ON LOEWNER AND KWONG MATRICES

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Abstract. Let $f(t)$ be an operator monotone function from the interval $(0,\infty)$ into itself. In this note, we show that for any positive integer m, the matrices

$$\left[\frac{\{f(t_i)\}^m + \{f(t_j)\}^m}{t_i^m + t_j^m}\right], \quad \left[\frac{\{f(t_i)\}^m - \{f(t_j)\}^m}{t_i^m - t_j^m}\right]$$

are positive semidefinite for all positive integers n and $t_1,\ldots,t_n$ in $(0,\infty)$; that is, the Kwong matrices $K_{\{f(t^{1/m})\}^m}(t_1,\ldots,t_n)$ and the Loewner matrices $L_{\{f(t^{1/m})\}^m}(t_1,\ldots,t_n)$ are positive semidefinite. The former is a generalization of Kwong's result, and the latter is an alternative proof for operator monotonicity of the function $t \mapsto \{f(t^{1/m})\}^m$.

1 Introduction

Let $f(t)$ be a continuously differentiable function from the interval $(0,\infty)$ into itself. The function $f(t)$ is said to be *operator monotone* on $(0,\infty)$ if for two positive definite matrices A and B of any size n the inequality $A \geqq B$ implies $f(A) \geqq f(B)$. Here $A \geqq B$ means that $A - B$ is positive semidefinite. For distinct $t_1,\ldots,t_n$ in $(0,\infty)$, we define the $n\times n$ matrix $L_{f(t)}(t_1,\ldots,t_n)$ as

$$L_{f(t)}(t_1,\ldots,t_n) := \left[\frac{f(t_i) - f(t_j)}{t_i - t_j}\right],$$

where the diagonal entries are understood as the first derivatives $f'(t_i)$. This matrix is called a *Loewner matrix*. Similarly we define the $n\times n$ matrix $K_{f(t)}(t_1,\ldots,t_n)$ as

$$K_{f(t)}(t_1,\ldots,t_n) := \left[\frac{f(t_i) + f(t_j)}{t_i + t_j}\right],$$

which we call an *Kwong matrix*. (In [2, 9] it is called an anti-Loewner matrix.)

We also define the $n\times n$ matrix $L^{(m)}_{f(t)}(t_1,\ldots,t_n)$ and $K^{(m)}_{f(t)}(t_1,\ldots,t_n)$ as

$$L^{(m)}_{f(t)}(t_1,\ldots,t_n) := \left[\frac{\{f(t_i)\}^m - \{f(t_j)\}^m}{t_i^m - t_j^m}\right], \quad K^{(m)}_{f(t)}(t_1,\ldots,t_n) := \left[\frac{\{f(t_i)\}^m + \{f(t_j)\}^m}{t_i^m + t_j^m}\right]$$

for a positive integer m.

It is well-known that $f(t)$ is operator monotone if and only if for all n and $t_1,\ldots,t_n$, the Loewner matrices $L_{f(t)}(t_1,\ldots,t_n)$ are positive semidefinite, which is one of principal results by Löwner [11]. If $f(t)$ is operator monotone, the Kwong matrices $K_{f(t)}(t_1,\ldots,t_n)$ are positive semidefinite; this was given by Kwong [10]. In fact, the latter is recently characterized by Audenaert [2]. On the other hand, it is known that if $f(t)$ is operator monotone, so is the

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function $t \mapsto \{f(t^{1/m})\}^m$ for any positive integer m. See [1, 8]. Hence, combining them, we conclude that if f is operator monotone, then the Loewner matrices $L_{\{f(t^{1/m})\}^m}(t_1, \dots, t_n)$ and the Kwong matrices $K_{\{f(t^{1/m})\}^m}(t_1, \dots, t_n)$ are positive semidefinite; therefore, so are $L^{(m)}_{f(t)}(t_1, \dots, t_n)$ and $K^{(m)}_{f(t)}(t_1, \dots, t_n)$.

In this note, we give an alternative proof for operator monotonicity of the function $t \mapsto \{f(t^{1/m})\}^m$ by showing in Theorem 2.6 that if f is operator monotone, then $L^{(m)}_{f(t)}(t_1, \dots, t_n)$ are positive semidefinite for all n and $t_1, \dots, t_n$ in $(0, \infty)$. We also show in Theorem 2.5 that if f is operator monotone, then $K^{(m)}_{f(t)}(t_1, \dots, t_n)$ are positive semidefinite for all n and $t_1, \dots, t_n$ in $(0, \infty)$; in the case of $n = 1$, this is just Kwong's result. We refer the reader to [3, 4, 7] for properties of operator monotone functions.

2 Main Theorems

We recall several facts as mentioned:

Theorem 2.1 (Löwner [11]) Let f be a C^1 function on $(0, \infty)$. Then f is operator monotone if and only if $L_{f(t)}(t_1, \dots, t_n)$ are positive semidefinite for all positive integers n and $t_1, \dots, t_n$ in $(0, \infty)$.

Theorem 2.2 (Kwong [10]) Let f be a positive C^1 function on $(0, \infty)$. If f is operator monotone, then $K_{f(t)}(t_1, \dots, t_n)$ are positive semidefinite for all positive integers n and $t_1, \dots, t_n$ in $(0, \infty)$.

Although the following characterization is not used in this note, but we review: **Theorem 2.3** (Audenaert [2]) Let f be a positive C^1 function on $(0, \infty)$. For all positive integers n and $t_1, \dots, t_n$ in $(0, \infty)$ $K_{f(t)}(t_1, \dots, t_n)$ are positive semidefinite if and only if $f(\sqrt{t})\sqrt{t}$ is operator monotone.

Theorem 2.4 (Ando [1], Fujii-Fujii [8]) Let f be an operator monotone function from $(0, \infty)$ into itself. Then so is the function $t \mapsto \{f(t^{1/m})\}^m$ for any positive integer m.

We will show the following theorems:

Theorem 2.5 Let f be an operator monotone function from $(0, \infty)$ into itself. Then for any positive integer m, $K^{(m)}_{f(t)}(t_1, \dots, t_n)$ are positive semidefinite for all positive integers n and $t_1, \dots, t_n$ in $(0, \infty)$: or $K_{\{f(t^{1/m})\}^m}(t_1, \dots, t_n)$ are positive semidefinite for all positive integers n and $t_1, \dots, t_n$ in $(0, \infty)$.

Theorem 2.5 is a generalization of Theorem 2.2.

Theorem 2.6 Let f be an operator monotone function from $(0, \infty)$ into itself. Then for any positive integer m, $L^{(m)}_{f(t)}(t_1, \dots, t_n)$ are positive semidefinite for all positive integers n and $t_1, \dots, t_n$ in $(0, \infty)$: or $L_{\{f(t^{1/m})\}^m}(t_1, \dots, t_n)$ are positive semidefinite for all positive integers n and $t_1, \dots, t_n$ in $(0, \infty)$.

Theorem 2.6 shows another proof of Theorem 2.4.

Proof of Theorem 2.5. It is known that the operator monotone function f is of the form

$$f(t) = \alpha + \beta t + \int_0^\infty \frac{t}{t+\lambda}\, d\mu(\lambda),$$

where α, β are non-negative numbers and μ is a positive measure on $(0, \infty)$. See [3, p.144]. Let $g(t) = \int_0^\infty t/(t+\lambda)\, d\mu(\lambda)$. Then the power $\{f(t)\}^m$ is represented as the sum of $t^k\{g(t)\}^l$

for non-negative integers k, l satisfying $k+l \leqq m$ with non-negative coefficients, and $t^k\{g(t)\}^l$ is the multi-integral of

$$h(t) := \frac{t^{k+l}}{(t+\lambda_1)(t+\lambda_2)\cdots(t+\lambda_l)}$$

over $d\mu(\lambda_1)\cdots d\mu(\lambda_l)$. Hence, for our purpose, it is sufficient to show that $X := \left[\dfrac{h(t_i)+h(t_j)}{t_i^m+t_j^m}\right]$ is positive semidefinite. Letting $p(t) = (t+\lambda_1)(t+\lambda_2)\cdots(t+\lambda_l)$, we have the expression

$$X = \left[\frac{1}{p(t_i)} \frac{t_i^{k+l}p(t_j)+t_j^{k+l}p(t_i)}{t_i^m+t_j^m} \frac{1}{p(t_j)}\right]$$

and using the expansion of $p(t)$: $p(t) = a_0t^l + a_1t^{l-1} + \cdots + a_{l-1}t + a_l$ for $a_0 = 1$ and non-negative integers $a_1, \ldots, a_l$,

$$X = \sum_{s=0}^{l} a_{l-s} \left[\frac{t_i^s}{p(t_i)} \frac{t_i^{k+l-s}+t_j^{k+l-s}}{t_i^m+t_j^m} \frac{t_j^s}{p(t_j)}\right] = \sum_{s=0}^{l} a_{l-s}\ D_s \left[\frac{t_i^{k+l-s}+t_j^{k+l-s}}{t_i^m+t_j^m}\right] D_s,$$

where D_s is the diagonal matrix given as $D_s = \operatorname{diag}\left(\dfrac{t_1^s}{p(t_1)}, \ldots, \dfrac{t_n^s}{p(t_n)}\right)$. By [5, 6] or Theorem 2.2, $\left[\dfrac{t_i^{k+l-s}+t_j^{k+l-s}}{t_i^m+t_j^m}\right]$ is positive semidefinite, so is $D_s \left[\dfrac{t_i^{k+l-s}+t_j^{k+l-s}}{t_i^m+t_j^m}\right] D_s$. Hence, we conclude that X is positive semidefinite; therefore, the proof is complete. ∎

Note that

$$\begin{aligned} L_{f(t)}^{(2)} &= \left[\frac{\{f(t_i)\}^2-\{f(t_j)\}^2}{t_i^2-t_j^2}\right] = \left[\frac{f(t_i)+f(t_j)}{t_i+t_j}\right] \circ \left[\frac{f(t_i)-f(t_j)}{t_i-t_j}\right] \\ &= K_{f(t)}(t_1, \ldots, t_n) \circ L_{f(t)}(t_1, \ldots, t_n), \end{aligned}$$

where $\circ$ stands for Hadamard or Schur product: the entrywise product. When f is operator monotone, both matrices are positive semidefinite by Theorems 2.1 and 2.2, by Schur's Theorem so is their Hadamard product $L_{f(t)}^{(2)}$. For a positive integer k, since

$$\begin{aligned} L_{f(t)}^{(2^k)} &= \left[\frac{\{f(t_i)\}^{2^k}-\{f(t_j)\}^{2^k}}{t_i^{2^k}-t_j^{2^k}}\right] \\ &= \left[\frac{\{f(t_i)\}^{2^{k-1}}+\{f(t_j)\}^{2^{k-1}}}{t_i^{2^{k-1}}+t_j^{2^{k-1}}}\right] \circ \left[\frac{\{f(t_i)\}^{2^{k-1}}-\{f(t_j)\}^{2^{k-1}}}{t_i^{2^{k-1}}-t_j^{2^{k-1}}}\right] \\ &= K_{f(t)}^{(2^{k-1})}(t_1, \ldots, t_n) \circ L_{f(t)}^{(2^{k-1})}(t_1, \ldots, t_n), \end{aligned}$$

we conclude by induction and Theorem 2.5 that $L_{f(t)}^{(2^k)}$ is positive semidefinite for all k. But in fact, we have Theorem 2.6:

Proof of Theorem 2.6. We use the same notation as in the proof of Theorem 2.5. By the similar argument, it is sufficient to show that $Y := \left[\dfrac{h(t_i)-h(t_j)}{t_i^m-t_j^m}\right]$ is positive semidefinite. This matrix is represented as

$$Y = \sum_{s=0}^{l} a_{l-s} \left[\frac{t_i^s}{p(t_i)} \frac{t_i^{k+l-s} - t_j^{k+l-s}}{t_i^m - t_j^m} \frac{t_j^s}{p(t_j)} \right] = \sum_{s=0}^{l} a_{l-s}\, D_s \left[\frac{t_i^{k+l-s} - t_j^{k+l-s}}{t_i^m - t_j^m} \right] D_s.$$

By [5, 6] or Theorem 2.1, $\left[\frac{t_i^{k+l-s} - t_j^{k+l-s}}{t_i^m - t_j^m} \right]$ is positive semidefinite, so is Y and the proof is complete. ■

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