N-FOLD COMMUTATIVE BCI-ALGEBRAS

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Abstract. In this article, we introduce the notion of n-fold commutative BCI-algebra, a generalization of commutative BCI-algebras. Furthermore, we generalize commutative BCI-ideals to n-fold commutative BCI-ideals and prove the extension property for n-fold commutative ideals. Finally, we use the extension property to obtain a characterization of n-fold commutative BCI-algebras in terms of n-fold commutative BCI-ideals. This work is the second part of the folding theory of BCI-algebras, the first par wast on positive implicativeness.

1 Introduction K. Iséki and S. Tanaka introduced the concept of commutative BCK-algebras [4] and Iséki introduced the concept of BCI-algebras as a generalization of BCK-algebras [3]. In many aspects, the theory of BCK/BCI-algebras is parallel to the classical ring theory and that of ordered structures. For instance, there is a notion of ideals in these algebras and their theory has been extensively studied. In addition each BCI/BCK algebra is naturally equipped with an order structure which is often used to study the algebra [9], [10]. As seen in [1], various concepts of lattice theory can also be formulated and studied for these algebras.

In [12], H. Xie and Y.S. Huang introduced and studied n-fold commutativity in BCK-algebras as a generalization of commutative BCK-algebras. In [8], J. Meng and X. L. Xin introduced the notion of commutative BCI-algebras as a generalization of commutative BCK-algebras and studied some of their properties. Further characterizations of commutative BCI-algebras were established in [5] and [7].

Along the same lines, we introduce in the present article the notion of n-fold commutative BCI-algebras. We show that the newly introduced class of BCI-algebras contains properly both the classes of commutative BCI-algebras and the class of n-fold commutative BCK-algebras. It is well known that in various logical systems, the theory of BCI-ideals plays a fundamental role, ideals correspond to sets of provable formulas and closed with respect to modus ponens. In the light of this, the notion of ideals is not just an abstract concept, but is a mathematically deep and significant concept with applications in various areas, and most notably in logic. So, we introduce the notion of n-fold commutative BCI-ideals and use it to characterize n-fold commutative BCI-algebras. It should be pointed out that our future work is to analyse the logical properties of such structures.

2 Preliminaries and Notations A *BCI-algebra* is a set X with a binary operation $\star$ and a constant 0 satisfying the following axioms:

(i) $((x \star y) \star (x \star z)) \star (z \star y) = 0$;

(ii) $(x \star (x \star y)) \star y = 0$;

(iii) $x \star x = 0$;

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(iv) $x \star y = 0$ and $y \star x = 0$ implies $x = y$.

Every BCI-algebra comes with a natural BCI-order $\leq$ given by $x \leq y$ if and only if $x \star y = 0$.
A BCI-algebra satisfying $0 \leq x$ for all $x \in X$ is called a BCK-algebra.
An ideal of a BCI-algebra X is a subset I of X satisfying:
(i) $0 \in I$ and,
(ii)For every $x, y \in X$, if I contains both $x \star y$ and y, then it contains x.
Note that if I is an ideal of X, and $x \in X$, $a, b \in I$ with $x \star a \leq b$, then $x \in I$. This observation will be used frequently in the proofs without further notice.
An ideal I of a BCI-algebra X is closed if $0 \star x \in I$ for all $x \in I$ or equivalently, $x, y \in I$ implies $x \star y \in I$ for all $x, y \in X$.
We denote the expression $(\dots(x \star a) \star a) \star \dots a)$ by $x \star a^n$, where n is the number of times a appears in the expression, in particular $x \star a^0 = x$.

Proposition 2.1. *[2], [6] In every BCI-algebra X, the following axioms hold for all $x, y, z \in X$.*

(i) $x \star 0 = x$;

(ii) $x \leq y$, *then* $x \star z \leq y \star z$ *and* $z \star y \leq z \star x$;

(iii) $(x \star y) \star z = (x \star z) \star y$;

(iv) $x \star (x \star (x \star y)) = x \star y$;

(v) $(x \star z) \star (y \star z) \leq x \star y$;

(vi) $0 \star (x \star y) = (0 \star x) \star (0 \star y)$;

(vii) $0 \star (x \star y) = 0 \star (0 \star (y \star x))$;

(viii) $(0 \star (x \star y)) \star (y \star x) = 0$;

(ix) *If* $x \leq y$, *then* $0 \star x = 0 \star y$;

(x) *If* $x \leq 0 \star y$, *then* $x = 0 \star y$;

(xi) $0 \leq x \star (y \star (y \star x))$;

(xii) $0 \star (x \star y)^n = (0 \star x^n) \star (0 \star y^n)$;

(xiii) $0 \star (0 \star x^n) = 0 \star (0 \star x)^n$.

Lemma 2.2. *Let X be a BCI-algebra, and $n \geq 1$;*
(i) *For every $x, y \in X$*

$$(0 \star (0 \star (y \star x))) \star x^n = 0 \star (0 \star (y \star x^{n+1})) = (0 \star (0 \star (y \star x^n))) \star x;$$

(ii) *For every $x, y \in X$,*

$$0 \star (x \star y^n) = (0 \star x) \star (0 \star y^n).$$

Proof. (i) By induction, it is enough to show that $(0 \star (0 \star (y \star x))) \star x = 0 \star (0 \star (y \star x^2))$. But

$$\begin{aligned}(0 \star (0 \star (y \star x))) \star x &= (0 \star x) \star (0 \star (y \star x)) \\ &= 0 \star (x \star (y \star x)) \\ &= 0 \star (0 \star (y \star x^2))\end{aligned}$$

(ii) This follows from Proposition 2.1(vi). $\square$

Definition 2.3. *[8] A BCI-algebra X is called commutative if for every $x, y \in X$: $x \star y = 0$ implies $x = y \star (y \star x)$.*

We have the following characterizations of commutative BCI-algebras.

Proposition 2.4. *[5],[8] For every BCI-algebra X, the following assertions are equivalent:*
(i) X is commutative;
(ii) For all $x, y \in X$, $x \star (x \star y) = y \star (y \star (x \star (x \star y)))$;
(iii) For all $x, y \in X$, $(x \star (x \star y)) \star (y \star (y \star x)) = 0 \star (x \star y)$.

Proposition 2.5. *A BCI-algebra is commutative if and only if for all $x, y \in X$, $x \star y = x \star (y \star (y \star (x \star (x \star y))))$.*

Proof. For simplicity, we will write $f(x, y)$ for $y \star (y \star (x \star (x \star y)))$. Note that by BCI-axiom (ii) $f(x, y) \leq x \star (x \star y) \leq y$.
Assume that X is a commutative BCI-algebra. Since $f(x, y) \leq y$, then by Proposition 2.1(ii), $x \star y \leq x \star f(x, y)$. On the other hand, by Proposition 2.4, $x \star f(x, y) = x \star (x \star (x \star y)) \leq x \star y$. Therefore, $x \star y = x \star f(x, y)$ as required.
Conversely, suppose that $x \star y = x \star f(x, y)$ for all $x, y \in X$. To show that X is a commutative, it is enough to show by Proposition 2.4 that $x \star (x \star y) = f(x, y)$ for all $x, y \in X$. Let $x, y \in X$, then as pointed out above $f(x, y) \leq x \star (x \star y)$. On the other hand, by assumption $x \star (x \star y) = x \star (x \star f(x, y)) \leq f(x, y)$, so $x \star (x \star y) \leq f(x, y)$. Hence, $x \star (x \star y) = f(x, y)$ as needed. □

3 n-fold Commutative BCI-algebras We recall the definition of the corresponding notion for BCK-algebras.

Definition 3.1. *[12] A BCK-algebra X is called n-fold commutative if for all $x, y \in X$, $x \star y = x \star (y \star (y \star x^n))$.*

We also have the following characterization of n-fold commutative BCK-algebras.

Proposition 3.2. *[12] A BCK-algebra X is n-fold commutative if and only if for all $x, y \in X$, $x \star (x \star y) \leq y \star (y \star x^n)$.*

We introduce the following definition:

Definition 3.3. *A BCI-algebra is called n-fold commutative ($n \geq 1$) if for all $x, y \in X$,*

$$u \star (y \star (y \star u^n)) = 0 \star (0 \star (y \star u^n))$$

where $u = x \star (x \star y)$.

In the entire work, we will use u to denote $x \star (x \star y)$. We have the following characterization of n-fold commutative BCI-algebras.

Proposition 3.4. *For every BCI-algebra X, the following assertions are equivalent:*
(i) X is n-fold commutative;
(ii) For all $x, y \in X$, $x \leq y$ implies $x \star (y \star (y \star x^n)) = 0 \star (0 \star (y \star x^n))$;
(iii) For all $x, y \in X$, $x \leq y$ implies $x \star (y \star (y \star x^n)) = 0 \star x^{n-1}$.

Proof. (i) ⇒ (ii): Assume that X is n-fold commutative and let $x, y \in X$ such that $x \leq y$. Then $u = x \star (x \star y) = x$, therefore, $x \star (y \star (y \star x^n)) = 0 \star (0 \star (y \star x^n))$ as required.
(ii) ⇒ (i): Assume for all $x, y \in X$, $x \leq y$ implies $x \star (y \star (y \star x^n)) = 0 \star (0 \star (y \star x^n))$ and let $x, y \in X$. Since $u \leq y$, then by assumption, $u \star (y \star (y \star u^n)) = 0 \star (0 \star (y \star u^n))$. That is, X is n-fold commutative.
To see that (ii) and (iii) are equivalent, suppose that $x, y \in X$ with $x \leq y$. Then by Lemma 2.2, $0 \star (0 \star (y \star x^n)) = (0 \star (0 \star (y \star x))) \star x^{n-1} = (0 \star (x \star y)) \star x^{n-1} = 0 \star x^{n-1}$. □

Remark 3.5. *1. It follows from Proposition 2.1(vii) and Proposition 3.4 that* 1*-fold commutative BCI-algebras coincide with commutative BCI-algebras.*

2. It follows from Proposition 3.2 and Proposition 3.4 that every n-fold commutative BCK-algebra is an n-fold commutative BCI-algebra. In fact, suppose that X is an n-fold commutative BCK-algebra and let $x, y \in X$ such that $x \leq y$. Then by Proposition 3.2, $x = x\star(x\star y) \leq y\star(y\star x^n)$. Hence $x \leq y\star(y\star x^n)$ and $x\star(y\star(y\star x^n)) = 0 = 0\star x^{n-1}$. Thus X is an n-fold commutative BCI-algebra by Proposition 3.4(iii).

The following examples of n-fold commutative BCI-algebras justify that the newly introduced class of BCI-algebras contains properly the classes known so far.

Example 3.6. *1. Consider the BCI-algebra X whose Cayley's table is given by:*

$\star$	0	1	a	b	c
0	0	0	a	a	a
1	1	0	a	a	a
a	a	a	0	0	0
b	b	a	1	0	0
c	c	a	1	1	0

Then $(X, \star, 0)$ is 2*-fold commutative but not commutative.*

2. Let $n \geq 1$ and $X = \{0, 1, \ldots, n, a, b, c\}$. We define the operation $\star$ on X by:

$$
\begin{aligned}
&k \star l = max(0, k - l) \ \ 0 \leq k, l \leq n \\
&a \star 0 = a, \ 0 \star a = a \star a = 0 \\
&a \star k = 1, \ k \star a = k - 1, \ \textit{if}\, k = 1, \ldots, n \\
&b \star a = n, \ \ b \star 0 = b \\
&b \star k = (n + 1) - k, \ \textit{if}\, k = 1, \ldots, n \\
&x \star b = 0, \ \textit{if} \ x \in X \setminus \{c\} \\
&c \star x = x \star c = c, \ \textit{if}\, x \in X \setminus \{c\} \\
&c \star c = 0
\end{aligned}
$$

Then X is an $(n + 1)$-fold commutative BCI-algebra that is not n-fold commutative. This can be seen from [2, Exercise 1.2.2 and Example 2.4.4] and few extra calculations.

Remark 3.7. *Suppose that X is a BCI-algebra satisfying: for every $x, y \in X$, $(x \star (x \star y)) \star (y \star (y \star x^n)) = 0 \star (0 \star (y \star x^n))$.*
Then X is n-fold commutative by Proposition 3.4(ii), but the converse is false as the following example shows.

Example 3.8. *Let X be the BCI-algebra in Example 3.6*(1) *and let $Y = X \cup \{d\}$ with $d \notin X$. Extend the operation of X to Y as follows:*
$x \star d = d \star x = d$ if $x \in X$ and $d \star d = 0$.
Then Y is a BCI-algebra, in fact $Y = X \cup_L Z_2^{ad}$, the Li's union of X and the adjoint algebra of the Abelian group Z_2. In addition, since X is 2*-fold commutative, it follows easily that Y is* 2*-fold commutative. On the other hand, $(d\star(d\star c))\star(c\star(c\star d^2)) = a \neq 0 = 0\star(0\star(c\star d^2))$.*

Proposition 3.9. *The following property holds in every commutative BCI-algebra X.*

$$(x \star (x \star y)) \star (y \star (y \star x^n)) = 0 \star (0 \star (y \star x^n))$$

for all $x, y \in X$ and all $n \geq 1$.

Proof. Suppose X is commutative, and let $x, y \in X$, then $x \star (x \star y) = y \star (y \star (x \star (x \star y)))$. So,

$$\begin{aligned}(x \star (x \star y)) \star (y \star (y \star x^n)) &= (y \star (y \star (x \star (x \star y))) \star (y \star (y \star x^n)) \\ &\leq (y \star x^n) \star (y \star (x \star (x \star y))) \\ &= (y \star (y \star (x \star (x \star y)))) \star x^n \\ &= (x \star (x \star y)) \star x^n \\ &= (0 \star (x \star y)) \star x^{n-1} \\ &= (0 \star (0 \star (y \star x))) \star x^{n-1} \\ &= 0 \star (0 \star (y \star x^n)) \text{ by Lemma 2.2}\end{aligned}$$

Hence $(x \star (x \star y)) \star (y \star (y \star x^n)) \leq 0 \star (0 \star (y \star x^n))$.
It follows by the minimality of $0\star(0\star(y\star x^n))$ that $(x\star(x\star y))\star(y\star(y\star x^n)) = 0\star(0\star(y\star x^n))$. □

Combining Proposition 3.9 and Remark 3.7, we obtain the following result.

Corollary 3.10. *Every commutative BCI-algebra is n-fold commutative for all $n \geq 1$.*

4 n-fold Commutative BCI-ideals

Definition 4.1. *[2] An ideal I of a BCI-algebra X is called commutative if $x \star y \in I$ implies that $x \star ((y \star (y \star x)) \star (0 \star (y \star x))) \in I$ for all $x, y \in X$.*

We have the following characterization of closed commutative ideals.

Proposition 4.2. *[2] A closed ideal I of a BCI-algebra X is a commutative ideal if and only if $x \star y \in I$ implies $x \star (y \star (y \star x)) \in I$ for all $x, y \in X$.*

Definition 4.3. *[2] An ideal I of a BCK-algebra X is called n-fold commutative if, $x \star y \in I$ implies that $x \star (y \star (y \star x^n)) \in I$ for all $x, y \in X$.*

Motivated by the above, we introduce the following class of ideals.

Definition 4.4. *Let I be an ideal of a BCI-algebra X. Then I is called an n-fold commutative ideal of X if for every $x, y \in X$, $x \star y \in I$ implies $(x \star (y \star (y \star x^n)) \star (0 \star (0 \star (y \star x^n)))) \in I$.*

The following result shows the relation between 1-fold commutative and commutative ideals in a BCI-algebra.

Proposition 4.5. *Let I be an ideal of a BCI-algebra X.*
(i) *If I is commutative, then I is 1-fold commutative.*
(ii) *If I is closed and 1-fold commutative, then I is commutative.*

Proof. (i) Suppose that I is commutative and let $x, y \in X$ with $x \star y \in I$. Then $x \star ((y \star (y \star x)) \star (0 \star (y \star x))) \in I$. If we denote $y \star (y \star x)$ by v, then $x \star (v \star (0 \star (y \star x))) \in I$. We have

$$\begin{aligned}&((x \star v) \star (0 \star (0 \star (y \star x)))) \star (x \star (v \star (0 \star (y \star x)))) \\ &\qquad = ((x \star (x \star (v \star (0 \star (y \star x))))) \star v) \star (0 \star (0 \star (y \star x))) \\ &\qquad \leq ((v \star (0 \star (y \star x))) \star v) \star (0 \star (0 \star (y \star x))) \\ &\qquad = (0 \star (0 \star (y \star x))) \star (0 \star (0 \star (y \star x))) \\ &\qquad = 0 \in I.\end{aligned}$$

Hence, $((x \star v) \star (0 \star (0 \star (y \star x)))) \in I$.
(ii) Suppose that I is closed and 1-fold commutative. Let $x, y \in X$ such that $x \star y \in I$, then by assumption $((x \star v) \star (0 \star (0 \star (y \star x)))) \in I$. On the other hand, since I is closed, $0 \star (0 \star (y \star x))) = 0 \star (x \star y) \in I$. Thus $x \star v = x \star (y \star (y \star x)) \in I$. It follows from Proposition 4.2 that I is commutative. $\square$

Remark 4.6. *Every n-fold commutative BCK-ideal is an n-fold commutative BCI-ideal.*

Example 4.7. 1. *Consider the BCI-algebra X of Example 3.6(1). It is easy to see that $I = \{0, 1\}$ is a 2-fold commutative ideal of X.*

2. *The Zero ideal of the BCI-algebra of Example 3.6(2) is not n-fold commutative for all $n \geq 1$.*

We have the following characterization for n-fold commutative ideals.

Proposition 4.8. *Let X be a BCI-algebra and I a closed ideal of X.*
Then I is n-fold commutative if and only if for every $x, y \in X$, $x \star y \in I$ implies $(x \star (y \star (y \star x^n)) \star (0 \star x^{n-1})) \in I$

Proof. For simplicity, we write w for $x \star (y \star (y \star x^n)$. Note that by Lemma 2.2 and Proposition 2.1, $0 \star (0 \star (y \star x^n))) = (0 \star (0 \star (y \star x)))) \star x^{n-1} = (0 \star (x \star y)) \star x^{n-1} = (0 \star x^{n-1}) \star (x \star y)$
$\Longrightarrow$) Suppose that I is n-fold commutative and let $x, y \in X$ such that $x \star y \in I$. Then $w \star (0 \star (0 \star (y \star x^n))) \in I$. Also, since I is closed, then $0 \star (x \star y) \in I$. On the other hand,

$$\begin{aligned}(w \star (0 \star x^{n-1})) \star (w \star (0 \star (0 \star (y \star x^n)))) &\leq (0 \star (0 \star (y \star x^n))) \star (0 \star x^{n-1}) \\ &= ((0 \star x^{n-1}) \star (x \star y)) \star (0 \star x^{n-1}) \\ &= 0 \star (x \star y) \in I\end{aligned}$$

Therefore, $v \star (0 \star x^{n-1}) \in I$ as needed.
$\Longleftarrow$) Suppose that I satisfies the condition of the Proposition, and let $x, y \in X$ with $x \star y \in I$. Then $w \star (0 \star x^{n-1}) \in I$. But since

$$\begin{aligned}(w \star (0 \star (0 \star (y \star x^n)))) \star (w \star (0 \star x^{n-1})) &\leq (0 \star x^{n-1}) \star (0 \star (0 \star (y \star x^n))) \\ &= (0 \star x^{n-1}) \star ((0 \star x^{n-1}) \star (x \star y)) \\ &\leq x \star y \in I\end{aligned}$$

Therefore, $w \star (0 \star (0 \star (y \star x^n))) \in I$ and I is n-fold commutative. $\square$

Our next goal is to establish the extension property for ideals. We will need the following lemma.

Lemma 4.9. *Let X be a BCI-algebra and A a closed ideal of X.*
Then $x \star y \in A$ implies $(y \star x^n) \star (y \star u^n) \in A$ for all $x, y \in X$ and all $n \geq 1$. (as stated earlier, $u = x \star (x \star y)$)

Proof. We argue by induction on n.
- For $n = 1$, let $x, y \in A$ such that $x \star y \in A$. Then $(y \star x) \star (y \star u) \leq u \star x = 0 \star (x \star y)$. But since $x \star y \in A$ and A is closed, then $0 \star (x \star y) \in A$. Hence, $(y \star x) \star (y \star u) \in A$ as needed.

- Suppose that $n > 1$ and assume that the result holds for $n-1$. Let $x, y \in X$ such that $x \star y \in A$, then $(y \star x^{n-1}) \star (y \star u^{n-1}) \in A$. But, since

$$\begin{aligned}
&((y \star x^n) \star (y \star u^n)) \star ((y \star x^{n-1}) \star (y \star u^{n-1})) \\
&\qquad = ((y \star x^{n-1}) \star (y \star u^n)) \star ((y \star x^{n-1}) \star (y \star u^{n-1})) \star x \\
&\qquad \leq ((y \star u^{n-1}) \star (y \star u^n)) \star x \\
&\qquad \leq u \star x \\
&\qquad = 0 \star (x \star y) \in A
\end{aligned}$$

It follows that $(y \star x^n) \star (y \star u^n) \in A$ as required, which completes the proof of the lemma. □

Theorem 4.10. *(The Extension Property for n-fold Commutative Ideals)*
Let X be a BCI-algebra and I an n-fold commutative ideal of X.
Then every closed ideal A of X containing I is also n-fold commutative.

Proof. We will use the characterization of n-commutative ideals given in Proposition 4.8. Let $x, y \in X$ such that $x \star y \in A$. Let $u = x \star (x \star y)$, then $u \star y = 0 \in I$, therefore $(u \star (y \star (y \star u^n)) \star (0 \star u^{n-1})) \in I \subseteq A$. Thus $(x \star (y \star (y \star u^n)) \star (0 \star u^{n-1})) \star (x \star y) \in A$, and since $x \star y \in A$, then $(x \star (y \star (y \star u^n)) \star (0 \star u^{n-1})) \in A$. On the other hand,

$$\begin{aligned}
&((x \star (y \star (y \star u^n)) \star (0 \star x^{n-1}))) \star ((x \star (y \star (y \star u^n)) \star (0 \star u^{n-1}))) \\
&\qquad \leq (0 \star u^{n-1}) \star (0 \star x^{n-1}) \\
&\qquad = 0 \star (u \star x)^{n-1} \\
&\qquad = 0 \star (0 \star (x \star y))^{n-1} \in A
\end{aligned}$$

Therefore, $(x \star (y \star (y \star u^n))) \star (0 \star x^{n-1}) \in A$.
In addition,

$$\begin{aligned}
&((x \star (y \star (y \star x^n))) \star (0 \star x^{n-1})) \star ((x \star (y \star (y \star u^n)) \star (0 \star x^{n-1})) \\
&\qquad \leq (x \star (y \star (y \star x^n))) \star (x \star (y \star (y \star u^n))) \\
&\qquad \leq (y \star x^n) \star (y \star u^n)
\end{aligned}$$

But, by Lemma 4.9, $(y \star x^n) \star (y \star u^n) \in A$. Thus $(x \star (y \star (y \star x^n))) \star (0 \star x^{n-1}) \in A$, and A is n-fold commutative. □

Proposition 4.11. *A BCI-algebra X is an n-fold commutative BCI-algebra if and only if the zero ideal of X is n-fold commutative*

Proof. $\Rightarrow$) Suppose that X is n-fold commutative. Let $x, y \in X$ such that $x \star y = 0$, then $x \leq y$. Therefore by Proposition 3.4, $x \star (y \star (y \star x^n)) = 0 \star (0 \star (y \star x^n))$. Hence $(x \star (y \star (y \star x^n))) \star (0 \star (0 \star (y \star x^n))) = 0$. Therefore the zero ideal is n-fold commutative.
$\Leftarrow$) Suppose that the zero ideal of X is n-fold commutative. To show that X is n-fold commutative, we will show that Proposition 3.4 holds. Let $x, y \in X$ with $x \leq y$, then $x \star y = 0$, that is $x \star y$ is in the zero ideal. Thus, as the zero ideal is n-fold commutative, then $(x \star (y \star (y \star x^n))) \star (0 \star (0 \star (y \star x^n)))$ is in the zero ideal, that is $(x \star (y \star (y \star x^n))) \star (0 \star (0 \star (y \star x^n))) = 0$. Hence, $x \star (y \star (y \star x^n)) \leq 0 \star (0 \star (y \star x^n))$, which implies that $x \star (y \star (y \star x^n)) = 0 \star (0 \star (y \star x^n))$ as required. □

Combining the Extension Property for ideals and Proposition 4.11, we obtain the following characterization of n-fold commutative BCI-algebras.

Corollary 4.12. *A BCI-algebra X is n-fold commutative if and only if all closed ideals of X are n-fold commutative.*

Recall the following construction of quotient BCI-algebras. Let X be a BCI-algebra, I be a closed ideal of X, define the relation Θ on X by: for all $x, y \in X$ $x\Theta y$ if and only if $x \star y \in I$ and $y \star x \in I$. Then Θ is a congruence on X whose factor set is denoted by X/Θ. It is known that $< \frac{X}{\Theta}; \star, [0]_\Theta >$ is a BCI-algebra where $[x]_\Theta \star [y]_\Theta = [x \star y]_\Theta$. This BCI-algebra is commonly denoted by X/I, the class $[x]_\Theta$ of x is denoted by I_x and the operation on X/I is given by $I_x \star I_y = I_{x\star y}$. In addition, one can verify that $I_0 = I$, and $x \in I$ if and only if $I_x = I$.

Proposition 4.13. *A closed ideal I of a BCI-algebra X is n-fold commutative if and only if the quotient BCI-algebra X/I is n-fold commutative.*

Proof. By Proposition 4.11, it is enough to show that I is an n-fold commutative ideal of X if and only if $\{I\}$ is an n-fold commutative ideal of X/I. But this follows easily from the description of the quotient BCI-algebra preceding the proposition.

□

The next result shows that the class of commutative ideals is a subclass of n-fold commutative ideals.

Proposition 4.14. *Let X be a BCI-algebra and I be a closed commutative ideal of X. Then I is an n-fold commutative ideal of X for all $n \geq 1$.*

Proof. We will use the characterization of n-fold commutative ideals given in Proposition 4.8.
Let $x, y \in X$ such that $x \star y \in I$, then by Proposition 4.2, $x \star (y \star (y \star x)) \in I$. On the other hand, we have

$$\begin{aligned}
((x \star (y \star (y \star x^n)))&\star(0 \star x^{n-1})) \star (x \star (y \star (y \star x))) \\
&= (((x \star (x \star (y \star (y \star x)))) \star (y \star (y \star x^n))) \star (0 \star x^{n-1})) \\
&\leq ((y \star (y \star x)) \star (y \star (y \star x^n))) \star (0 \star x^{n-1})) \\
&\leq ((y \star x^n) \star (y \star x)) \star (0 \star x^{n-1}) \\
&= (0 \star x^{n-1}) \star (0 \star x^{n-1}) \\
&= 0 \in I
\end{aligned}$$

Hence, $(x \star (y \star (y \star x^n))) \star (0 \star x^{n-1}) \in I$ as needed. Therefore, I is n-fold commutative. □

Example 4.7(1) is a special case of the following result.

Proposition 4.15. *The BCK-part P of every BCI-algebra X is an n-fold commutative ideal of X for all $n \geq 1$.*

Proof. Let $x, y \in X$ such that $x \star y \in P$, that is $0 \star (x \star y) = 0$. Then using Lemma 2.2, $0 \star (0 \star (y \star x^n)) = (0 \star (0 \star (y \star x))) \star x^{n-1} = (0 \star (x \star y)) \star x^{n-1} = 0 \star x^{n-1}$.

Therefore a combination of Lemma 2.2 and Proposition 2.1 yields,

$$\begin{aligned}
0\star((x\star(y\star(y\star x^n))\star(0\star(0\star(y\star x^n))))) &= 0\star((x\star(y\star(y\star x^n)))\star(0\star x^{n-1}))\\
&= ((0\star x)\star(0\star(y\star(y\star x^n))))\star(0\star(0\star x^{n-1}))\\
&= ((0\star x)\star(0\star(0\star(0\star x^n))))\star(0\star(0\star x^{n-1}))\\
&= ((0\star x)\star((0\star x^n))\star(0\star(0\star x^{n-1}))\\
&= (0\star(x\star x^n))\star(0\star(0\star x^{n-1}))\\
&= (0\star(0\star x^{n-1}))\star(0\star(0\star x^{n-1}))\\
&= 0
\end{aligned}$$

Hence $(x\star(y\star(y\star x^n))\star(0\star(0\star(y\star x^n)))) \in P$. Thus, P is n-fold commutative as claimed. □

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