ON ZERO-DIMENSIONALITY OF WIJSMAN TOPOLOGIES ON DISCRETE SPACES

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Abstract. In this paper, we give an example of a uniformly discrete metric space whose Wijsman hyperspace is not zero-dimensional, which answers a question posed by Cao, Junnila and Moors [3] negatively.

1. Introduction

Let (X, d) be a nonempty metric space and $CL(X)$ the set of nonempty closed subsets of X . For each $x \in X$ and $A \in CL(X)$, put $d(x, A) = \inf\{d(x, y) : y \in A\}$, and let $d(x, \cdot)$ denote the real-valued function on $CL(X)$ assigning $A \in CL(X)$ to $d(x, A)$. The Wijsman hyperspace $(CL(X), \tau_{w(d)})$ is equipped with the Wijsman topology $\tau_{w(d)}$ which is the weak topology determined by the family $\{d(x, \cdot) : x \in X\}$. This topology is suggested by the set convergence introduced by Wijsman [4] (see also [2]). For properties of Wijsman hyperspaces, we refer to [1] and [2].

In [3], Cao, Junnila and Moors proved some theorems on Wijsman hyperspaces of discrete spaces. In particular, they proved that if (X, d) is a finite-valued discrete metric space, then $(CL(X), \tau_{w(d)})$ is zero-dimensional. On the other hand, it is also proved in [3] that if (X, d) is a discrete metric space, then $(CL(X), \tau_{w(d)})$ is totally disconnected. Concerning these theorems, they asked the following question.

Question 1 (Cao, Junnila and Moors [3])**.** Is $(CL(X), \tau_{w(d)})$ *zero-dimensional* even if (X, d) is a uniformly discrete metric space or a discrete metric space?

In this note, we give a counterexample which answers Question 1 negatively.

2. A counterexample

A topological space X is said to be *zero-dimensional* if it has a clopen base. A metric space (X, d) is said to be *uniformly discrete* if there exists $\varepsilon > 0$ such that $d(x, y) > \varepsilon$ for every $x, y \in X$ with $x \neq y$. Let $\mathbb{R}$ denote the set of all real numbers and set $\mathbb{R}^+ = \{x \in \mathbb{R} : x \geq 0\}$.

Example 2.1. There exists a uniformly discrete metric d on $\mathbb{R}^+$ such that the Wijsman topology $\tau_{w(d)}$ on $CL(\mathbb{R}^+)$ is not zero-dimensional.

Proof. Define $d : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ by setting

$$d(x, y) = \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{if } 0 < |x - y| \leq 1, \text{and} \\ |x - y| & \text{if } |x - y| > 1. \end{cases}$$

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Then d is a uniformly discrete metric on $\mathbb{R}^+$. For $x \in \mathbb{R}^+$ and $a, b \in \mathbb{R}$, put $S^x_{(a,b)} = \{F \in CL(\mathbb{R}^+) : a < d(x, F) < b\}$. Then $\{S^x_{(a,b)} : x \in \mathbb{R}^+, a, b \in \mathbb{R}\}$ is a subbase for the Wijsman topology $\tau_{w(d)}$ on $CL(\mathbb{R}^+)$ (see [1, §2.1]). To see that $\tau_{w(d)}$ is not zero-dimensional, it suffices to show that every nonempty open subset $\mathcal{U}$ of $(CL(\mathbb{R}^+), \tau_{w(d)})$ with $\mathcal{U} \subset S^0_{(1,3)}$ is not closed in $(CL(\mathbb{R}^+), \tau_{w(d)})$.

Let $\mathcal{U}$ be a nonempty open subset of $(CL(\mathbb{R}^+), \tau_{w(d)})$ with $\mathcal{U} \subset S^0_{(1,3)}$. First we claim that the partial ordered set $(\overline{\mathcal{U}}, \subset)$ has a maximal element where $\overline{\mathcal{U}}$ denotes the closure of $\mathcal{U}$ in $(CL(\mathbb{R}^+), \tau_{w(d)})$. Let $\mathcal{E}$ be a chain in $\overline{\mathcal{U}}$. We show that $\bigcup \mathcal{E} \in \overline{\mathcal{U}}$.

Indeed, assume $\bigcup \mathcal{E} \notin \overline{\mathcal{U}}$. Then there exist $x_i \in \mathbb{R}^+$ and $a_i, b_i, \in \mathbb{R}$, $i \in \{1, 2, \dots n\}$, such that $\bigcup \mathcal{E} \in \bigcap_{i=1}^n S^{x_i}_{(a_i,b_i)}$ and

$$\bigcap_{i=1}^{n} S^{x_i}_{(a_i,b_i)} \cap \overline{\mathcal{U}} = \varnothing. \tag{1}$$

For every $i \in \{1, \dots, n\}$, choose $e_i \in \bigcup \mathcal{E}$ such that $a_i < d(x_i, e_i) < b_i$. Since $\mathcal{E}$ is a chain, there exists $F \in \mathcal{E}$ such that $\{e_1, e_2, \dots, e_n\} \subset F$. Since $\{e_1, e_2, \dots, e_n\} \subset F \subset \bigcup \mathcal{E}$, we have $a_i < d(x_i, \bigcup \mathcal{E}) \le d(x_i, F) \le d(x_i, e_i) < b_i$ for every $i \in \{1, \cdots, n\}$, which implies $F \in \bigcap_{i=1}^n S^{x_i}_{(a_i,b_i)}$. Because $F \in \mathcal{E} \subset \overline{\mathcal{U}}$, we have $\bigcap_{i=1}^n S^{x_i}_{(a_i,b_i)} \cap \overline{\mathcal{U}} \neq \varnothing$. This contradicts (1). Therefore $\bigcup \mathcal{E} \in \overline{\mathcal{U}}$.

Thus, $\bigcup \mathcal{E}$ is an upper bound of $\mathcal{E}$ in $\overline{\mathcal{U}}$. By Zorn's lemma, there exists a maximal element $E_0 \in \overline{\mathcal{U}}$.

Next, we claim that $E_0 \notin \mathcal{U}$, which implies $\mathcal{U}$ is not closed. Suppose $E_0 \in \mathcal{U}$. Then $E_0 \in S^0_{(1,3)}$, and we have $\inf E_0 > 1$. Since $\mathcal{U}$ is open in $(CL(\mathbb{R}^+), \tau_{w(d)})$, there exist $x_i \in \mathbb{R}^+$ and $a_i, b_i, \in \mathbb{R}$, $i \in \{1, 2, \dots n\}$, such that $E_0 \in \bigcap_{i=1}^n S^{x_i}_{(a_i,b_i)} \subset \mathcal{U}$. Without loss of generality, we may assume that $x_1 = 0$, $a_1 = 1$, and $b_1 = 3$. Put

$I_1 = \{i \in \{1, 2, \dots, n\} : d(x_i, E_0) > 1, x_i \le \inf E_0\}$,
$I_2 = \{i \in \{1, 2, \dots, n\} : d(x_i, E_0) = 1, x_i \le \inf E_0\}$,
$I_3 = \{i \in \{1, 2, \dots, n\} : d(x_i, E_0) = 0, x_i \le \inf E_0\}$,
$I_4 = \{i \in \{1, 2, \dots, n\} : x_i > \inf E_0\}$,

and

$c = \max\{1 + x_i, a_i + x_i : i \in I_1\}$.

Since $\max\{a_i, 1\} < d(x_i, E_0) = \inf E_0 - x_i$ for every $i \in I_1$, we have $c < \inf E_0$. Take $z \in (c, \inf E_0) \setminus \{x_i : i \in I_2\}$ and let $E_1 = \{z\} \cup E_0$. To show that $E_1 \in \bigcap_{i=1}^n S^{x_i}_{(a_i,b_i)}$, let $i \in \{1, 2, \dots, n\}$. Since $E_0 \subset E_1$, we have $d(x_i, E_1) \le d(x_i, E_0) < b_i$. To see $a_i < d(x_i, E_1)$, we consider four cases.

If $i \in I_1$, then $d(x_i, z) = d(x_i, E_1)$ since $x_i < z < \inf E_0$ and $E_1 = \{z\} \cup E_0$. Therefore $a_i \le c - x_i < z - x_i \le d(x_i, z) = d(x_i, E_1)$.

If $i \in I_2$, then $1 = d(x_i, E_1)$ since $x_i \neq z$ and $d(x_i, E_0) = 1$. Hence $a_i < d(x_i, E_0) = 1 = d(x_i, E_1)$.

If $i \in I_3$, then $x_i \in E_1$ since $x_i \in E_0 \subset E_1$. Thus $a_i < d(x_i, E_0) = 0 = d(x_i, E_1)$.

If $i \in I_4$, then $a_i < d(x_i, E_0) = d(x_i, E_1)$ since $z < \inf E_0 < x_i$.

Therefore we have $E_1 \in \bigcap_{i=1}^n S^{x_i}_{(a_i,b_i)}$, which implies $E_1 \in \bigcap_{i=1}^n S^{x_i}_{(a_i,b_i)} \subset \mathcal{U} \subset \overline{\mathcal{U}}$. This contradicts with the maximality of E_0 in $(\overline{\mathcal{U}}, \subset)$. Hence we have $E_0 \notin \mathcal{U}$. □

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