\ r/CCCp00CCq00\ *\ C37—C38—39,$

39 $CCpqCCCp00CCq00.$

$DD16 \cdot 39 \cdot 11 = 40,$

40 $CCpqCCq0Cp0.$

$12\ q/p,\ p/q,\ r/0 * 41,$

41 $CCpCq0CqCp0.$

$15\ p/Cp0,\ r/CCq00\ *\ C35\ p/q—42,$

42 $CCCp0qCCp0CCq00.$

16 $p/CCp0q,\ q/CCp0CCq00,\ r/CCq0p\ *\ C42$—$C11\ q/Cq0$—43,

43 $CCCp0qCCq0p$.

02 $Apq = CCp0q$
43 $*$ 02 **I** $\cdot$ 02 $p/q,\ q/p$ **II** $*$ 44,

44 $CApqAqp$.

15 $q/CqCr0,\ r/CrCq0\ *\ C41\ p/q,\ q/r$—45,

45 $CCpCqCr0CpCrCq0$.

45 $q/CpCq0,\ r/q\ *\ C25\ q/Cq0$—46,

46 $CpCqCCpCq00$.

03 $Kpq = CCpCq00$.
46 $*$ 03 II $\cdot$ 47,

47 $CpCqKpq$.

47 $p/CApqAqp,\ q/CAqpApq\ *\ C44$—$C44\ p/q,\ q/p$—48,

48 $KCApqAqpCAqpApq$.

12 $p/q,\ q/p,\ r/Kpq\ *\ C47$—49,

49 $CqCpKpq$.

43 $q/Cpq\ *\ C27$—50,

50 $CCCpq0p$.

40 $p/q, q/Cpq\ *\ C9\ r/q,\ q/p$—51,

51 $CCCpq0Cq0$.

50 $q/Cq0\ *$ 03 I $\cdot$ 52,

52 $CKpqp$.

43 $p/q,\ q/CpCq0\ *\ C9\ r/Cq0,\ q/p$—53,

53 $CCCpCq00q$.

53 $*$ 03 I $\cdot$ 54,

54 $CKpqq$.

16 $p/Kpq,\ q/p\ *\ C52$—55,

55 $CCprCKpqr$.

16 $p/Cpr,\ q/CKpqr,\ r/s\ *\ C55$—56,

56 $CCCKpqrsCCprs$.

12 $q/CKpqCqr,\ p/CKpqq,\ r/CKpqr\ *\ C24\ p/Kpq$—$C54$—57,

57 $CCKpqCqrCKpqr$.

56 $r/Cqr,\ s/CKpqr\ *\ C57$—58,

58 $CCpCqrCKpqr$.

58 $r/Kqp\ *\ C49\ p/q,\ q/p$—59,

59 $CKpqKqp$.

$27\ p/0\ *\ C4\ p/0$—60,

60 $C0p$.

$D16 \cdot 35 = \mathsf{i} : CCCCp00qCpq$.
$\mathsf{i} * 02 = \mathsf{ii} : CACp0qCpq$.
$D40 \cdot \mathsf{ii} = \mathsf{iii} : CCCpq0CACp0q0$.
$\mathsf{iii} * 03 = \mathsf{iv} : CKpqCACp0Cq00$.
$D41 \cdot \mathsf{iv} = 61$,

61 $CACp0Cq0CKpq0$.
$37 * 02 = \mathsf{i} : CCpqACp0q$.
$D40 \cdot \mathsf{i} = \mathsf{ii} : CCACp0q0CCpq0$.
$\mathsf{ii} * 03 = \mathsf{iii} : CCACp0Cq00Kpq$.
$D43 \cdot \mathsf{iii} = 62$,

62 $CCKpq0ACp0Cq0$. □

3. Deduction of laws.

From i:Cpq and ii:Cqp, we can deduce Epq and Eqp[0)]. Then we denote: $CCE(\mathsf{i}, \mathsf{ii}) * Epq,\ Eqp$.

$CCE(34, 35) * 101$,

101 $EpNNp, ENNpp$. Law of double negation.

$CCE(44, 44p/q, q/p)\ *\ 102$,

102 $EApqAqp,\ EAqpApq$. Law of commutativity of alternation.

$CCE(59, 59p/q, q/p) * 103$,

103 $EKpqKqp, EKqpKpq$. Law of commutativity of conjunction.

$CCE(61, 62) * 104$,

104 $ENKpqANpNq,\ EANpNqNKpq$. De Morgan's law (1).

$D49 \cdot 4 = \mathsf{i} : CpKpC00$.
$52\ q/C00 = \mathsf{ii} : CKC00p$.
$CCE(\mathsf{i}, \mathsf{ii}) * 105$,

105 $EpKpC00,\ EKpC00p$. Law of elimination of Conjunct C00.

$60\ p/KpNp = \mathsf{i} : C0KpNp$.
$52\ q/C00 = \mathsf{ii} : CKpC00p$.
$CCE(\mathsf{i}, \mathsf{ii}) * 106$,

106 $EKNpp0,\ \ E0KNpp$. Law of contradiction.

$05\ \delta/',\ \ 1\ (= N0) = \mathsf{i} : CN0C00,\ \mathsf{ii} : CC00N0$,
$CCE\ (\mathsf{i},\mathsf{ii}) * 107$,

107 $EN0C00,\ \ EC00N0$. Law of negation of 0.
$EN01,\ \ E1N0$.

$05\ \delta/N',\ NC00 = 0 * \mathsf{i} : CN1NC00, \mathsf{ii} : CNC00N1$.
$CCE\ (\mathsf{i}, \mathsf{ii}) * 108$,

108 $ENC000,\ \ E0NC00$. Law of negation of $C00$.

$EN10,\ E0N1.$

We shall call 107 and 108 the "Law of two poles".

$26\ q/0 = \mathsf{i} : CpAp0; 34\ 02 = \mathsf{ii} : CAp0p.$
$CCE(\mathsf{i},\ \mathsf{ii}) * 109,$

109 $EpAp0,\ \ EAp0p.$ Law of elimination of Alternat 0.

$5\ q/0 *\ 02 = \mathsf{i} :\ CApNpC00.$
$D58 \cdot 4 = \mathsf{ii} : ApCp0.$
$\mathsf{ii} * 01 = \mathsf{iii} : ApNp.$
$D9 \cdot \mathsf{iii} = \mathsf{iv} : CC00ApNp.$
$CCE(\mathsf{i},\ \mathsf{iv}) * 110,$

110 $EApNpC00,\ EC00ApNp.$ Law of the excluded middle.

$11 * 02'.2, 03'.2\ \mathrm{I} = \mathsf{i} :\ CApKqrCCqNrp.$
$DD16 \cdot \mathsf{i} \cdot 33 = \mathsf{ii} :\ CApKqrCNqp.$
$DD \cdot 43 \cdot \mathsf{ii} = \mathsf{iii} :\ CApKqrCNpq.$
$D16 \cdot 12 * 02.1 = \mathsf{iv} : CCCqNrpCCrNqp.$
$DD16 \cdot \mathsf{i} \cdot \mathsf{iv} = \mathsf{v} : CApKqrCCrNqp.$
$DD16 \cdot \mathsf{v} \cdot 33 = \mathsf{vi} : CApKqrCNrp.$
$DD15 \cdot 43 \cdot \mathsf{vi} = \mathsf{vii} : CApKqrCNpr.$
$DD16 \cdot \mathsf{iii} \cdot 47 = \mathsf{viii} : CApKqrCCNprKCNpqCNpr.$
$DD24 \cdot \mathsf{viii} \cdot \mathsf{vii} = \mathsf{ix} :\ CApKqrKCNpqCNpr.$
$\mathsf{ix} * 02'.2\ \mathrm{II} \times 2 = 111,$

111 $CApKqrKApqApr.$ The first law of distribution (1).

$D15 \cdot 47 = \mathsf{i} :\ \ CCsqCsCrKqr.$
$DD16 \cdot \mathsf{i} \cdot 24 = \mathsf{ii} :\ \ CCsqCCsrCsKqr.$
$D63 \cdot \mathsf{ii} = \mathsf{iii} :\ \ CKCsqCsrCsKqr.$
$\mathsf{iii} * 02.2,\ 1 \times 2,\ \mathsf{II} = 112,$

112 $CKApqAprApKqr.$ The first law of distribution (2).

$CCE(111, 112) * 113,$

113 $EApKqrKApqApr,\ EKApqAprApKqr.$ The first law of distribution.

$113 * 02.1,\ \mathsf{I} \times 2,\ \mathsf{II},\ 101 \times 3 = 114,$

114 $ECpKqrKCpqCpr,\ EKCpqCprCpKqr.$ Distributive law for C and K

$4 * 02'.1 \cdot 03'1\ \mathsf{II} = \mathsf{i} :\ CAKpqKprCNNCpNqNCpNr.$
$DD16 \cdot \mathsf{i} \cdot 11 = \mathsf{ii} :\ \ CAKpqKprCCpNrNCpNq.$
$DD6 \cdot \mathsf{ii} \cdot 63 = \mathsf{iii} :\ \ CAKpqKprNKCpNrCpNq.$
$D40 \cdot 64 = \mathsf{iv} :\ \ CNKCpNrCpNqNKCpNqCpNr.$
$DD16 \cdot \mathsf{iii} \cdot \mathsf{iv} = \mathsf{v} :\ \ CAKpqKprNKCpNqCpNr.$
$D40 \cdot 36 = \mathsf{vi} :\ \ CNKCpNqCpNrNCpKNqNr.$
$DD16 \cdot \mathsf{v} \cdot \mathsf{vi} = \mathsf{vii} :\ CAKpqKprNCpKNqNr.$
$03'.2 = \mathsf{viii} :\ \ CNCpKNqNrKpNKNqNr.$
$DD16 \cdot \mathsf{vii} \cdot \mathsf{viii} = \mathsf{ix} :\ \ CAKpqKprKpNKNqNr.$
$D104 \cdot 101 = \mathsf{x} :\ \ CKpNKNqNrKpAqr.$
$DD16 \cdot \mathsf{ix} \cdot \mathsf{x} = 115,$

115 $CAKpqKprKpAqr$. The second law of distribution (1).

$4\ p/kpAqr\ *\ 03'.1\ \mathsf{II} = \mathsf{i}:\quad CKpAqrNCpNAqr.$
$DD16 \cdot \mathsf{i} \cdot 27 = \mathsf{ii}:\quad CKpAqrCCpNAqr0.$
$\mathsf{ii} * 02'.1 = \mathsf{iii}:\quad CKpAqrCCpNCNqr0.$
$\mathsf{iii} * 101\ p/r = \mathsf{iv}:\quad CKpAqrCCpNCNqNNr0.$
$\mathsf{iv} * 03'.2 = \mathsf{v}:\quad CKpAqrCCpKNqNr0.$
$D16 \cdot 114 = \mathsf{vi}:\quad CCCpKNqNr0CKCpNqCpNr0.$
$DD16 \cdot \mathsf{v} \cdot \mathsf{vi} = \mathsf{vii}:\quad CKpAqrCKCpNqCpNr0.$
$\mathsf{vii} * 0.3'1,\ \mathsf{II} = \mathsf{viii}:\quad CKpAqrCNCCpNqNCpNr0.$
$\mathsf{viii} * 0.1, \mathsf{II}, 101 = \mathsf{ix}:\quad CKpAqrCNNCpNqNCpNr.$
$\mathsf{ix} * 03'.2 \times 2 = \mathsf{x}:\quad CKpAqrCNKpqKpr.$
$\mathsf{x} * 02'.2 = 116,$

116 $CKpAqrAKpqKpr$. The second law of distribution (2).

$CCE(115, 116) * 117,$

117 $EAKpqKprKpAqr$. $EKpAqrAKpqKpr$. The second law of distribution. □

4. Huntington's first axiom system for Boolean algebra (1904).

[1] Huntington's first axiom system :

We take as the fundamental concepts a class, K, with two rules of combination, + and ·, and as the fundamental propositions, the following ten postulates.

I a. $a + b$ is in the class whenever a and b are in the class.
I b. $a \cdot b$ is in the class whenever a and b are in the class.
II a. There is an element $\wedge$ such that $a + \wedge = a$ for every element a.
II b. There is an element $\vee$ such that $a \cdot \vee = a$ for every element a.
III a. $a + b = b + a$ whenever $a,\ b,\ a + b$, and $b + a$ are in the class.
III b. $a \cdot b = b \cdot a$ whenever $a,\ b,\ a \cdot b$, and $b \cdot a$ are in the class.
IV a. $a + (b \cdot c) = (a + b) \cdot (a + c)$ whenever $a,\ b,\ c,\ a + b,\ a + c,\ b \cdot c,\ a + (b \cdot c)$, and $(a + b) \cdot (a + c)$ are in the class.
IV b. $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$ whenever $a,\ b,\ c,\ a \cdot b,\ a \cdot c,\ b + c,\ a \cdot (b + c)$, and $(a \cdot b) + (a \cdot c)$ are in the class.
V. If the elements $\wedge$ and $\vee$ in the postulates II a and II b exist and unique, then for every element a there is an element $\bar{a}$ such that $a + \bar{a} = \vee$ and $a \cdot \bar{a} = \wedge$.
VI . There are at least two elements, x and y, in the class such that $x \neq y$. □

[2] Dictionary of K and M

K	Boole	$p = q$	$p \neq q$	$\wedge$	$\vee$	$\bar{p}$	$p + q$	$p \cdot q$	$p \subset q$	$\in$
M	Logic	$\boldsymbol{I}pq$	$N\boldsymbol{I}pq$	0	$C00$	Np	Apq	Kpq	Cpq	$\in$

5. Deduction of Huntington's system from $C\delta pC\delta Cp0\delta q$.

[1] We use the following Basics and Metatheoem:

Basics

(1) Primitive constant $0 \in \mathrm{M}$ and 0 is unique[1)].
(2) Primitive functor $C \in \mathrm{M}$ and C is unique.
(3) fundamental proposition $C00 \in \mathrm{M}$ and $C00$ is unique[2)].
(4) Formulas constructed by a functor-variable δ exist.
That is, Np, Apq, Kpq, Epq and $\boldsymbol{I}pq \in \mathrm{M}$.

Metatheorem.

If $P, Q \in \mathrm{M}$, then $\boldsymbol{I}PQ$ is deduced from EPQ[3)],

[2] Deduction of Huntington's system.

(1) Postulate Ia.
From Basics (4) when $p \in \mathrm{M}$, $q \in \mathrm{M}$, then $Apq \in \mathrm{M}$. That is to say,
When $p \in \mathrm{K}, q \in \mathrm{K}$, then $p + q \in \mathrm{K}$.

(2) Postulate Ib.
From Basics (4) when $p \in \mathrm{M}$, $q \in \mathrm{M}$, then $Kpq \in \mathrm{M}$. That is to say,
When $p \in \mathrm{K}, q \in \mathrm{K}$, then $p \cdot q \in \mathrm{K}$.

(3) Postulate IIa.
From thesis 112 : Law of elimination of alternat 0, $EpAp0$. By Metatheorem, $\boldsymbol{I}pAp0$. By Dictionary we obtain,
$\exists \wedge,\ \wedge \in \mathrm{K}.\ p = p + \wedge$.

(4) Postulate IIb.
From thesis 106 : Law of elimination of conjunct $C00$, $EpKpC00$. By Metatheorem, $\boldsymbol{I}pKpC00$. By Dictionary we obtain,
$\exists \vee, \vee \in \mathrm{K}.\ \ p = p \cdot \vee$.

(5) Postulate IIIa.
From thesis 106 : Law of commutativity of alternation, $EApqAqp$. By Metatheorem, $\boldsymbol{I}ApqAqp$. By Dictionary we obtain,
$p + q = q + p$

(6) Postulate IIIb.
From thesis 103 : Law of commutativity of conjunction, $EKpqKqp$. By Metheorem, $\boldsymbol{I}KpqKqp$. By Ditionary we obtain,
$p \cdot q = q \cdot p$.

(7) Postulate IVa.
From 126 : The second law of distribution, $EApKqrKApqApr$. By Metatheorem, $\boldsymbol{I}ApKqrKApqApr$. By Dictionary we obtain,
$p + (q \cdot r) = (p + q) \cdot (p + r)$.

(8) Postulate IVb.
From thesis 129 : The first law of distribution, $EAKpqKprKpAqr$. By Metatheorem, $\boldsymbol{I}AKpqKprKpAqr$. By Dictionary we obtain,
$p \cdot (q + r) = (p \cdot q) + (p \cdot r)$.

(9) – 1 Postulate V-1.
From thesis 113 : Law of the excluded middle, $EApNpC00$. By Metatheorem, $\boldsymbol{I}ApNpC00$. From Basic (3), $C00 \in \mathrm{M}$ and $C00$ is unique. Hence by Dictionary we have,
When $\vee \in \mathrm{K}$. $\vee$ is unique, $\bar{p} \in \mathrm{K}$ such that $p + \bar{p} = \vee$.

(9) – 2 Postulate V-2.

From thesis 107 : Law of contradiction, $EKNpp0$. By Metatheorem, $\boldsymbol{I}KNpp0$. From Basic (2), M $\ni 0$ is unique. Hence by Dictionary we have,

When $\wedge \in$ K. $\wedge$ is unique, $\bar{p} \in$ K such that $p + \bar{p} = \wedge$.

$*$ These two postulates V-1 and V-2 are one statement in Huntington's system.

(10) Postulate VI.

From thesis 110 : Law of negation of 0 and thesis 111 : Law of negation of $C00$, $EN0C00$ and $ENC000$ (we call Law of two poles). By basic (1) and (2), M $\ni 0$ is unique and M $\ni C00$ is unique. And further $N\boldsymbol{I}0C00$[4]. Hence we have

$\exists x, \exists y;\ x \in \mathrm{K},\ y \in \mathrm{K},\ x \neq y.$

Notes.

0) Eqp follows from Epq.

i Epq.

$04'.1\delta/' * C$i — ii,

ii $KCpqCqp$.

$55\ p/Cpq,\ q/Cqp * C$ii — iii,

iii $KCqpCpq$.

$04'.2\ \ \delta/', p/q,\ q/p\ * C$ iii — iv,

iv Eqp. □

1) Proof : Term 0 is unique. Assume that there are 0_1 and 0_2.

i $0_1 \in$ M, $0_2 \in$ M.

$60 *$ ii,

ii $C0_1p,\ C0_2p$.

$9\ r/C0_2p,\ q/C0_1p * C$ii — iii,

iii $CC0_1pC0_2p$.

iii $p/0_1 * C4p/0_1$—iv,

iv $C0_20_1$.

Similarly,

v $C0_10_2$.

$47\ p/C0_10_2,\ q/C0_20_1\ *\ C$v — Civ — vi,

vi $KC0_10_2C0_20_1$

$04'.2,\ \delta/' *$ vii,

vii $E0_20_1$.

Similarly,

viii $E0_20_1$.

From vii and viii, we have,

ix $\boldsymbol{I}0_10_2,\ \ \boldsymbol{I}0_20_1$. This shows 0 is uniquie. □

2), 4) Proof : 0 and $C00$ are not equal, that is 0 and $C00$ are different.

Since that $0 \in$ M, $C00 \in$ M, then

$4\ p/0 *$ i,

i $C00$.

$47\ p/C00,\ q/C00 * C$i — Ci — ii,

ii $KC00C00$.
$01.2\ \delta/KC00',\ p/0 * C$ii — iii,

iii $KC00N0$.
$06'.2$ II $\delta/',\ q/C00,\ p/0\ *$ iii — iv,

iv $N\boldsymbol{I}0C00$.
This shows that 0 and $C00$ are not equal. That is 0 and $C00$ are different. □

3) $\boldsymbol{I}PQ$ is deduced from EPQ.

i EPQ.
Doing likely as 0), we obtain,

ii EQP.
$47\ p/EPQ,\ q/EQP * C$i — Cii — iii,

iii $KEPQEQP$.
$06.2\ \delta/', p/P, q/Q * C$ iii — iv,

iv $\boldsymbol{I}PQ$. □

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Shotaro Tanaka
3-4-8 Takatadai Kamigori
Ako, Hyogo 678-1226 Japan
E-mail: `xymn47698@iris.eonet.ne.jp`