INTERPOLATIONALITY FOR SYMMETRIC OPERATOR MEANS

Jun Ichi Fujii

Received September 24, 2012

Abstract. Interpolational path of an operator mean has importance in the geometric sense. Here we give an equivalent condition that a symmetric operator mean in the sense of Kubo-Ando forms an interpolational path. Also we discuss some properties for interpolational means including those of another type of operator means. In addition, integral means induced by interpolational ones are introduced.

1 Introduction. Recall that an operator mean m in the sense of Kubo-Ando [12] is defined by a positive operator monotone function f_{m} on the half interval $(0,\infty)$ with $f_{\mathrm{m}}(1)=1$;

$$A\,\mathrm{m}\,B = A^{\frac{1}{2}} f_{\mathrm{m}}\left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}\right) A^{\frac{1}{2}}$$

for positive invertible operators A and B on a Hilbert space. Then the *monotonicity* holds,

$$A \le C \text{ and } B \le D \text{ imply } A\,\mathrm{m}\,B \le C\,\mathrm{m}\,D.$$

One of important examples is the geometric operator mean $\#$ defined by

$$A\#B = A^{\frac{1}{2}} \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}\right)^{\frac{1}{2}} A^{\frac{1}{2}}.$$

Another typical property for Kubo-Ando means is the 'transformer equality',

$$C^*(A\,\mathrm{m}\,B)C = C^*AC\,\mathrm{m}\,C^*BC$$

for any invertible operator C. In particular, the *homogeneity*

$$c(A\,\mathrm{m}\,B) = cA\,\mathrm{m}\,cB$$

holds for all positive numbers c. Thus this mean can be constructed by a numerical function $f_{\mathrm{m}}(x) = 1\,\mathrm{m}\,x$ which is called the representing function of m.

Let m be a symmetric operator mean; $A\,\mathrm{m}\,B = B\,\mathrm{m}\,A$, or equivalently

$$f_{\mathrm{m}}(x) = 1\,\mathrm{m}\,x = x\left(\frac{1}{x}\,\mathrm{m}\,1\right) = x\left(1\,\mathrm{m}\,\frac{1}{x}\right) = x f_{\mathrm{m}}\left(\frac{1}{x}\right). \tag{1}$$

Then, the initial conditions

$$A\,\mathrm{m}_0 B = A, \quad A\,\mathrm{m}_{1/2} B = A\,\mathrm{m}\,B, \quad A\,\mathrm{m}_1 B = B$$

and the following inductive relation

$$A\,\mathrm{m}_{(2k+1)/2^{n+1}} B = (A\,\mathrm{m}_{k/2^n} B)\,\mathrm{m}\,(A\,\mathrm{m}_{(k+1)/2^n} B) = (A\,\mathrm{m}_{(k+1)/2^n} B)\,\mathrm{m}\,(A\,\mathrm{m}_{k/2^n} B) \tag{2}$$

2000 *Mathematics Subject Classification.* 47A64, 47A63, 26E60.
Key words and phrases. operator mean, operator monotone function, path, geodesic, quasi-arithmetic mean, integral mean.

for natural numbers n and k with $2k+1 < 2^{n+1}$ determines the continuous path $A\,\mathrm{m}_t B$ from A to B of operator means. (Note that this argument can be reduced to the numerical one by the transformer equality.) If this path satisfies

$$(A\,\mathrm{m}_r B)\,\mathrm{m}_t (A\,\mathrm{m}_s B) = A\,\mathrm{m}_{(1-t)r+ts} B$$

for all weights r, s, $t \in [0,1]$, then we call it an *interporational path* and also call the original mean an *interpolational* one as in [7, 8, 11]. For $r \in [-1,1]$, the following parametrized operator means $\#_t^{(r)}$, which are also called the *quasi-arithmetic* ones,

$$A \#_t^{(r)} B = A^{\frac{1}{2}} \left((1-t)I + \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right)^r \right)^{\frac{1}{r}} A^{\frac{1}{2}}$$

are interpolational. The path $\#_t^{(0)} = \lim_{\varepsilon \downarrow 0} \#_t^{(\varepsilon)}$ is that of the geometric operator mean and it is also the geodesic of the Finsler manifold of the positive invertible operators by Corach-Porta-Recht [1], see also [2]. It is shown in [8] that all interpolational paths are differentiable. As Corach-Porta-Recht [1] pointed out, the derivatives of an interpolational path at the end points are closely related to the relative operator entropy, cf. [7, 8, 11, 15, 16]. Thus interpolational paths are significant in the geometric sense.

In this paper, we give an equivalent condition to the interpolationality for symmetric operator means and discuss the related properties for interpolational means.

2 Mixing property. Now we give an equivalent condition that a symmetric operator mean forms an interpolational path,

Theorem 1. *A symmetric operator mean* m *is interpolational if and only if*

mixing property, $\qquad (a\,\mathrm{m}\,b)\,\mathrm{m}\,(c\,\mathrm{m}\,d) = (a\,\mathrm{m}\,c)\,\mathrm{m}\,(b\,\mathrm{m}\,d)$

holds for all positive numbers a, b, c and d.

Proof. Suppose m_t is an interpolational path. By the homogeneity, we may assume that $d = 1$, $a > b, c > 1$. Then there exist $r, s > 0$ with $b = 1\,\mathrm{m}_r a$ and $c = 1\,\mathrm{m}_s a$. It follows that

$$\begin{aligned}(a\,\mathrm{m}\,b)\,\mathrm{m}\,(c\,\mathrm{m}\,1) &= (a\,\mathrm{m}\,(1\,\mathrm{m}_r a))\,\mathrm{m}\,((1\,\mathrm{m}_s a)\,\mathrm{m}\,1)\\ &= (1\,\mathrm{m}_{(r+1)/2} a)\,\mathrm{m}\,(1\,\mathrm{m}_{s/2} a) = 1\,\mathrm{m}_{(r+s+1)/4} a\\ &= (1\,\mathrm{m}_{(s+1)/2} a)\,\mathrm{m}\,(1\,\mathrm{m}_{r/2} a)\\ &= (a\,\mathrm{m}\,(1\,\mathrm{m}_s a))\,\mathrm{m}\,((1\,\mathrm{m}_r a)\,\mathrm{m}\,1) = (a\,\mathrm{m}\,c)\,\mathrm{m}\,(b\,\mathrm{m}\,1).\end{aligned}$$

Conversely suppose m satisfies the mixing property. First we show

$$(1\,\mathrm{m}_{k/2^n} a)\,\mathrm{m}\,(1\,\mathrm{m}_{\ell/2^n} a) = 1\,\mathrm{m}_{(k+\ell)/2^{n+1}} a \tag{3}$$

inductively. It holds for $n = 1$. Suppose it holds for not greater than n. We may assume that the k and ℓ are odd numbers $2k+1$ and $2\ell+1$ respectively. Then, by the definition (2), the mixing property and symmetry, we get

$$\begin{aligned}&(1\,\mathrm{m}_{(2k+1)/2^{n+1}} a)\,\mathrm{m}\,(1\,\mathrm{m}_{(2\ell+1)/2^{n+1}} a)\\ &\quad = \big((1\,\mathrm{m}_{k/2^n} a)\,\mathrm{m}\,(1\,\mathrm{m}_{(k+1)/2^n} a)\big)\ \mathrm{m}\ \big((1\,\mathrm{m}_{(\ell+1)/2^n} a)\,\mathrm{m}\,(1\,\mathrm{m}_{\ell/2^n} a)\big)\\ &\quad = \big((1\,\mathrm{m}_{k/2^n} a)\,\mathrm{m}\,(1\,\mathrm{m}_{(\ell+1)/2^n} a)\big)\ \mathrm{m}\ \big((1\,\mathrm{m}_{(k+1)/2^n} a)\,\mathrm{m}\,(1\,\mathrm{m}_{\ell/2^n} a)\big)\\ &\quad = (1\,\mathrm{m}_{(k+\ell+1)/2^{n+1}} a)\,\mathrm{m}\,(1\,\mathrm{m}_{(k+\ell+1)/2^{n+1}} a) = 1\,\mathrm{m}_{(k+\ell+1)/2^{n+1}} a,\end{aligned}$$

so that (3) holds for all n. By the continuity, we have

$$(1\,\mathrm{m}_r a)\,\mathrm{m}\,(1\,\mathrm{m}_s a) = 1\,\mathrm{m}_{(r+s)/2} a \tag{3$'$}$$

for all $r, s \in [0,1]$. Similarly we show

$$(1\,\mathrm{m}_r a)\,\mathrm{m}_{k/2^n}(1\,\mathrm{m}_s a) = 1\,\mathrm{m}_{(1-k/2^n)r+(k/2^n)s} a \tag{4}$$

inductively. In fact,

$$\begin{aligned}
&(1\,\mathrm{m}_r a)\,\mathrm{m}_{(2k+1)/2^{n+1}}(1\,\mathrm{m}_s a)\\
&= \big((1\,\mathrm{m}_r a)\,\mathrm{m}_{k/2^n}(1\,\mathrm{m}_s a)\big)\ \mathrm{m}\ \big((1\,\mathrm{m}_r a)\,\mathrm{m}_{(k+1)/2^n}(1\,\mathrm{m}_s a)\big)\\
&= \left((1\,\mathrm{m}_{\big((2^n-k)r+ks\big)/2^n} a)\right)\ \mathrm{m}\ \left((1\,\mathrm{m}_{\big((2^n-(k+1))r+(k+1)s\big)/2^n} a)\right)\\
&= 1\,\mathrm{m}_{\big((2^{n+1}-(2k+1))r+(2k+1)s\big)/2^{n+1}} a = 1\,\mathrm{m}_{\big(1-(2k+1)/2^{n+1}\big)r+\big((2k+1)/2^{n+1}\big)s} a,
\end{aligned}$$

so that (4) holds, and hence we have the interpolationality by the continuity. □

Remark 1. Essentially the interpolationality would be obtained by the special case;

$$(A\,\mathrm{m}_{1/4}B)\,\mathrm{m}_{1/2}(A\,\mathrm{m}_{3/4}B) = A\,\mathrm{m}_{1/2}B, \quad \text{that is,} \quad (1\,\mathrm{m}\,f(x))\,\mathrm{m}\,(f(x)\,\mathrm{m}\,x) = f(x)$$

holds for all $x > 0$. Note that two equations

$$\begin{aligned}
&(f(x)\,\mathrm{m}\,f(x))\,\mathrm{m}\,(1\,\mathrm{m}\,x) = f(x)\,\mathrm{m}\,f(x) = f(x) \qquad \text{and}\\
&(f(x)\,\mathrm{m}\,1)\,\mathrm{m}\,(f(x)\,\mathrm{m}\,x) = (f(x)\,\mathrm{m}\,1)\,\mathrm{m}\,(f(x)\,\mathrm{m}\,x) = (1\,\mathrm{m}\,f(x))\,\mathrm{m}\,(f(x)\,\mathrm{m}\,x)
\end{aligned}$$

hold. Then the first terms of them are equal if m has the mixing property and the last ones are equal if m is interpolational. Thus the equivalence follows immediately from the above two equations.

Putting $b = x$, $c = y$ and $a = d = 1$, we immediately obtain,

Corollary 2. *If* m *is interpolational, then*

$$f(x\,\mathrm{m}\,y) = f(x)\,\mathrm{m}\,f(y), \quad \textit{or equivalently} \quad x\,\mathrm{m}\,y = f^{-1}\big(f(x)\,\mathrm{m}\,f(y)\big). \tag{5}$$

The above theorem shows that the mean of two interpolational ones is not always interpolational. Though $\frac{\frac{A+B}{2}+A\#B}{2}$ is interpolational since it coincides with $A\#^{(1/2)}B$, the following example does not satisfy even the condition (5),

Example 1. Put a symmetric operator mean $\mathbf{M}$ defined by the representing operator monotone function

$$f(x) = \frac{\frac{1+x}{2} + \left(\frac{1+\sqrt{x}}{2}\right)^2}{2} = \frac{3+3x+2\sqrt{x}}{8},$$

which is the mean for two quasi-arithmetic means. Since $a\,\mathbf{M}\,b = \frac{3a+3b+2\sqrt{ab}}{8}$, we have

$$\begin{aligned}
f(a\,\mathbf{M}\,b) &= \frac{24+9a+9b+6\sqrt{ab}+2\sqrt{8(3a+3b+2\sqrt{ab})}}{64} \quad \text{and}\\
f(a)\,\mathbf{M}\,f(b) &= \frac{3f(a)+3f(b)+2\sqrt{f(a)f(b)}}{8}\\
&= \frac{18+9a+9b+6\sqrt{a}+6\sqrt{b}+2\sqrt{(3+3a+2\sqrt{a})(3+3b+2\sqrt{b})}}{64}
\end{aligned}$$

are not equal.

One of the useful operation among operator means is the *adjoint* m^* for m,

$$A\,\mathrm{m}^*B = \left(A^{-1}\,\mathrm{m}\,B^{-1}\right)^{-1}, \quad \text{and} \quad f^*(x) = \frac{1}{f\left(\frac{1}{x}\right)}$$

for invertible operators A and B and $x > 0$. The arithmetic mean ∇ and the harmonic one **!** are adjoint each other and the geometric one $\#$ is 'self-adjoint'. Then it follows immediately from Theorem 1 that the interpolationality preserves this operation,

Corollary 3. *The adjoint* m^* *is interpolational if and only if* m *is interpolational.*

3 Operator mixing property. Here we discuss the following property stronger than the mixing one

operator mixing property, $\quad (A\,\mathrm{m}\,B)\,\mathrm{m}\,(C\,\mathrm{m}\,D) = (A\,\mathrm{m}\,C)\,\mathrm{m}\,(B\,\mathrm{m}\,D).$

The arithmetic mean ∇ and the harmonic one **!** satisfy it. Immediately we have this is invariant for the adjoint,

Lemma 4. *If* m *satisfies the operator mixing property, then so does* m^**.*

We will show later in this section that other operator means do not.

Let m be a symmetric operator mean with the representing function f. When m satisfies the above, it coincides with the quasi-mean for f and m like (5),

Lemma 5. *If* m *satisfies the operator mixing property, then*

$$(5') \qquad f(A\,\mathrm{m}\,B) = f(A)\,\mathrm{m}\,f(B), \quad \textit{or equivalently} \quad A\,\mathrm{m}\,B = f^{-1}\big(f(A)\,\mathrm{m}\,f(B)\big).$$

It is shown in [12, 6] that the maximum (resp. minimum) is the arithmetic (resp. harmonic) mean ∇ (resp. **!**) in the symmetric operator ones. Note that $a(0) \equiv \lim_{\varepsilon\downarrow 0} a(\varepsilon) = 1/2$ and $h(0) = 0$ for the representing functions for ∇ and **!** respectively, $a(x) = (1+x)/2$ and $h(x) = 2x/(1+x)$. Here we also use the following result for the adjoint,

Lemma 6. *If* f *is a positive operator monotone function for symmetric operator mean, then* $f^*(0) = 0$ *or* $f(0) = 0$*.*

Proof. Suppose $f(0) > 0$. Then (1) shows that f must diverge at infinity, so that

$$f^*(0) = \lim_{\varepsilon\downarrow 0} f^*(\varepsilon) = \lim_{\varepsilon\downarrow 0} \frac{1}{f\left(\frac{1}{\varepsilon}\right)} = 0.$$

Thus we completes the proof. $\square$

Noting that $f(P) = P$ and $f(tP) = f(t)P$ for a projection P if $f(0) = 0$, we easily obtain the operator mean $A\,\mathrm{m}\,P$ if $\operatorname{rank} P = 1$,

Lemma 7. *Let* m *be an operator mean whose representiong function* f *satisfies* $f(0) = 0$*. If* A *is invertible and* $B = \xi \otimes \xi^*$ *is a projection of rank 1 (*ξ *is a unit vector), then*

$$A\,\mathrm{m}\,B = f^*\left(\frac{1}{\|A^{-\frac{1}{2}}\xi\|^2}\right)\xi\otimes\xi^* = f^*\left(\frac{1}{\|BA^{-1}B\|}\right)B.$$

Proof. Let η be the unit vector $A^{-\frac{1}{2}}\xi/\|A^{-\frac{1}{2}}\xi\|$. Then

$$A^{-\frac{1}{2}}(\xi \otimes \xi^*)A^{-\frac{1}{2}} = A^{-\frac{1}{2}}\xi \otimes (A^{-\frac{1}{2}}\xi)^* = \|A^{-\frac{1}{2}}\xi\|^2 \eta \otimes \eta^*,$$

so that $f(A^{-\frac{1}{2}}(\xi \otimes \xi^*)A^{-\frac{1}{2}}) = f(\|A^{-\frac{1}{2}}\xi\|^2)\eta \otimes \eta^*$. Since $f^*(y) = yf(1/y)$, we have

$$\begin{aligned} A \,\mathrm{m}\, B &= A^{\frac{1}{2}} f(\|A^{-\frac{1}{2}}\xi\|^2)\eta \otimes \eta^* A^{\frac{1}{2}} = f(\|A^{-\frac{1}{2}}\xi\|^2) A^{\frac{1}{2}}\eta \otimes (A^{\frac{1}{2}}\eta)^* \\ &= f(\|A^{-\frac{1}{2}}\xi\|^2) \frac{1}{\|A^{-\frac{1}{2}}\xi\|^2} \xi \otimes \xi^* = f^*\left(\frac{1}{\|A^{-\frac{1}{2}}\xi\|^2}\right)(\xi \otimes \xi^*). \end{aligned}$$

It is well-known that $\|A^{-\frac{1}{2}}\xi\|^2 = \|BA^{-1}B\|$. □

Now we show that the operator mixing property holds only for ∇ and $\mathbf{!}$,

Theorem 8. *If a symmetric operator mean* m *satisfies the operator mixing property, then it coincides with the arithmetic mean* ∇ *or the harmonic one* $\mathbf{!}$.

Proof. For the representing function f of m, we may assume $f(0) = 0$ by Corollary 2 and Lemmas 4 and 6. Now we will show m is the harmonic mean. For $x > 0$ and projections of rank one

$$P = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad Q = \frac{1}{1+x}\begin{pmatrix} x & \sqrt{x} \\ \sqrt{x} & 1 \end{pmatrix} \quad \text{and} \quad R = \frac{1}{1+f(x)}\begin{pmatrix} f(x) & \sqrt{f(x)} \\ \sqrt{f(x)} & 1 \end{pmatrix},$$

put

$$A = \begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix} \quad \text{and} \quad B = P.$$

Since $\|BA^{-1}B\| = \frac{1}{2}(1 + 1/x) = \frac{1+x}{2x}$ and $f^* = f$, we have by the above lemma

$$\begin{aligned} A \,\mathrm{m}\, B &= f\left(\frac{1+x}{2x}\right)\frac{2x}{1+x}P = f\left(\frac{2x}{1+x}\right)P, \\ f\left(A \,\mathrm{m}\, B\right) &= f\left(f\left(\frac{2x}{1+x}\right)\right)P = f(f(h(x)))P \quad \text{and} \\ f(A) \,\mathrm{m}\, f(B) = f(A) \,\mathrm{m}\, B &= f\left(\frac{2f(x)}{1+f(x)}\right)P = f(h(f(x)))P, \end{aligned}$$

so that $f(h(x)) = h(f(x))$, or equivalently, $f^*(a(x)) = a(f^*(x))$. Since f^* is concave, then f^* is affine on the interval between 1 and x, and hence $f^*(x) = \alpha + \beta x$ for all $x > 0$. By the symmetric condition (1), we have

$$\alpha + \beta x = f^*(x) = xf^*(1/x) = \alpha x + \beta$$

for all $x > 0$. Thus $\alpha = \beta = 1/2$ by $f^*(1) = 1$, that is, m^* is the arithmetic mean. Therefore m is the harmonic one. □

For the numerical case, it is uncertain whether (5) implies the mixing property or not. But, by the proof of Theorem 8, we have that (5') implies the operator mixing property since the means ∇ and $\mathbf{!}$ have its property.

Corollary 9. *If a symmetric operator mean satisfies* $(5')$*, then it has the operator mixing property.*

Remark 2. As a generalization of the Kubo-Ando mean, we introduced a *chaotic* operator mean $\mathfrak{m}$ in [10] which satisfies the monotonicity for the chaotic order $\ll$,

$$A \le C \text{ and } B \le D \text{ imply } A\,\mathfrak{m}\,B \ll C\,\mathfrak{m}\,D$$

where $X \ll Y$ stands for $\log X \le \log Y$. A typical example is the path of the quasi-arithmetic mean,

$$A\,\mathfrak{m}_t^{(r)}B = ((1-t)A^r + tB^r)^{1/r}$$

which is 'interpolational' and also the geodesic of the Finsler manifold of positive invertible operators shown in [5], see also [14, 4]. Also it has the mixing property for operators. More generally the quasi-arithmetic mean defined by

$$A\,\mathfrak{m}\,B = f^{-1}\left(\frac{f(A)+f(B)}{2}\right)$$

satisfies the mixing property,

$$\begin{aligned}(A\,\mathfrak{m}\,B)\,\mathfrak{m}\,(C\,\mathfrak{m}\,D) &= f^{-1}\left(\frac{f(A\,\mathfrak{m}\,B)+f(C\,\mathfrak{m}\,D)}{2}\right)\\ &= f^{-1}\left(\frac{\frac{f(A)+f(B)}{2}+\frac{f(C)+f(D)}{2}}{2}\right)\\ &= f^{-1}\left(\frac{f(A)+f(B)+f(C)+f(D)}{4}\right) = (A\,\mathfrak{m}\,C)\,\mathfrak{m}\,(B\,\mathfrak{m}\,D).\end{aligned}$$

4 Integral means.

Recall that the *transpose* m° of a mean is defined by $A\,\mathrm{m}^\circ B = B\,\mathrm{m}\,A$. Then we show symmetric property for every path generated by (2),

Lemma 10. $A\,\mathrm{m}_t^\circ B = B\,\mathrm{m}_t A = A\,\mathfrak{m}_{1-t}B$.

Proof. The above relation holds for $t = 0, 1/2, 1$. Suppose it holds for $t = k/2^n$ for some n and all $k < 2^n$. Then the inductive relation (2) implies

$$\begin{aligned}B\,\mathrm{m}_{(2k+1)/2^{n+1}}A &= (B\,\mathrm{m}_{k/2^n}A)\,\mathrm{m}\,(B\,\mathrm{m}_{(k+1)/2^n}A) = (A\,\mathrm{m}_{1-k/2^n}B)\,\mathrm{m}\,(A\,\mathrm{m}_{1-(k+1)/2^n}B)\\ &= (A\,\mathrm{m}_{(2^n-k)/2^n}B)\,\mathrm{m}\,(A\,\mathrm{m}_{(2^n-k-1)/2^n}B) = (A\,\mathrm{m}_{(2^n-k-1)/2^n}B)\,\mathrm{m}\,(A\,\mathrm{m}_{(2^n-k)/2^n}B)\\ &= A\,\mathrm{m}_{(2^{n+1}-2k-1)/2^{n+1}}B = A\,\mathrm{m}_{1-(2k+1)/2^{n+1}}B,\end{aligned}$$

which implies the required relation. □

The logarithmic mean $\mathbf{L}$ defined by the function $f_{\mathbf{L}}(x) = (x-1)/\log x$ is an example that is not interpolational. In fact, by $e^2\,\mathbf{L}\,e = \frac{e^2-e}{\log e^2 - \log e} = e(e-1)$, the values

$$\begin{aligned}f(e^2\,\mathbf{L}\,e) &= \frac{e^2-e-1}{\log e(e-1)} = \frac{e^2-e-1}{1+\log(e-1)} \approx 2.38157 \quad \text{and}\\ f(e^2)\,\mathbf{L}\,f(e) &= \frac{e^2-1}{2}\,\mathbf{L}\,(e-1) = \frac{\frac{e^2-1}{2}-(e-1)}{\log\frac{e+1}{2}} = \frac{(e-1)^2}{2\log\frac{e+1}{2}} \approx 2.38060\end{aligned}$$

are not equal and hence it is not interpolational by Corollary 2. It is also known that $A\,\mathbf{L}\,B = \int_0^1 A\#_t B\,dt$ and $A\,\mathbf{L}\,B \ge A\#B$, cf.[9]. In this section, we generalize this result. For a path m_t generated by (2), we define the *integral mean* $\widetilde{\mathrm{m}}$ *induced by* m_t as

$$A\,\widetilde{\mathrm{m}}\,B = \int_0^1 A\,\mathrm{m}_t B\,dt.$$

Then we extend the previous result,

Theorem 11. *The integral mean* $\widetilde{\mathrm{m}}$ *induced by an interpolational path* m_t *for a symmetric operator mean* m *is symmetric and not less than* m*.*

Proof. The above lemma shows

$$A\,\widetilde{\mathrm{m}}\,B = \int_0^1 A\,\mathrm{m}_t B\,dt = \int_0^1 B\,\mathrm{m}_{1-t} A\,dt = \int_0^1 B\,\mathrm{m}_s A\,ds = B\,\widetilde{\mathrm{m}}\,A.$$

By the maximality of the arithmetic mean, we have

$$\begin{aligned} A\,\widetilde{\mathrm{m}}\,B &= \int_0^1 A\,\mathrm{m}_t B\,dt = \frac{\int_0^1 A\,\mathrm{m}_t B\,dt + \int_0^1 B\,\mathrm{m}_t A\,dt}{2} = \int_0^1 \frac{A\,\mathrm{m}_t B + A\,\mathrm{m}_{1-t} B}{2}\,dt \\ &\geq \int_0^1 (A\,\mathrm{m}_t B)\,\mathrm{m}\,(A\,\mathrm{m}_{1-t} B)\,dt = \int_0^1 A\,\mathrm{m}\,B\,dt = A\,\mathrm{m}\,B. \end{aligned}$$

□

Finally we give a fruitful example of the integral mean for parametrized power one,

Example 2. Let $\#_t^{(r)}$ be the quasi-arithmetic mean with the representing function

$$f_t^{(r)}(x) = (1 - t + tx^r)^{\frac{1}{r}}$$

for $-1 \leqq r \leqq 1$. Then, the representing function $\widetilde{f}^{(r)}$ of the integral mean $\widetilde{\#}^{(r)}$ is obtained by

$$\widetilde{f}^{(r)}(x) = \int_0^1 (1 - t + tx^r)^{\frac{1}{r}}\,dt = \left[\frac{(1 - t + tx^r)^{\frac{1+r}{r}}}{(x^r - 1)\frac{1+r}{r}}\right]_0^1 = \frac{r}{1+r}\,\frac{x^{r+1} - 1}{x^r - 1},$$

which is known as the power difference means, cf. [13]. Typical operator means we obtain here are,

$$\begin{aligned} (r = 1) &\quad \text{arithmetic mean, } \widetilde{f}^{(1)}(x) = \frac{1+x}{2}, \\ (r = 0) &\quad \text{logarithmic mean, } \widetilde{f}^{(0)}(x) \equiv \lim_{\varepsilon \downarrow 0} \widetilde{f}^{(\varepsilon)}(x) = \frac{x-1}{\log x}, \\ (r = -1/2) &\quad \text{geometric mean, } \widetilde{f}^{(-1/2)}(x) = \sqrt{x}, \\ (r = -1) &\quad \text{adjoint logarithmic mean, } \widetilde{f}^{(-1)}(x) \equiv \lim_{\varepsilon \downarrow 0} \widetilde{f}^{(\varepsilon - 1)}(x) = \frac{x \log x}{x - 1}. \end{aligned}$$

Even for $-2 \leqq r < -1$, we can formally obtain operator means, for example, $\#^{(-2)}$ yields the harmonic one,

$$\widetilde{f}^{(-2)}(x) = \frac{2x}{x+1}.$$

But the original $\#^{(-2)}$ itself is not operator mean in the sense of Kubo-Ando since $f^{(-2)}$ is not operator monotone.

Acknowledgement. This study is partially supported by the Ministry of Education, Science, Sports and Culture, Grant-in-Aid for Scientific Research (C), 23540200, 2011. Also the author would like to express his hearty thanks to members of Nakamura seminar.

References

[1] G.Corach, H.Porta and L.Recht, *Geodesics and operator means in the space of positive operators*, Internat. J. Math., **4** (1993), 193–202.

[2] G.Corach and A.L.Maestripieri, *Differential and metrical structure of positive operators*, Positivity, **3** (1999), 297–315.

[3] J.I.Fujii, *Initial conditions on operator monotone functions*, Math. Japon., **24** (1979), 459–462.

[4] J.I.Fujii, *Path of quasi-means as a geodesic*, Linear Alg. Appl., **434**(2011), 542–558.

[5] J.I.Fujii, *Structure of Hiai-Petz parametrized geometry for positive definite matrices*, Linear Alg. Appl., **432**(2010), 318–326.

[6] J.I.Fujii and M.Fujii, *Some remarks on operator means*, Math. Japon., **24** (1979), 335–339.

[7] J.I.Fujii and E.Kamei, *Uhlmann's interpolational method for operator means*, Math. Japon., **34** (1989), 541–547.

[8] J.I.Fujii and E.Kamei, *Interpolational paths and their derivatives*, Math. Japon., **39** (1994), 557–560.

[9] J.I.Fujii and Y.Seo, *On parametrized operator means dominated by power ones*, Sci. Math., **1** (1998), 301–306.

[10] J.I.Fujii, M.Nakamura and S.-E.Takahasi, *Cooper's approach to chaotic operator means*, Sci. Math. Japon., **39** (1994), 557–560.

[11] E.Kamei, *Paths of operators parametrized by operator means,* Math. Japon., **39**(1994), 395–400.

[12] F.Kubo and T.Ando, *Means of positive linear operators*, Math. Ann., **248** (1980) 205–224.

[13] F.Hiai and H.Kosaki, Means of Hilbert Space Operators, Springer-Verlag, 2003.

[14] F.Hiai, D.Petz, *Riemannian metrics on positive definite matrices related to means*, Linear Alg. Appl., **430** (2009), 3105–3130.

[15] A.Uhlmann, *Relative entropy and the Wigner-Yanase-Dyson-Lieb concavity in an interpolation theory*, Commun. Math. Phys., **54**(1977), 22–32.

[16] K.Yanagi, K.Kuriyama and S.Furuichi, *Generalized Shannon inequalities based on Tsallis relative operator entropy*, Linear Alg. Appl., **394**(2005), 109–118.

Communicated by *Masatoshi Fujii*

Department of Art and Sciences (Information Science), Osaka Kyoiku University, Asahigaoka, Kashiwara, Osaka 582-8582, Japan.

E-mail address, fujii@cc.osaka-kyoiku.ac.jp