FUZZY TOPOLOGICAL PROPERTIES OF FUZZY POINTS AND ITS APPLICATIONS

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Received July 28, 2012

ABSTRACT. The present paper studies new properties of the concept of fuzzy points in the sense of Pu Pao-Ming and Liu Ying-Ming (Definition 2.3 , Theorem 3.1). We first prove that, for an arbitrary Chang's fuzzy topological space (Y, τ_Y), every fuzzy set λ in Y with $\lambda \neq 0_Y$ is decomposed by at most three fuzzy sets: $\lambda = \lambda_1 \vee \lambda_2 \vee \lambda_3$ with $\lambda_i \wedge \lambda_j = 0_Y$ for each distinct integers i and $j (1 \leq i, j \leq 3)$ (Theorem 2.10); moreover λ_1 is fuzzy preopen in (Y, τ_Y) (Theorem 2.9(i)). Especially, if τ_Y is a fuzzy topology, say σ^f (cf. Example II in Section 3) which is induced from an ordinary topology σ of Y, then every fuzzy set λ in Y is decomposed by at most two fuzzy sets: $\lambda = \lambda_1 \vee \lambda_2$ with $\lambda_1 \wedge \lambda_2 = 0_Y$ (Corollary 3.7(i)); and λ_1 is fuzzy preopen in the fuzzy topological space (Y, σ^f) (Corollary 3.7(ii)). Moreover, every fuzzy point (in the sense of Pu Pao-Ming and Lin Ying-Ming) is fuzzy open or fuzzy nowhere dense in (Y, σ^f) (Theorem 3.1). As applications, the results are applied to the case where $Y := \mathbb{Z}^2$ and $\sigma = \kappa^2$ (=the Khalimsky topology), i.e., $(Y, \sigma) = (\mathbb{Z}^2, \kappa^2)$ is the digital plane. So, every digital image($\neq 0$) on $\mathbb{Z}^2$ is decomposed by at most two digital images and they have such fuzzy topological properties (Theorem 3.9 in Section 3(III-5)).

1 Introduction and preliminaries In 1965, L.A. Zadeh [29] introduced and investigated the fundamental notion of *fuzzy sets* and fuzzy sets operations. Subsequently several authors applied various basic concepts from general topology to fuzzy sets and developed the theory of Fuzzy topological spaces. In 1968, C.L. Chang [6] introduced and investigated the concept of *fuzzy topological spaces* (cf. Definition 1.1 below). In 1974, K.K. Wong [27, Definition 3.1] introduced and investigated the notion of *fuzzy points* (cf.[27, Theorem 3.1 and p.319]). In 1980, Pu Pao-Ming and Liu Ying-Ming [24, Definition 2.1] redefined the concept of *fuzzy points*; it takes a crisp singleton, equivalentely, an ordinary point as a special case. In the present paper, we adopte and use the definition

*2000 Math. Subject Classification: 54A40.
Key words and phrases: Topology, Chang's fuzzy topological spaces, Fuzzy points, Fuzzy preopen sets, Fuzzy nowhere dense sets, Decompositions of fuzzy sets, Digital planes, Digital images.

of fuzzy points in the sense of Pu Pao-Ming and Lin Ying-Ming [24] (cf. Definition 1.3 below). For a fuzzy set in a fuzzy topological space, the concept of *fuzzy preopen sets* and *fuzzy preclosed sets* were introduced by A.S. Bin Shahna [2] in 1991.

We recall some terminologies on fuzzy sets: throughout the present paper, let Y be a nonempty set and I be the unit interval [0, 1]. For Y, I^Y denotes the collection of all functions from Y into I; the equality $\lambda = \mu$ in I^Y if and only if $\lambda(x) = \mu(x)$ for every point $x \in Y$. A member λ of I^Y is called a fuzzy set in Y [29]. For $\lambda \in I^Y$ and $\mu \in I^Y$, λ is *contained in* μ, denoted by $\lambda \leq \mu$, if $\lambda(x) \leq \mu(x)$ for every point $x \in Y$. Let $\lambda, \mu \in I^Y$, the following fuzzy sets $\lambda \vee \mu \in I^Y$ and $\lambda \wedge \mu \in I^Y$ are defined by $(\lambda \vee \mu)(x) :=\max\{\lambda(x), \mu(x)\}$ for every point $x \in Y$ and $(\lambda \wedge \mu)(x) :=\min\{\lambda(x), \mu(x)\}$ for every point $x \in Y$, respectively. The fuzzy sets 0_Y is defined by $0_Y(x) = 0$ for every point $x \in Y$.

The purpose of the present paper is to study decomposition of a given fuzzy set in Y by at most three fuzzy sets using an arbitrary fuzzy topology on Y; for $\lambda \in I^Y$ with $\lambda \neq 0$, λ is equal to $\lambda_{\mathcal{FPO}(Y,\tau_Y)} \vee \lambda_{\mathcal{FPO}(Y,\tau_Y)} \vee \lambda_{\mathcal{REST}(Y,\tau_Y)}$ and $\lambda_{\mathcal{FPO}(Y,\tau_Y)}$ is a fuzzy preopen set of the arbitrary fuzzy topological space (Y, τ_Y); and $\lambda_{\mathcal{FPO}(Y,\tau_Y)} \wedge \lambda_{\mathcal{FPO}(Y,\tau_Y)} = 0_Y$ etc. hold, (cf. Theorem 2.10 and Theorem 2.9 below). In Section 2 we prove the main results (Theorem 2.9 and Theorem 2.10); some fuzzy topological propertirs on fuzzy points are used fot their proofs. In Section 3, we give three kinds of examples of the decomposition of a fuzzy set; Example I shows the decomposition of a fuzzy set in a finite fuzzy topological space; Example II shows the decomposition of an arbtrary fuzzy set in a specified fuzzy topological space induced from an arbitrary topological space (cf. Theorem 3.1, Corollary 3.7 below); in Example III we apply the results of Section 2 and Example II to decomposition of an arbitrary fuzzy set in a specified fuzzy topological space induced from the digital plane and so we investigate deomposition problem of grey pictures (digital images) on the digital planes (cf. Theorem 3.9 in (III-5), three figures in (III-6, 7) below). We need some concepts and notation in the digital planes (resp. digital line) (cf. (III-9) (resp. (III-10)) to prove Theorem 3.9 in (III-13) below.

In the end of this section, we need more detail and fundamental concepts as follows. For a family of fuzzy sets, $\{\lambda_j | j \in J\}$, the *union* $\bigvee\{\lambda_j | \ j \in J\}$, and the *intersection*, $\bigwedge\{\lambda_j | \ j \in J\}$, are defined by

$\bigvee\{\lambda_j | \ j \in J\}(x):=\sup\{\lambda_j(x) | \ j \in J\}, \ x \in Y;$

$\bigwedge\{\lambda_j | j \in J\}(x):=\inf\{\lambda_j(x) | j \in J\}, \ x \in Y;$

where J denotes arbitrary index set [6]. The fuzzy set 1_Y is defined by $1_Y(x) = 1$ for every point $x \in Y$. The *complement* λ^c of a fuzzy set λ is defined by $\lambda^c(x) := 1 - \lambda(x)$ for every point $x \in Y$.

Definition 1.1 (C.L. Chang [6, Defimition 2.2]) A *fuzzy topological space* [6] is a pair (Y, τ_Y), where Y is a non-empty set and τ_Y is a *fuzzy topology* on it, i.e., a family τ_Y of fuzzy sets satisfying the following three axioms:

(1) $0_Y, 1_Y \in \tau_Y$;

(2) If $\lambda, \mu \in \tau_Y$, then $\lambda \wedge \mu \in \tau_Y$;

(3) Let J be an index set. If $\lambda_j \in \tau_Y$ for each $j \in J$, then $\bigvee\{\lambda_j | j \in J\} \in \tau_Y$.
The elements of τ_Y are called *fuzzy open sets* of (X, τ_Y). A fuzzy set λ is called a *fuzzy closed set* of (Y, τ_Y) if the fuzzy complement $\lambda^c \in \tau_Y$.

Sometimes, the fuzzy topological space of Definiton 1.1 is called *Chang's fuzzy topological space.* For a fuzzy set $\lambda \in I^Y$ and a fuzzy topological space (Y, τ_Y), the fuzzy closure and fuzzy interior of λ are defined by $\mathrm{Cl}(\lambda) := \bigwedge\{\mu \in I^Y \mid \lambda \leq \mu, \mu^c \in \tau_Y\}$ and $\mathrm{Int}(\lambda) := \bigvee\{\nu \in I^Y \mid \nu \leq \lambda, \nu \in \tau_Y\}$, respectively.

Definition 1.2 Let (Y, τ_Y) be a fuzzy topological space.

(i) ([26]) A fuzzy set λ is called *fuzzy preopen* (resp. *fuzzy preclosed*) in (Y, τ_Y), if $\lambda \leq \mathrm{Int}(\mathrm{Cl}(\tau_Y))$ (resp. λ^c is fuzzy preopen in (Y, τ_Y)).

(ii) A fuzzy set λ is called *fuzzy nowhere dense* in (Y, τ_Y) if $\mathrm{Int}(\mathrm{Cl}(\lambda)) = 0_Y$ holds.

(iii) ([1, Definition 4.1, Theorem 4.2]) A fuzzy set $\lambda \in I^Y$ is called *fuzzy semi-open* (resp. *fuzzy semi-closed*) in (Y, τ_Y), if $\lambda \leq \mathrm{Cl}(\mathrm{Int}(\lambda))$ (resp. λ^c is fuzzy semi-open in (Y, τ_Y)).

Moreover, we recall the following well known definition of *fuzzy points* in Y [24, Definition 2.1] (e.g.,[11, p.120]).

Definition 1.3 ([24, Definition 2.1]) A fuzzy set in Y is said to be *fuzzy point* if it takes the value 0 for all $y \in Y$ except one, say $x \in Y$. If its value at x is a $(0 < a \leq 1)$, we denote this fuzzy point by x_a, where this point x is called *its support.* Namely, for a point $x \in Y$ and a real number $a \in I$ such that $0 < a \leq 1$, a *fuzzy point* $x_a \in I^Y$ is a fuzzy set defined as, for any point $y \in Y, x_a(y) = a$ if $y = x; x_a(y) = 0$ if $y \neq x$.

We recall the following concepts on fuzzy set $\lambda \in I^Y$: The set $\{y \in X | \lambda(y) > 0\}$ is called the *support of* λ and it is denoted by $\mathrm{supp}(\lambda)$ (e.g.,[24, Definition 1.1]). For examples, for a point $x \in Y$ and a subset A of Y, we have $\mathrm{supp}(x_a) = \{x\}$ and $\mathrm{supp}(\chi_A) = A$, where χ_A is the characteristic function of A defined by $\chi_A(y) := 1$ for every point $y \in A$ and $\chi_A(y) := 0$ for every point $y \notin A$.

2 A decomposition of fuzzy sets from a fuzzy topological space point of view In the present section, we investigate a decomposition theorem of a given fuzzy set λ by three specified fuzzy sets $\lambda_{\mathcal{FPO}(Y,\tau_Y)}$, $\lambda_{\mathcal{FND}(Y,\tau_Y)}$ and $\lambda_{\mathcal{REST}(Y,\tau_Y)}$, where (Y, τ_Y) is an arbitrary fuzzy topological space (cf. Theorem 2.9, Theorem 2.10, Definition 2.6 below). First we show that, in the following example (Y, τ_Y) of Chang's fuzzy topological space, there exists a fuzzy point x_a such that x_a is not fuzzy preopen in (Y, τ_Y) and also x_a is not fuzzy nowhere dense in (Y, τ_Y).

Example 2.1 The following (Y, τ_Y) is an example of a fuzzy topological space (cf. this space (Y, τ_Y) is given in [25, Example 3.1]). Let $Y := I$ (the unit interval [0, 1]) and $\tau_Y := \{0_Y, \mu, \nu, \mu \vee \nu, 1_Y\}$, where $\mu(y){:=}0$ for every $y \in Y$

with $0 \leq y \leq 1/2, \mu(y) := 2y-1$ for every $y \in Y$ with $1/2 \leq y \leq 1$; and $\nu(y) := 1$ for every $y \in Y$ with $0 \leq y \leq 1/4, \nu(y) := -4y+2$ for every $y \in Y$ with $1/4 \leq y \leq 1/2, \nu(y) := 0$ for every $y \in Y$ with $1/2 \leq y \leq 1$.

(i) For a point $x := 3/4 \in Y$ and a real number $a \in I$ with $1/2 < a \leq 1$, we take a fuzzy point x_a; then x_a is not fuzzy preopen in (Y, τ_Y), because $\mathrm{Int}(\mathrm{Cl}(x_a))(3/4) = \mathrm{Int}(\nu^c)(3/4) = \mu(3/4) = 1/2 \ngeq a = x_a(3/4)$. And so the fuzzy point x_a is not fuzzy nowhere dense.

(ii) For a point $x := 3/4 \in Y$ and a real number $b \in I$ with $0 < b \leq 1/2$, we have x_b is not fuzzy preopen in (Y, τ_Y); but x_b is fuzzy nowhere dense, because $\mathrm{Int}(\mathrm{Cl}(x_b)) = \mathrm{Int}((\mu \vee \nu)^c) = 0_Y$.

(iii) For a point $x := 1/4 \in Y$ and a real number $d \in I$ with $0 < d \leq 1$, we have x_d is fuzzy preopen in (Y, τ_Y); but x_d is not fuzzy nowhere dense, because $\mathrm{Int}(\mathrm{Cl}(x_d)) = \mathrm{Int}(\mu^c) = \nu$.

Remark 2.2 Example 2.1(i) above shows that it is not true that every fuzzy topological space satisfies the following property:
$(*)$ every fuzzy point is fuzzy preopen or fuzzy nowhere dense in the fuzzy topological space.
However, it is well known that in an arbitrary topological space (Y, σ) every singleton $\{x\}$ is preopen (i.e., $\{x\} \subset \mathrm{Int}(\mathrm{Cl}(\{x\}))$ or nowhere dense (i.e.,$\mathrm{Int}(\mathrm{Cl}(\{x\})) = \emptyset$) in (Y, σ) (Janković-Reilly's lemma [14, Lemma 2], e.g., [5, Lemma 2.4], [4, Observation 3.1(b)]). Thus, by Example 2.1 above, it is shown that a fuzzy version of Janković-Reilly's lemma does not hold for arbitrary fuzzy topological space.

It follows from Remark 2.2 that we can define the following concept of a fuzzy topological space which satisfies the above property $(*)$.

Definition 2.3 A fuzzy topological space (Y, τ_Y) is said to *satisfy the Janković-Reilly condition*, if every fuzzy point x_a in Y is fuzzy preopen or fuzzy nowhere dense in (Y, τ_Y), where the point $x \in Y$ and the real number $a \in I$ with $0 < a \leq 1$.

Lemma 2.4 *For an arbitrary fuzzy topological space (Y, τ_Y) and a fuzzy point $x_a \in I^Y$, we have the following properties.*

(i) *If x_a is fuzzy preopen in (Y, τ_Y), then x_a is not fuzzy nowhere dense.*

(ii) *If x_a is fuzzy nowhere dense in (Y, τ_Y), then x_a is not fuzzy preopen.*

Proof. (i) Since $x_a \leq \mathrm{Int}(\mathrm{Cl}(x_a))$ holds and $x_a \neq 0_Y$, we have $\mathrm{Int}(\mathrm{Cl}(x_a)) \neq 0_Y$, i.e., x_a is not fuzzy nowhere dense.

(ii) Suppose the fuzzy point x_a is fuzzy preopen in (Y, τ_Y). Then, we have $x_a \leq \mathrm{Int}(\mathrm{Cl}(x_a))$ and so $x_a = 0_Y$, because x_a is fuzzy nowhere dense. This contradicts the definition of fuzzy points. □

We need a lemma and some notation below.

Lemma 2.5 *Let* $\lambda \in I^Y$ *be a fuzzy set with* $\lambda \neq 0_Y$ *and* A *and* B *two subsets of* $\text{supp}(\lambda)$. *Then, the following properties hold.*

(i) (cf. [24, Definition 2.2], e.g., [19, Lemma 2.2]) $\lambda = \bigvee\{x_{\lambda(x)} \in I^Y \mid x \in \text{supp}(\lambda)\}$ *holds.*

(ii) $\bigvee\{x_{\lambda(x)} \in I^Y \mid x \in A \cup B\} = (\bigvee\{x_{\lambda(x)} \in I^Y \mid x \in A\}) \vee (\bigvee\{x_{\lambda(x)} \in I^Y \mid x \in B\})$ *holds.*

(iii) *Further to* (ii), *suppose* $A \cap B = \emptyset$, *then* $(\bigvee\{x_{\lambda(x)} \in I^Y \mid x \in A\}) \wedge (\bigvee\{x_{\lambda(x)} \in I^Y \mid x \in B\}) = 0_Y$. □

(**Notation I**): For an arbitrary fuzzy topological space (Y, τ_Y), we denote the following three families of fuzzy sets as follows (cf. Definition 1.2):

(1) $FPO(Y, \tau_Y) := \{\lambda \in I^Y \mid \lambda \text{ is fuzzy preopen in } (Y, \tau_Y)\}$;
(2) $FND(Y, \tau_Y) := \{\lambda \in I^Y \mid \lambda \text{ is fuzzy nowhere dense in } (Y, \tau_Y)\}$;
(3) $FSO(Y, \tau_Y) := \{\lambda \in I^Y \mid \lambda \text{ is fuzzy semi-open in } (Y, \tau_Y)\}$.

(**Notation II**): For an arbitrary fuzzy topological space (Y, τ_Y) and a fuzzy set $\lambda \in I^Y$ with $\lambda \neq 0_Y$ (i.e., $\text{supp}(\lambda) \neq \emptyset$), we define three subsets of $\text{supp}(\lambda)$:
(1) $\text{supp}(\lambda)^{FPO} := \{x \in \text{supp}(\lambda) \mid x_{\lambda(x)} \in FPO(Y, \tau_Y)\}$;
(2) $\text{supp}(\lambda)^{FND} := \{x \in \text{supp}(\lambda) \mid x_{\lambda(x)} \in FND(Y, \tau_Y)\}$;
(3) $\text{supp}(\lambda)^{REST} := \{x \in \text{supp}(\lambda) \mid x_{\lambda(x)} \notin FPO(Y, \tau_Y) \cup FND(Y, \tau_Y)\}$.

Definition 2.6 Let (Y, τ_Y) be an arbitrary fuzzy topological space and $\lambda \in I^Y$ be a fuzzy set with $\lambda \neq 0_Y$. The following fuzzy sets are well defined:

(i) $\lambda_{\mathcal{FPO}(Y,\tau_Y)} := \bigvee\{x_{\lambda(x)} \in I^Y \mid x_{\lambda(x)} \in FPO(Y, \tau_Y), x \in \text{supp}(\lambda)\}$ if $\text{supp}(\lambda)^{FPO} \neq \emptyset$; $\lambda_{\mathcal{FPO}(Y,\tau_Y)} := 0_Y$ if $\text{supp}(\lambda)^{FPO} = \emptyset$;

(ii) $\lambda_{\mathcal{FND}(Y,\tau_Y)} := \bigvee\{x_{\lambda(x)} \in I^Y \mid x_{\lambda(x)} \in FND(Y, \tau_Y), x \in \text{supp}(\lambda)\}$ if $\text{supp}(\lambda)^{FND} \neq \emptyset$; $\lambda_{\mathcal{FND}(Y,\tau_Y)} := 0_Y$ if $\text{supp}(\lambda)^{FND} = \emptyset$;

(iii) $\lambda_{\mathcal{REST}(Y,\tau_Y)} := \bigvee\{x_{\lambda(x)} \in I^Y \mid x_{\lambda(x)} \notin FPO(Y, \tau_Y) \cup FND(Y, \tau_Y), x \in \text{supp}(\lambda)\}$ if $\text{supp}(\lambda)^{REST} \neq \emptyset$; $\lambda_{\mathcal{REST}(Y,\tau_Y)} := 0_Y$ if $\text{supp}(\lambda)^{REST} = \emptyset$.

Remark 2.7 Let (Y, τ_Y) be an arbitrary fuzzy topological space (cf. Notaions I, II above).

(i) $\lambda_{\mathcal{FPO}(Y,\tau_Y)} = 0$ if and only if $\text{supp}(\lambda)^{FPO} = \emptyset$.
(ii) $\lambda_{\mathcal{FND}(Y,\tau_Y)} = 0$ if and only if $\text{supp}(\lambda)^{FND} = \emptyset$.
(iii) $\lambda_{\mathcal{REST}(Y,\tau_Y)} = 0$ if and only if $\text{supp}(\lambda)^{REST} = \emptyset$.

Proof of (i): (Necessity) Suppose that $\text{supp}(\lambda)^{FPO} \neq \emptyset$, i.e., there exists a point $z \in \text{supp}(\lambda)$ such that $z_{\lambda(z)} \in FPO(Y, \tau_Y)$. For the point z we set $\mathcal{A}_z := \{x_{\lambda(x)}(z) \in I \mid x_{\lambda(x)} \in FPO(Y, \tau_Y), x \in \text{supp}(\lambda)\}$. By Definition 2.6 (i), $\sup \mathcal{A}_z = \lambda_{\mathcal{FPO}(Y,\tau_Y)}(z)$. For any element $\alpha \in \mathcal{A}_z, \alpha \leq \sup \mathcal{A}_z$. Then, we have $z_{\lambda(z)}(z) \leq \sup \mathcal{A}_z$, because $z_{\lambda(z)}(z) \in \mathcal{A}_z$. Since $0 < \lambda(z) = z_{\lambda(z)}(z)$, we have $0 < \sup \mathcal{A}_z = \lambda_{\mathcal{FPO}(Y,\tau_Y)}(z)$. We conclude that $\lambda_{\mathcal{FPO}(Y,\tau_Y)} \neq 0_Y$ in (Y, τ_Y).
(Sufficiency) The proof is obtained directly by Definition 2.6 (i).
Proof of (ii) (resp. (iii)): this is proved by the same argument as that in (i) (resp. (iii)) using Definition 2.6 (ii) (resp. (iii)).

The following theorem is a characterization of $\lambda_{\mathcal{FPO}(Y,\tau_Y)}$ and $\lambda_{\mathcal{FND}(Y,\tau_Y)}$.

Theorem 2.8 *Let (Y, τ_Y) be an arbitrary fuzzy topological space and $\lambda \in I^Y$ be a fuzzy point with $\lambda \neq 0_Y$.*

(i-1) *If $\lambda = \lambda_{\mathcal{FND}(Y,\tau_Y)}$ holds, then $\lambda_{\mathcal{FPO}(Y,\tau_Y)} = 0_Y$ and $\lambda_{\mathcal{REST}(Y,\tau_Y)} = 0_Y$.*

(i-2) *Conversely, if $\lambda_{\mathcal{FPO}(Y,\tau_Y)} = 0_Y$ and $\lambda_{\mathcal{REST}(Y,\tau_Y)} = 0_Y$, then $\lambda = \lambda_{\mathcal{FND}(Y,\tau_Y)}$ holds.*

(ii-1) *If $\lambda = \lambda_{\mathcal{FPO}(Y,\tau_Y)}$ holds, then $\lambda_{\mathcal{FND}(Y,\tau_Y)} = 0_Y$ and $\lambda_{\mathcal{REST}(Y,\tau_Y)} = 0_Y$.*

(ii-2) *Conversely, if $\lambda_{\mathcal{FND}(Y,\tau_Y)} = 0_Y$ and $\lambda_{\mathcal{REST}(Y,\tau_Y)} = 0_Y$, then $\lambda = \lambda_{\mathcal{FPO}(Y,\tau_Y)}$ holds.*

(iii) *Especially, suppose that (Y, τ_Y) satisfies the Janković-Reilly condition* (cf. Definition 2.3).

(iii-1) *If $\lambda_{\mathcal{FPO}(Y,\tau_Y)} = 0_Y$, then $\lambda = \lambda_{\mathcal{FND}(Y,\tau_Y)}$ holds.*

(iii-2) *If $\lambda_{\mathcal{FND}(Y,\tau_Y)} = 0_Y$, then $\lambda = \lambda_{\mathcal{FPO}(Y,\tau_Y)}$ holds.*

Proof. **(i-1)** Assume $\lambda = \lambda_{\mathcal{FND}(Y,\tau_Y)}$ holds in (Y, τ_Y). We recall that $\lambda_{\mathcal{FND}(Y,\tau_Y)} = \bigvee \mathcal{B}$, where $\mathcal{B} := \{x_{\lambda(x)} | \, x_{\lambda(x)} \in FND(Y, \tau_Y), x \in \text{supp}(\lambda)\}$ and $\lambda_{\mathcal{FND}(Y,\tau_Y)}(z) = \sup \mathcal{B}_z$, where $z \in Y$ and $\mathcal{B}_z := \{x_{\lambda(x)}(z) \in I | \, x_{\lambda(x)} \in FND(Y, \tau_Y), x \in \text{supp}(\lambda)\}$. We claim that
$(*)$ $w_{\lambda(w)} \in FND(Y, \tau_Y)$ for each point $w \in$supp(λ).
Proof of $(*)$. Let $w \in$supp(λ). Then, by assumption, $\lambda(w) = \lambda_{\mathcal{FND}(Y,\tau_Y)}(w) = \sup \mathcal{B}_w$. For any positive real number ε with $0 < \sup \mathcal{B}_w - \varepsilon$, there exists a real number $y \in \mathcal{B}_w$ such that $0 < \sup \mathcal{B}_w - \varepsilon < y \leq \sup \mathcal{B}_w$. Hence, $y = x_{\lambda(x)}(w)$ for some $x \in$supp(λ) and $x_{\lambda(x)} \in FND(Y, \tau_Y)$. Since $y = x_{\lambda(x)}(w) > 0$, we have $x_{\lambda(x)}(w) = \lambda(x)$ and so $x = w$. Thus, we show $(*)$ above, i.e., $w_{\lambda(w)} \in FND(Y, \tau_Y)$ for each point $w \in$supp(λ). $\Diamond$
Namely, $w_{\lambda(w)}$ is fuzzy nowhere dense in (Y, τ_Y) for each point $w \in$supp(λ). Using Lemma 2.4 (ii) for each point $w \in$supp(λ), $w_{\lambda(w)}$ is not fuzzy preopen in (Y, τ_Y). By Remark 2.7(i) (resp. (iii)), it is shown that $\lambda_{\mathcal{FPO}(Y,\tau_Y)} = 0_Y$ (resp. $\lambda_{\mathcal{REST}(Y,\tau_Y)} = 0_Y$ because $w_{\lambda(w)} \in FPO(Y, \tau_Y) \cup FND(Y, \tau_Y)$ for each $w \in$supp(λ)) in (Y, τ_Y).

(i-2) Assume $\lambda_{\mathcal{FPO}(Y,\tau_Y)} = 0_Y$ and $\lambda_{\mathcal{REST}(Y,\tau_Y)} = 0_Y$. By Remark 2.7(i) and (iii), respectively, it is obtained that supp$(\lambda)^{FPO} := \{x \in$supp$(\lambda) | \, x_{\lambda(x)} \in FPO(Y, \tau_Y)\} = \emptyset$ and supp$(\lambda)^{REST} := \{x \in$supp$(\lambda) | \, x_{\lambda(x)} \notin FPO(Y, \tau_Y) \cup FND(Y, \tau_H)\} = \emptyset$. Thus, we have supp$(\lambda) = \{x \in$supp$(\lambda) | \, x_{\lambda(x)} \in FND(Y, \tau_Y)\}$ and so $\lambda = \bigvee\{x_{\lambda(x)} | \, x_{\lambda(x)} \in FND(Y, \tau_Y), x \in \text{supp}(\lambda)\} = \lambda_{\mathcal{FND}(Y,\tau_Y)}$ (cf. Lemma 2.5(i)).

(ii-1) (ii-2) Their proofs are similar to that of (i-1) and (i-2) above, respectively.

(iii) (iii-1) Assume $\lambda_{\mathcal{FPO}(Y,\tau_Y)} = 0_Y$. By Remark 2.7(i), it is obtained that every fuzzy point $x_{\lambda(x)}$ is not fuzzy preopen in (Y, τ_Y) for each point $x \in$supp(λ). Since (Y, τ_Y) satisfies the Janković-Reilly condition (cf. Definition 2.3), every fuzzy point $x_{\lambda(x)}$ is fuzzy nowhere dense in (Y, τ_Y) for each point $x \in$supp(λ); and so $\{x_{\lambda(x)} | \, x \in$supp$(\lambda)\}$= $\{x_{\lambda(x)} | \, x_{\lambda(x)} \in FND(Y, \tau_Y), x \in$supp$(\lambda)\}$. Therefore, using Lemma 2.5(i), we have $\lambda = \bigvee\{x_{\lambda(x)} | \, x \in$supp$(\lambda)\}$ $=\bigvee\{x_{\lambda(x)} | \, x_{\lambda(x)} \in FND(Y, \tau_Y),$
$x \in$supp$(\lambda)\} = \lambda_{\mathcal{FND}(Y,\tau_Y)}$.

(iii-2) The proof is similar to that of (iii-1) above. □

In the end of this section we shall prove two theorems of main results (cf. Theorem 2.9, Theorem 2.10 below).

Theorem 2.9 *For an arbitrary fuzzy topological space* (Y, τ_Y) *and* $\lambda \in I^Y$ *a fuzzy set with* $\lambda \neq 0_Y$*, we have the following properties on* $\lambda_{\mathcal{FPO}(Y,\tau_Y)}$ *and* $\lambda_{\mathcal{FND}(Y,\tau_Y)}$*, respectively.*

(i) $\lambda_{\mathcal{FPO}(Y,\tau_Y)}$ *is fuzzy preopen.*

(ii) *If the characteristc function* χ_B *is fuzzy nowhere dense in* (Y, τ_Y)*, then* $\lambda_{\mathcal{FND}(Y,\tau_Y)}$ *is fuzzy nowhere dense in* (Y, τ_Y)*, where* $B := \{x \in \mathrm{supp}(\lambda) | x_{\lambda(x)} \in FND(Y, \tau_Y)\}$.

Proof. **(i)** If $\lambda_{\mathcal{FPO}(Y,\tau_Y)} \neq 0_Y$, then $\lambda_{\mathcal{FPO}(Y,\tau_Y)} := \bigvee\{x_{\lambda(x)} \in I^Y |\ x_{\lambda(x)} \in FPO(Y, \tau_Y), x \in\mathrm{supp}(\lambda)\}$ is the fuzzy union of fuzzy preopen fuzzy points $x_{\lambda(x)}$ for some points $x \in\mathrm{supp}(\lambda)$ (cf. Definition 2.6(1)). Then, $\lambda_{\mathcal{FPO}(Y,\tau_Y)}$ is fuzzy preopen in (Y, τ_Y), because $FPO(Y, \tau_Y)$ is closed under an arbitrary fuzzy union. If $\lambda_{\mathcal{FPO}(Y,\tau_Y)} = 0_Y$, then it is obviously fuzzy preopen in (Y, τ_Y).

(ii) We recall $B = \mathrm{supp}(\lambda)^{FND} := \{x \in \mathrm{supp}(\lambda) | x_{\lambda(x)} \in FND(Y, \tau_Y)\}$ (cf. (Notation II)). For the case where $B \neq \emptyset$, it follows from assumption and Definition 2.6(2) that $\mathrm{Int}(\mathrm{Cl}(\lambda_{\mathcal{FND}(Y,\tau_Y)}))= \mathrm{Int}(\mathrm{Cl}(\bigvee\{x_{\lambda(x)} | x \in B\})) \leq \mathrm{Int}(\mathrm{Cl}(\bigvee\{\chi_{\{x\}} | x \in B\}))=\mathrm{Int}(\mathrm{Cl}(\chi_{\cup\{\{x\}|x\in B\}}))= \mathrm{Int}(\mathrm{Cl}(\chi_B)) = 0_Y$. Namely, we have $\mathrm{Int}(\mathrm{Cl}(\lambda_{\mathcal{FND}(Y,\tau_Y)})) = 0_Y$; and so $\lambda_{\mathcal{FND}(Y,\tau_Y)}$ is fuzzy nowhere dense in (Y, τ_Y). For the case where $B = \emptyset$, we have $\lambda_{\mathcal{FND}(Y,\tau_Y)} := 0_Y$; and so $\lambda_{\mathcal{FND}(Y,\tau_Y)}$ is also fuzzy nowhere dense for the case where $B = \emptyset$. □

Theorem 2.10 *Let* (Y, τ_Y) *be an arbitrary fuzzy topological space and* $\lambda \in I^Y$ *a fuzzy set with* $\lambda \neq 0_Y$*. We have the following properties on* λ*.*

(i) $\lambda = \lambda_{\mathcal{FPO}(Y,\tau_Y)} \vee \lambda_{\mathcal{FND}(Y,\tau_Y)} \vee \lambda_{\mathcal{REST}(Y,\tau_Y)}$ *holds.*

(ii) $\lambda_{\mathcal{FPO}(Y,\tau_Y)} \wedge \lambda_{\mathcal{FND}(Y,\tau_Y)} = 0_Y$*;* $\lambda_{\mathcal{FPO}(Y,\tau_Y)} \wedge \lambda_{\mathcal{REST}(Y,\tau_Y)} = 0_Y$*;* $\lambda_{\mathcal{FND}(Y,\tau_Y)} \wedge \lambda_{\mathcal{REST}(Y,\tau_Y)} = 0_Y$.

(iii) *Especially, if* (Y, τ_Y) *satisfies the Janković-Reilly condition* (cf. Definition 2.3), *then* λ *has a decomposition:* $\lambda = \lambda_{\mathcal{FPO}(Y,\tau_Y)} \vee \lambda_{\mathcal{FND}(Y,\tau_Y)}$ *with* $\lambda_{\mathcal{FPO}(Y,\tau_Y)} \wedge \lambda_{\mathcal{FND}(Y,\tau_Y)} = 0_Y$.

Proof. **(i)** First, we claim that for an arbitrary fuzzy topological space (Y, τ_Y)
$(*)$ $\mathrm{supp}(\lambda) = \mathrm{supp}(\lambda)^{FPO} \cup \mathrm{supp}(\lambda)^{FND} \cup \mathrm{supp}(\lambda)^{REST}$ holds.
Let $x \in \mathrm{supp}(\lambda)$ and $x_{\lambda(x)}$ a fuzzy point. We consider the following all cases on the values on $(\mathrm{Int}(\mathrm{Cl}(x_{\lambda(x)})))(x) \in I$ and $x_{\lambda(x)}(x) \in I$.

Case 1. $x_{\lambda(x)}(x) \leq (\mathrm{Int}(\mathrm{Cl}(x_{\lambda(x)})))(x)$ in I: for this case we have $x_{\lambda(x)} \leq \mathrm{Int}(\mathrm{Cl}(x_{\lambda(x)}))$ in I^Y, because $x_{\lambda(x)}(y) = 0$ for every point $y \in\mathrm{supp}(\lambda)$ with $y \neq x$. Thus, for this case, we have $x_{\lambda(x)} \in FPO(Y, \tau_Y)$ and so $x \in\mathrm{supp}(\lambda)^{FPO}$.

Case 2. $(\mathrm{Int}(\mathrm{Cl}(x_{\lambda(x)})))(x) = 0$ in I: for this case for a point $y \in Y$ with $y \neq x$, we consider the following two cases:
Case (2-1). $(\mathrm{Int}(\mathrm{Cl}(x_{\lambda(x)})))(y) = 0$ for every point $y \in Y$ with $y \neq x$: for this case we have $\mathrm{Int}(\mathrm{Cl}(x_{\lambda(x)})) = 0_Y$, i.e., $x_{\lambda(x)} \in FND(Y, \tau_Y)$ and so $x \in\mathrm{supp}(\lambda)^{FND}$.

Case (2-2). $(\mathrm{Int}(\mathrm{Cl}(x_{\lambda(x)})))(y) \neq 0$ for some point $y \in Y$ with $y \neq x$: for this case, we have $\mathrm{Int}(\mathrm{Cl}(x_{\lambda(x)})) \neq 0_Y$ in I^Y; and so $x_{\lambda(x)}$ is not fuzzy nowhere dense in (Y, τ_Y), i.e., $x_{\lambda(x)} \notin FND(Y, \tau_Y)$. Moreover, at the point y, we have $x_{\lambda(x)}(y) = 0 < (\mathrm{Int}(\mathrm{Cl}(x_{\lambda(x)})))(y)$; however at the point x for the Case 2, we have $x_{\lambda(x)}(x) \not\leq (\mathrm{Int}(\mathrm{Cl}(x_{\lambda(x)})))(x) = 0$, because $0 < x_{\lambda(x)}(x) = \lambda(x)$. Thus, for this Case (2-2), $x_{\lambda(x)}$ is not fuzzy preopen in (Y, τ_Y), i.e., $x \notin FPO(Y, \tau_Y)$. Namely, for the point x where the Case (2-2), we have $x \notin FND(Y, \tau_Y) \cup FPO(Y, \tau_Y)$. i.e., $x \in \mathrm{supp}(\lambda)^{REST}$.
Thus, for the point x in this Case 2, we show that $x \in \mathrm{supp}(\lambda)^{FND} \cup \mathrm{supp}(\lambda)^{REST}$.

Case 3. $0 < (\mathrm{Int}(\mathrm{Cl}(x_{\lambda(x)})))(x) < x_{\lambda(x)}(x)$ in I: for this case, we have $\mathrm{Int}(\mathrm{Cl}(x_{\lambda(x)})) \neq 0$ and $x_{\lambda(x)} \not\leq \mathrm{Int}(\mathrm{Cl}(x_{\lambda(x)}))$. Namely, $x_{\lambda(x)} \notin FND(Y, \tau_Y) \cup FPO(Y, \tau_Y)$, i.e., $x \in \mathrm{supp}(\lambda)^{REST}$.

By all possible cases above, it is shown that
$x \in \mathrm{supp}(\lambda)^{FPO} \cup \mathrm{supp}(\lambda)^{FND} \cup \mathrm{supp}(\lambda)^{REST}$ holds for any point $x \in \mathrm{supp}(\lambda)$. Namely, we show one of the inequalities of $(*)$:
$\mathrm{supp}(\lambda) \subset \mathrm{supp}(\lambda)^{FPO} \cup \mathrm{supp}(\lambda)^{FND} \cup \mathrm{supp}(\lambda)^{REST}$. Since the converse implication is obvious, we conclude the proof of the equality $(*)$ for the arbitrary fuzzy topological space (Y, τ_Y).

Therefore, by Lemma 2.5 and $(*)$ above, it is concluded that
$\lambda = \bigvee\{x_{\lambda(x)} | x \in \mathrm{supp}(\lambda)\}$
$= (\bigvee\{x_{\lambda(x)} | x \in \mathrm{supp}(\lambda)^{FPO}\}) \vee (\bigvee\{x_{\lambda(x)} | x \in \mathrm{supp}(\lambda)^{FND}\})$
$\vee (\bigvee\{x_{\lambda(x)} | x \in \mathrm{supp}(\lambda)^{REST}\}) = \lambda_{\mathcal{FPO}(Y,\tau_Y)} \vee \lambda_{\mathcal{FND}(Y,\tau_Y)} \vee \lambda_{\mathcal{REST}(Y,\tau_Y)}$.

(ii) By Lemma 2.4 and definitions, it is shown that $\mathrm{supp}(\lambda)^{FPO} \cap \mathrm{supp}(\lambda)^{FND} = \emptyset$; and so, by Lemma 2.5(iii), it is obtained that $\lambda_{\mathcal{FPO}(Y,\tau_Y)} \wedge \lambda_{\mathcal{FND}(Y,\tau_Y)} = 0_Y$. It is similarly shown that the other equalities hold, because $\mathrm{supp}(\lambda)^{FPO} \cap \mathrm{supp}(\lambda)^{REST} = \emptyset$ and $\mathrm{supp}(\lambda)^{REST} \cap \mathrm{supp}(\lambda)^{FND} = \emptyset$, respectively.

(iii) Let $x \in \mathrm{supp}(\lambda)$. Then, it follow from assumption that every fuzzy point $x_{\lambda(x)}$ is fuzzy preopen or fuzzy nowhere dense in (Y, τ_Y),i.e., $x \in \mathrm{supp}(\lambda)^{FPO} \cup \mathrm{supp}(\lambda)^{FND}$. Namely, we have $\mathrm{supp}(\lambda) = \mathrm{supp}(\lambda)^{FPO} \cup \mathrm{supp}(\lambda)^{FND}$.
By Lemma 2.5(ii)(iii) and definitions, it is shown that $\lambda = \lambda_{\mathcal{FPO}(Y,\tau_Y)} \vee \lambda_{\mathcal{FND}(Y,\tau_Y)}$ and $\lambda_{\mathcal{FPO}(Y,\tau_Y)} \wedge \lambda_{\mathcal{FND}(Y,\tau_Y)} = 0_Y$. □

3 Examples and applications In the present section, we give some examples on results (cf. Theorem 2.10) of Section 2 above. With respect to the decomposition of a fuzzy set, Example I below shows one for the finite fuzzy topological space of Example 2.1; Example II below shows detailly one for a specific fuzzy topological space (Y, σ^f) which is induced from a topological space (Y, σ); in Example III below, some applications for a digital image λ with grey scale $\lambda(x)$ are given; they are done for a fuzzy set on the digital planes and the decomposition of the images is given exactly for the simple fuzzy topological space $(\mathbb{Z}^2, (\kappa^2)^f)$ (cf. Theorem 3.9 in (III-5), (III-10) below). The proof of Theorem 3.9 is done in (III-13) after recalling properties on the digital planes (cf. (III-8,9) below). In (III-6) and (III-7) below, simple illustrations of a decomposition of image λ on $\mathbb{Z}^2$ with grey scale $\lambda(x)$, where $x \in \mathbb{Z}^2$, are illustrated.

(Example I).

We consider the fuzzy topological space (Y, τ_Y) in Example 2.1; $Y := I$ and $\tau_Y := \{0_Y, \mu, \nu, \mu \vee \nu, 1_Y\}$ (cf. Example 2.1). By Theorem 2.10, it is obtained that any fuzzy set λ on Y has a decomposition: $\lambda = \lambda_{\mathcal{FPO}(Y,\tau_Y)} \vee \lambda_{\mathcal{FND}(Y,\tau_Y)} \vee \lambda_{\mathcal{REST}(Y,\tau_Y)}$. For this fuzzy topological space (Y, τ_Y), we can see the precise forms of them. First, in order to investigate fuzzy topological properties of any fuzzy point on (Y, τ_Y), we should decompose $(Y \times I) \setminus Y \times \{0\}$ as follows:
$(Y \times I) \setminus Y \times \{0\}= \bigcup\{(Y \times I)_i |\ i \in \{1, 2, 3, 4, 5\}\}$,
where $(Y \times I)_1 := \{(x, t)|0 \leq x < 3/8, 0 < t \leq -4x + 2, 4x - 1 < t \leq 1\}$;
$(Y \times I)_2 := \{(x, t)|1/4 \leq x < 1/2, -4x - 1 < t \leq 1, -4x + 2 < t \leq 1\}$;
$(Y \times I)_3 := \{(x, t)|1/4 \leq x < 1, 0 < t \leq -2x + 2, 0 < t \leq 4x - 1\}$;
$(Y \times I)_4 := \{(x, t)|1/2 < x < 1, -2x + 2 < t \leq 1, 2x - 1 < t \leq 1\}$;
$(Y \times I)_5 := \{(x, t)|3/4 \leq x \leq 1, 0 < t \leq 2x - 1, -2x + 2 < t \leq 1\}$.

Secondly, we have the following properties on a fuzzy point x_a on Y: for a pair $(x, a) \in Y \times I \setminus Y \times \{0\}$,
· if $(x, a) \in (Y \times I)_1 \cup (Y \times I)_5$, then x_a is fuzzy preopen in (Y, τ_Y) and so, by Lemma 2.4(i), x_a is not fuzzy nowhere dense;
· if $(x, a) \in (Y \times I)_2 \cup (Y \times I)_4$, then x_a is not fuzzy preopen and x_a is not fuzzy nowhere dense in (Y, τ_Y);
· if $(x, a) \in (Y \times I)_3$, then x_a is fuzzy nowhere dense in (Y, τ_Y); and so, by Lemma 2.4(ii), x_a is not fuzzy preopen in (Y, τ_Y).

Finally, let λ be a fuzzy set on Y. Then, λ has a decomposition as follows (cf. Theorem 2.10(i)(ii)): $\lambda = \lambda_{\mathcal{FPO}(Y,\tau_Y)} \vee \lambda_{\mathcal{FND}(Y,\tau_Y)} \vee \lambda_{\mathcal{REST}(Y,\tau_Y)}$. The precise forms are obtained by using above decopmpstion of $Y \times I \setminus Y \times \{0\}$:
$\lambda_{\mathcal{FPO}(Y,\tau_Y)} = \bigvee\{x_a|(x, a) \in G(\lambda) \cap ((Y \times I)_1 \cup (Y \times I)_5)\}$; $\lambda_{\mathcal{FND}(Y,\tau_Y)} = \bigvee\{x_a|(x, a) \in G(\lambda) \cap (Y \times I)_3\}$; $\lambda_{\mathcal{REFT}(Y,\tau_Y)} = \bigvee\{x_a|(x, a) \in G(\lambda) \cap ((Y \times I)_2 \cup (Y \times I)_4)\}$;
where $G(\lambda)$ denotes the graph of λ (i.e., $G(\lambda) := \{(x, \lambda(x)) \in Y \times I|x \in Y\}$).

In additions, we give two simple examples of fuzzy sets on Y to see the more exact decompositions.

• For exmaple, let $\lambda \in I^Y$ be a fuzzy set defined by $\lambda(x) := 3/4$ for every point $x \in Y$. Then, we have $G(\lambda) \cap (Y \times I)_i \neq \emptyset$ for each integer i with $1 \leq i \leq 5$ and $\text{supp}(\lambda)^{FPO} = [0, 5/16] \cup [7/8, 1]$, $\text{supp}(\lambda)^{FND} = [7/16, 5/8]$ and $\text{supp}(\lambda)^{REST} = (5/16, 7/16) \cup (5/8, 7/8)$, where $[a, b]$ and (c, d) denote a closed interval and an open interval in I for real numbers $a, b, c, d \in I$ with $a < b$ and $c < d$, respectively; and so
$\lambda_{\mathcal{FPO}(Y,\tau_Y)} = \bigvee\{x_{3/4} \in I^Y|x \in [0, 5/16] \cup [7/8, 1]\}$; $\lambda_{\mathcal{FND}(Y,\tau_Y)} = \bigvee\{x_{3/4} \in I^Y|x \in [7/16, 5/8]\}$; $\lambda_{\mathcal{REST}(Y,\tau_Y)} = \bigvee\{x_{3/4} \in I^Y|x \in (5/16, 7/16) \cup (5/8, 7/8)\}$. This example shows that this fuzzy space (Y, τ_Y) does not satisfy the Janković-Reilly condition.

• For the following example λ, we shows that $\lambda_{\mathcal{REST}(Y,\tau_Y)} = 0_Y$: let $\lambda \in I^Y$ be a fuzzy set on Y defined by $\lambda(x) := 1/2$ for every point $x \in Y$; $\lambda_{\mathcal{FPO}(Y,\tau_Y)} = \bigvee\{x_{1/2} \in I^Y|x \in [0, 3/8) \cup (3/4, 1]\}$ and $\lambda_{\mathcal{FND}(Y,\tau_Y)} = \bigvee\{x_{1/2} \in I^Y|x \in [3/8, 3/4]\}$. This example shows that the converse of Therorem 2.10(iii) is not true.

(**Example II**).

The following fuzzy topological space (Y, σ^f) is a typical example of a fuzzy toplogical space satisfying the Janković-Reilly condition (cf. Definition 2.3, Theorem 2.10(ii), Theorem 3.1 and its proof below). In the present Example II, for a nonempty set Y, we assume that Y has an ordinary topology, say σ, and on Y we define an induced fuzzy topology, say σ^f, as follows: $\sigma^f := \{\chi_U | U \in \sigma\}$. Then, it is obviously shown that σ^f forms a fuzzy topology on Y and following fuzzy topological properties are shown straigtforwardly except Theorem 3.1 below. Since σ^f is crisp, there is a bijection between the topology σ and the fuzzy topology σ^f, say $f : \sigma \to \sigma^f$, which is defined by $f(U) := \chi_U$ for every set $U \in \sigma$. However, the fuzzy topology σ^f has some interesting properties (cf. Theorem 3.1, (3.3) and (3.5) below). Throughout the present Example II, $\mathrm{Cl}(\bullet)$ (resp. $\mathrm{Int}(\bullet)$) denotes the fuzzy closure (resp. fuzzy interior) in this example (Y, σ^f).

Theorem 3.1 *This example, say (Y, σ^f), of a fuzzy topological space, satisfies the Janković-Reilly condition* (cf. Definition 2.3; the proof is shown in the end of this (Example II)).

(3.2) *For a fuzzy set λ on Y,* $\mathrm{Cl}(\lambda) = \chi_{\mathrm{Cl}(\mathrm{supp}(\lambda))}$ *holds in (Y, σ^f); and* $\mathrm{Int}(\lambda) = \chi_{\mathrm{Int}(\lambda^{-1}(\{1\}))}$ *holds in (Y, σ^f).*

A family $PO(Y, \sigma)$ denotes the family of all preopen subsets of (Y, σ), i.e., $PO(Y, \sigma) := \{V \subset Y | \ V \subset \mathrm{Int}(\mathrm{Cl}(V)) \text{ holds in } (Y, \sigma)\}$.

(3.3) The following property shows that the function $f : \sigma \to \sigma^f$ is extended to the family $PO(Y, \sigma)$ by $f_p(E) := \chi_E$ for every $E \in PO(Y, \sigma)$, say $f_p : PO(Y, \sigma) \to FPO(Y, \sigma^f)$. *And, in spite of the bijection of $f : \sigma \to \sigma^f$ above, we note that the function f_p is not bijective; indeed it is not surjective in general* (cf. (III-11) in (Example III) below for the digital plane $(Y, \sigma) = (\mathbb{Z}^2, \kappa^2)$).

(3.4) (i) *A subset E is preopen in (Y, σ) if and only if χ_E is fuzzy preopen in (Y, σ^f).*
(ii) *For a subset E of (Y, σ) and a fuzzy set $\lambda_E \in I^Y$ with $E =$*supp(λ_E),
E is nowhere dense (resp. *preopen*) *in (Y, σ) if and only if λ_E is fuzzy nowhere dense* (resp. *fuzzy preopen*) *in (Y, σ^f).*
(iii) *Let λ_1 and λ_2 be fuzzy sets such that* supp(λ_1)=supp(λ_2). *Then, λ_1 is fuzzy nowhere dense* (resp. *fuzzy preopen) in (Y, σ^f) if and only if λ_2 is fuzzy nowhere dense* (resp. *fuzzy preopen*) *in (Y, σ^f).*

(3.5) Also, an extended function $f_s : SO(Y, \sigma) \to FSO(Y, \sigma^f)$ is well defined by $f_s(E) := \chi_E$ for every set $E \in SO(Y, \sigma)$, where $SO(Y, \sigma) := \{U \subset Y | \ U$ is semi-open in (Y, σ), i.e., $V \subset \mathrm{Cl}(\mathrm{Int}(V))$ holds$\}$. *The function f_s is not surjective in general* (cf. (III-4), (III-12) in (Example III) for the digital plane $(Y, \sigma) = (\mathbb{Z}^2, \kappa^2)$ below).

(3.6) *For a fuzzy point x_a, where $x \in Y$ and $a \in I(a \neq 0)$, the following properties holds in (Y, σ^f).*

(i) *Every fuzzy point x_a is fuzzy open or fuzzy preclosed in (Y, σ^f).*

(ii) *A fuzzy point x_a is fuzzy open in (Y, σ^f) if and only if $a = 1$ and $\{x\}$ is open in (Y, σ).*

(iii) (a) *If $a = 1$ and $\{x\}$ is preclosed in (Y, σ), then x_a is fuzzy preclosed in (Y, σ^f).*

(b) *If $0 < a < 1$ or $\{x\} \notin \sigma$, then x_a is fuzzy preclosed in (Y, σ^f).*

(iv) *Suppose that x_a is fuzzy preclosed in (Y, σ^f).*

(a) *If $(\mathrm{Cl}(\mathrm{Int}(x_a)))(x) = 1$ holds, then $a = 1$ and $\{x\}$ is preclosed in (Y, σ).*

(b) *If $(\mathrm{Cl}(\mathrm{Int}(x_a)))(x) \neq 1$ holds, then $0 < a < 1$ or $\{x\} \notin \sigma$.*

Proof of Theorem 3.1. We should prove that every fuzzy point in Y is fuzzy preopen or fuzzy nowhere dense in this example (Y, σ^f) of the fuzzy topological space. Let x_a be a fuzzy set on Y. We have the following two cases on a subset $\mathrm{Int}(\mathrm{Cl}(x_a))$ in (Y, σ^f).

Case 1. $\mathrm{Int}(\mathrm{Cl}(x_a)) = 0_Y$: for this case, x_a is fuzzy nowhere dense in (Y, σ^f).

Case 2. $\mathrm{Int}(\mathrm{Cl}(x_a)) \neq 0_Y$: since $\mathrm{Int}(\mathrm{Cl}(x_a)) = \chi_{\mathrm{Int}(\mathrm{Cl}(\{x\}))}$ (cf. (3.2)), for this case, we have: $(*)$ there exists a point $y \in Y$ such that $\mathrm{Int}(\mathrm{Cl}(x_a))(y) = 1$.

We recall definitions in general as follows: for a fuzzy set $\nu \in I^Y$, $\mathrm{Int}(\nu) := \bigvee\{\mu \in I^Y | \mu \leq \nu, \mu$ is fuzzy open in $(Y, \sigma^f)\}$; for a point $y \in Y$, $\mathrm{Int}(\nu)(y) := \sup\{\mu(y) \in I | \mu \in \mathcal{P}_\nu\}$, where $\mathcal{P}_\nu := \{\mu \in I^Y | \mu \leq \nu, \mu$ is fuzzy open in $(Y, \sigma^f)\}$. Then, for the fuzzy set $\mathrm{Cl}(x_a)$, put $\mathcal{P}_{\mathrm{Cl}(x_a)} := \{\mu \in I^Y | \mu \leq \mathrm{Cl}(x_a), \mu$ is fuzzy open in $(Y, \sigma^f)\}$. It follows from $(*)$ that $\mathrm{Int}(\mathrm{Cl}(x_a))(y) = \sup\{\mu(y) \in I | \mu \in \mathcal{P}_{\mathrm{Cl}(x_a)}\} = 1$. For any positive real number ε such that $0 < \varepsilon < 1$, there exists a fuzzy set $\mu_0 \in \mathcal{P}_{\mathrm{Cl}(x_a)}$ such that $1 - \varepsilon < \mu_0(y) \leq 1$. Namely, there exists an open set $U \in \sigma$ such that $\mu_0 = \chi_U \leq \mathrm{Cl}(x_a), \mu_0 \leq \mathrm{Int}(\mathrm{Cl}(x_a))$ and $1 - \varepsilon < \chi_U(y) \leq 1$. Then, we have that $\chi_U(y) = 1$ because $1 - \varepsilon$ is any positive real number and $\chi_U(y) \in \{0, 1\}$. Thus, we conclude that $\mu_0 = \chi_U \leq \mathrm{Cl}(x_a) = \chi_{\mathrm{Cl}(\{x\})}$ hold in (Y, σ^f) (cf. (3.2)) and $y \in U \in \sigma$ in (Y, σ). Thus, we have that, in (Y, σ^f), $\chi_{\{y\}} \leq \chi_U \leq \chi_{\mathrm{Cl}(\{x\})}$. Namely, $y \in \mathrm{Cl}(\{x\})$ holds in (Y, σ) and so $U \cap \{x\} \neq \emptyset$, i.e., $x \in U$. It is shown that $x_a \leq \chi_{\{x\}} \leq \chi_U = \mu_0 \leq \mathrm{Int}(\mathrm{Cl}(x_a))$. Therefore, we have that $x_a \leq \mathrm{Int}(\mathrm{Cl}(x_a))$, i.e., x_a is fuzzy preopen in (Y, σ^f).

Therefore, (Y, σ^f) satisfies the Janković-Reilly condition. □

Using Theorem 3.1, Theorem 2.10(iii) and Theorem 2.9, we prove the following corollary.

Corollary 3.7 *For this example (Y, σ^f) of a fuzzy topological space, we have the following properties.*

(i) *Every fuzzy set $\lambda \in I^Y$ with $\lambda \neq 0_Y$ has a decomposition: $\lambda = \lambda_{\mathcal{FPO}(X,\sigma^f)} \vee \lambda_{\mathcal{FND}(X,\sigma^f)}$ with $\lambda_{\mathcal{FPO}(X,\sigma^f)} \wedge \lambda_{\mathcal{FND}(X,\sigma^f)} = 0_Y$.*

(ii) *The fuzzy set $\lambda_{\mathcal{FPO}(X,\sigma^f)}$ in* (i) *is fuzzy preopen in (Y, σ^f).*

Remark 3.8 By using Theorem 3.1 for the fuzzy point $\chi_{\{x\}}$, where $x \in Y$, it is proved that every singleton is preopen or nowhere dense in a topological space (Y, σ). This result was showned by Janković and Reilly [14, Lemma 2] and it is called the Janković-Reilly lemma ([14, Lemma 2], eg., [3, p.40]).

(Example III and applications).

We apply results in Section 2 to decompositions of a fuzzy set (grey picture) on the digital plane $(\mathbb{Z}^2, \kappa^2)$ (cf. Theorem 3.9 in (III-5) below).

(III-1) The 2-dimensional images on a nonempty set Y are investigated mathematically or digitally by regarding Y as $\mathbb{Z}^2$ or a subset of $\mathbb{Z}^2$. In 1990 a topology on $\mathbb{Z}^2$ convenient for the study of digital images was introduced by Khalimsky, Kopperman and Meyer in [16]. That topology, called the Khalimsky Topology, is one of the most importante concepts of the theory called digital topology. It has been studied and used by many persons (cf. (III-8) and (III-9) below) ; a topological space $(\mathbb{Z}^2, \kappa^2)$ is called *Khalimsky plane* or *the digital plane.*

(III-2) On the other hand, the concept of grey pictures relates to the concept of fuzzy sets. We recall the concept of *grey pictures* (e.g., [28]). Let $\mathcal{F} : \mathbb{Z}^2 \to \mathbb{R}$ be a function and $\mathcal{F}(i,j) := f_{i,j} \geq 0 (f_{i,j} \in \mathbb{R})$ for each $(i,j) \in \mathbb{Z}^2$. The function $\mathcal{F}$ is called a *digital grey picture* with a grey scale $\{f_{i,j}\}$ and it is denoted by $\mathcal{F} = \{f_{i,j}\}$ (e.g., [28, Definition 1]). Conveniently, it is assumed that $f_{i,j} \neq 0$ only for finite numbers of pairs (i,j) and $f_{i,j} = 0$ for other pairs. We set $Y := \mathbb{Z}^2$ and $\lambda((i,j)) := f_{i,j}/M$ for each pair $(i,j) \in \mathbb{Z}^2$, where $M := Max\{f_{i,j} | f_{i,j} \neq 0, (i,j) \in \mathbb{Z}^2\}$. Thus, we have a fuzzy set on Y, $\lambda \in I^Y$, where $Y = \mathbb{Z}^2$. Conversely, with the grey level $\lambda(x)$, the points $x \in \mathbb{Z}^2$ forms a grey picture; they form an image on $Y = \mathbb{Z}^2$ (cf. a figure in (III-6) below).

(III-3) When we can introduce an arbitrary fuzzy topology τ_Y on $Y = \mathbb{Z}^2$, by Theorem 2.10 in Section 2, it is obtained that every fuzzy set λ on $Y := \mathbb{Z}^2$ has a decomposition using at most three fuzzy sets $\lambda_{\mathcal{FPO}(Y,\tau_Y)}, \lambda_{\mathcal{FND}(Y,\tau_Y)}$ and $\lambda_{\mathcal{REST}(Y,\tau_Y)}$. As an application, we have that: for an arbitrary fuzzy topology τ_Y on $Y := \mathbb{Z}^2$,
(*1) every 2-dimensional digital image (or so called, grey picture) λ has been decomposed by at most three digital images : $\lambda = \lambda_{\mathcal{FPO}(Y,\tau_Y)} \vee \lambda_{\mathcal{FND}(Y,\tau_Y)} \vee \lambda_{\mathcal{REST}(Y,\tau_Y)}$ (cf. Theorem 2.10(i)); and
(*2) the digital image $\lambda_{\mathcal{FPO}(Y,\tau_Y)}$ is really a fuzzy preopen set in the fuzzy topological space $(Y, \tau_Y) := (\mathbb{Z}^2, \tau_Y)$ (cf. Theorem 2.9(i)); and moreover,
(*3) each two digital images of them are fuzzy disjoint in $I^Y = I^{\mathbb{Z}^2}$ (cf. Theorem 2.10(ii)); and so even if we do not know the content of the original digital image λ and we receive separatedly these three digital images from any ones, we collect them and put together them as one sheet along the x-axis, y-axis and the origin $(0,0)$ on each three digital images; then we can get the fuzzy set $\lambda_{\mathcal{FPO}(Y,\tau_Y)} \vee \lambda_{\mathcal{FND}(Y,\tau_Y)} \vee \lambda_{\mathcal{REST}(Y,\tau_Y)}$. By Theorem 2.10(i), it is shown that this fuzzy set is equal to the original digital image λ.

Therefore, using Theorem 2.10 in Section 2, we find a method of decomposition of 2-dimensional digital image using a fuzzy topology and we explain the reasons of the decomposition by purely fuzzy topological tools and we have an application above on digital images.

(III-4) For a most simple example, as the fuzzy topology τ_Y on $Y := \mathbb{Z}^2$, now we consider an induced fuzzy topology $(\kappa^2)^f$ from the Khalimsky topology κ^2 on $Y := \mathbb{Z}^2$. Even though there exists a bijection $f : \kappa^2 \to (\kappa^2)^f$, the extended function $f_s : SO(\mathbb{Z}^2, \kappa^2) \to FSO(\mathbb{Z}^2, (\kappa^2)^f)$ is not bijective; indeed, it is not surjective (cf. (III-12) below and also (III-11) for $PO(\mathbb{Z}^2, \kappa^2)$). Thus, $(\mathbb{Z}^2, (\kappa^2)^f)$ can not be identified to the digital plane $(\mathbb{Z}^2, \kappa^2)$; and $(\mathbb{Z}^2, (\kappa^2)^f)$ is a simple example of fuzzy topological space and also the topological properties of $(\mathbb{Z}^2, \kappa^2)$ influences fuzzy topological ones of $(\mathbb{Z}^2, (\kappa^2)^f)$; that are nice influences for their studies (cf. (III-5, 6, 7) below). By Theorem 3.1 of (Example II), we have the following property: an arbitrary fuzzy set $\lambda \in I^{\mathbb{Z}^2}$ with $\lambda \neq 0_{\mathbb{Z}^2}$, has a decomposition by at most two fuzzy sets $\lambda_{\mathcal{FPO}(\mathbb{Z}^2,(\kappa^2)^f)}$ and $\lambda_{\mathcal{FND}(\mathbb{Z}^2,(\kappa^2)^f)}$ (cf. Theorem 3.9(iii) in (III-5) below); and they have exactly the following forms of Theorem 3.9(i)(ii) in (III-5) below.

(III-5) The proof of the following theorem shall be done in (III-13) below, after observing an example of the decomposition of a digital image (fuzzy set) on $\mathbb{Z}^2$ (cf. (III-6,7) below) and preparing some new terminologies and notation (cf. (III-8,9,10) below).

Theorem 3.9 *Let $\lambda \in I^{\mathbb{Z}^2}$ be a fuzzy set with $\lambda \neq 0_{\mathbb{Z}^2}$. Let $(\mathbb{Z}^2, (\kappa^2)^f)$ be a special fuzzy topological space induced by the digital plane $(\mathbb{Z}^2, \kappa^2)$.*

(i) (a) *If* $\text{supp}(\lambda) \cap (\mathbb{Z}^2)_{\kappa^2} \neq \emptyset$, *then* $\lambda_{\mathcal{FPO}(\mathbb{Z}^2,(\kappa^2)^f)} = \bigvee\{x_{\lambda(x)} \in I^{\mathbb{Z}^2} \mid x \in \text{supp}(\lambda) \cap (\mathbb{Z}^2)_{\kappa^2}\}$; *if* $\text{supp}(\lambda) \cap (\mathbb{Z}^2)_{\kappa^2} = \emptyset$, *then* $\lambda_{\mathcal{FPO}(\mathbb{Z}^2,(\kappa^2)^f)} = 0_{\mathbb{Z}^2}$.

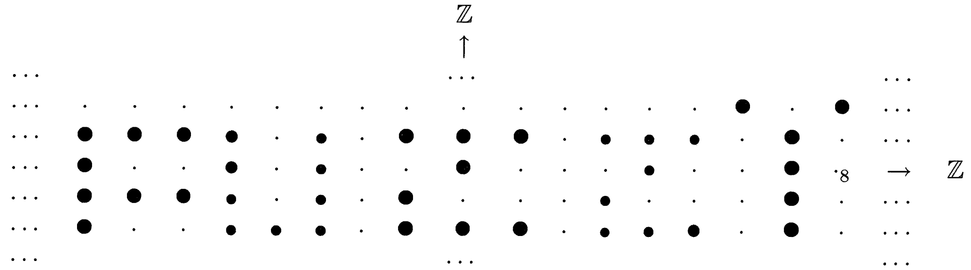
(b) *The fuzzy set $\lambda_{\mathcal{FPO}(\mathbb{Z}^2,(\kappa^2)^f)}$ is fuzzy preopen in $(\mathbb{Z}^2, (\kappa^2)^f)$.*

(ii) (a) *If* $\text{supp}(\lambda) \cap ((\mathbb{Z}^2)_{\mathcal{F}^2} \cup (\mathbb{Z}^2)_{mix}) \neq \emptyset$, *then* $\lambda_{\mathcal{FND}(\mathbb{Z}^2,(\kappa^2)^f)} = \bigvee\{x_{\lambda(x)} \in I^{\mathbb{Z}^2} \mid x \in \text{supp}(\lambda) \cap ((\mathbb{Z}^2)_{\mathcal{F}^2} \cup (\mathbb{Z}^2)_{mix})\}$. *If* $\text{supp}(\lambda) \cap ((\mathbb{Z}^2)_{\mathcal{F}^2} \cup (\mathbb{Z}^2)_{mix}) = \emptyset$, *then* $\lambda_{\mathcal{FND}(\mathbb{Z}^2,(\kappa^2)^f)} = 0_{\mathbb{Z}^2}$.

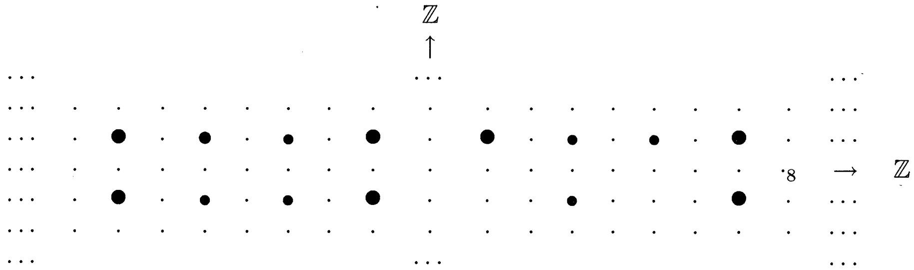
(b) *The fuzzy set $\lambda_{\mathcal{FND}(\mathbb{Z}^2,(\kappa^2)^f)}$ is fuzzy nowhere dense in $(\mathbb{Z}^2, (\kappa^2)^f)$.*

(iii) *A fuzzy set λ has a decomposition: $\lambda = \lambda_{\mathcal{FPO}(\mathbb{Z}^2,(\kappa^2)^f)} \vee \lambda_{\mathcal{FND}(\mathbb{Z}^2,(\kappa^2)^f)}$ with $\lambda_{\mathcal{FPO}(\mathbb{Z}^2,(\kappa^2)^f)} \wedge \lambda_{\mathcal{FND}(\mathbb{Z}^2,(\kappa^2)^f)} = 0_{\mathbb{Z}^2}$* (cf. (i) and (ii) above).

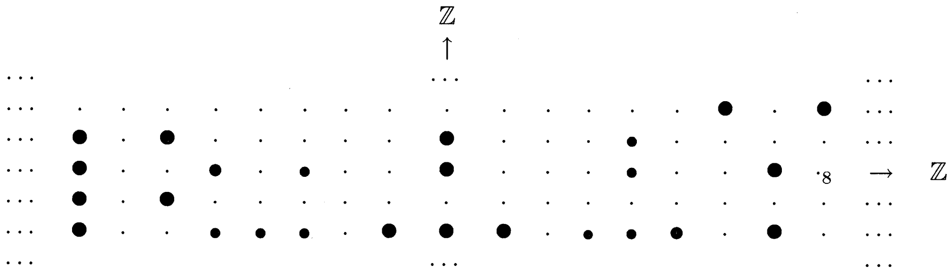
(III-6) For a simple example of digital images, we consider the following one which illustrated by the fuzzy set (image) λ on $\mathbb{Z}^2$ with some grey scale $\lambda(x)$. With the grey level $\lambda(x)$, the points $x \in \mathbb{Z}^2$ forms a grey picture; they form an image on X. In the figure, the grey levels are illustrated by ●, ●, • or •.



(III-7) (1) For the example of digital image of (III-6), say λ, the first fuzzy preopen set $\lambda_{\mathcal{PO}(\mathbb{Z}^2,(\kappa^2)^f)}$ in Theorem 3.9(i) above is illustrated as follows:



(2) And, the other fuzzy nowhere dense set (image) $\lambda_{\mathcal{FND}(\mathbb{Z}^2,(\kappa^2)^f)}$ in Theorem 3.9(ii) is illustrated as follows:



From now we recall the concept of the digital plane and related notation etc (cf. (III-8), (III-9) below); and after them we mention the proof of Theorem 3.9 (cf. (III-12) below).

(III-8) We should first recall the concept of the *digital line* or so-called *Khalimsky line*; it was originally introduced by Khalimsky (see Khalimsky et al.[16] and references there; [15]; e.g., [17, p.905]). We shall recall the following definition of the *digital line*, or so called the *Khalimsky line* (cf. [17, p.908] [18, Definition 2] [13, line −5 - −1 in page 1034] [12, line +1 - +13 after Proposition 2.1 in p.926] [22, Example 4 in Section 2.3] [8, line +2 in p.123] [20, Section 3]); the definition is more direct than the original definition given [16] etc.

• The *digital line* or so-called *Khalimsky line* $(\mathbb{Z}, \kappa)$ is the set of the integers, $\mathbb{Z}$, equipped with the topology κ, having $\{\{2m-1, 2m, 2m+1\} | m \in \mathbb{Z}\}$ as a subbase. The topology κ is called the *digital topology* or the *Khalimsky topology* on $\mathbb{Z}$. For a point $2m \in \mathbb{Z}$, $\{2m\}$ is closed and not open in $(\mathbb{Z}, \kappa)$ and for a point $2m+1 \in \mathbb{Z}, \{2m+1\}$ is open and not closed in $(\mathbb{Z}, \kappa)$, where $m \in \mathbb{Z}$. For any open subset V of $(\mathbb{Z}, \kappa)$ and a point $x \in V$, if $x = 2m$ then there exists the smallest open set $\{2m-1, 2m, 2m+1\}$ containing x such that $\{2m-1, 2m, 2m+1\} \subset V$, and if $x = 2m+1$ then there exists the smallest open set $\{2m+1\}$ containing x such that $\{2m+1\} \subset V$, where $m \in \mathbb{Z}$. We recall : a subset A of a topological space (X, τ) is called the *smallest open set containing* x if $x \in A, A \in \kappa$ and $G = A$ holds for any open sets G such that $x \in G$ and $G \subset A$. By [7, Example 4.6], it is shown that $(\mathbb{Z}, \kappa)$ is a $\mathrm{T}_{1/2}$-topological space; but an induced fuzzy topological space $(\mathbb{Z}, \kappa^f)$ is not fuzzy $T_{1/2}$ ([10, Example 4.8]).

(III-9) Next we recall definitions, fundamental properties and notations on the *digital plane*. In the present paper,
• the *digital plane* $(\mathbb{Z}^2, \kappa^2)$ is the topological product of two copies of the digital line $(\mathbb{Z}, \kappa)$ (cf. (III-8) above), where $\mathbb{Z}^2 = \mathbb{Z} \times \mathbb{Z}$ and $\kappa^2 = \kappa \times \kappa$ (e.g., [17, p.907 - p.909] [16, p.10 - p.12] [12, line +14 - +16 after Proposition 2.1] [13, p.1035] [21, Section 3] [20, Section 3]). Because of the topology κ^2, for each point $x \in \mathbb{Z}^2$, there exists the open set $U(x)$ containing x, it is called, in the present paper, the *smallest open set* $U(x)$ *containing* x (e.g.,[9]; in [9, line -7 in p.38], the set $U(x)$ is called the *basic open neighbourhood of* x); let $m, s \in \mathbb{Z}$,
· $U(x) = \{(2m+1, 2s+1)\}$ if $x = (2m+1, 2s+1)$;
· $U(x) = \{2m-1, 2m, 2m+1\} \times \{2s-1, 2s, 2s+1\}$ if $x = (2m, 2s)$;
· $U(x) = \{2m+1\} \times \{2s-1, 2s, 2s+1\}$ if $x = (2m+1, 2s)$;
· $U(x) = \{2m-1, 2m, 2m+1\} \times \{2s+1\}$ if $x = (2m, 2s+1)$.
It is well known that $x \in U(x) \subset V$ for a point $x \in \mathbb{Z}$ and any open set V containing x. Every singleton $\{(2m, 2s)\}$ is closed in $(\mathbb{Z}^2, \kappa^2)$ and it is not open, where $m, s \in \mathbb{Z}$. Every singleton $\{(2m'+1, 2s'+1)\}$ is open in $(\mathbb{Z}^2, \kappa^2)$ and it is not closed, where $m', s' \in \mathbb{Z}$. We have the explicite forms on the closures and interiors of singletons as follows: let $m, s \in \mathbb{Z}$,
· $\mathrm{Cl}(\{(2m, 2m)\}) = \{(2m, 2m)\}$; · $\mathrm{Cl}(\{(2m+1, 2s+1)\}) = \{2m, 2m+1, 2m+2\} \times \{2s, 2s+1, 2s+2\}$; · $\mathrm{Cl}(\{(2m+1, 2s)\}) = \{2m, 2m+1, 2m+2\} \times \{2s\}$; · $\mathrm{Cl}(\{(2m, 2s+1)\}) = \{2m\} \times \{2s, 2s+1, 2s+2\}$; and · $\mathrm{Int}(\{(2m+1, 2s+1)\}) = \{(2m+1, 2s+1)\}$; · $\mathrm{Int}(\{(2m+1, 2s)\}) = \mathrm{Int}(\{(2m, 2s+1)\}) = \mathrm{Int}(\{(2m, 2s)\}) = \emptyset$.
We use the following notation:
· $(\mathbb{Z}^2)_{\mathcal{F}^2} := \{x \in \mathbb{Z}^2 | \{x\}$ is closed in $(\mathbb{Z}^2, \kappa^2)\} = \{(2m, 2s) | m, s \in \mathbb{Z}\}$;
· $(\mathbb{Z}^2)_{\kappa^2} := \{x \in \mathbb{Z}^2 | \{x\}$ is open in $(\mathbb{Z}^2, \kappa^2)\} = \{(2m+1, 2s+1) | m, s \in \mathbb{Z}\}$;
· $(\mathbb{Z}^2)_{mix} := \mathbb{Z}^2 \setminus ((\mathbb{Z}^2)_{\kappa^2} \cup (\mathbb{Z}^2)_{\mathcal{F}^2}) = \{(2m+1, 2s) | m, s \in \mathbb{Z}\} \cup \{(2m, 2s+1) | m, s \in \mathbb{Z}^2\}$.
$(*)$ For a point $x \in \mathbb{Z}^2, x \in (\mathbb{Z}^2)_{mix} \cup (\mathbb{Z}^2)_{\mathcal{F}^2}$ if and only if $\mathrm{Int}(\mathrm{Cl}(\{x\})) = \emptyset$ in $(\mathbb{Z}^2, \kappa^2)$ (i.e., $\{x\}$ is nowhere dense in $(\mathbb{Z}^2, \kappa^2)$); moreover, a singleton $\{x\}$ is preopen if and only if $\{x\}$ is open if and only if $x \in (\mathbb{Z}^2)_{\kappa^2}$. We have a decomposition of $\mathbb{Z}^2$ as follows:
· $\mathbb{Z}^2 = (\mathbb{Z}^2)_{\kappa^2} \cup (\mathbb{Z}^2)_{\mathcal{F}^2} \cup (\mathbb{Z}^2)_{mix}$ (a disjoint union); and
· $(\mathbb{Z}^2)_{\mathcal{F}^2} \cup (\mathbb{Z}^2)_{mix}$ is a nowhere dense subset of $(\mathbb{Z}^2, \kappa^2)$ (e.g., [20, Lemma 3.1 for $n = 2$ and $B = \mathbb{Z}^2$], [23, Lemma 6.1 (i)(ii)]); these facts are used in Proof of Theorem 3.9.

(III-10) We define a special fuzzy topology $(\kappa^2)^f$ from κ^2 as follows (cf. (Example II) for σ^f, where σ is a topology):
• Let $(\kappa^2)^f := \{\chi_U | \ U \in \kappa^2\}$; we have a special fuzzy topological space $(\mathbb{Z}^2, (\kappa^2)^f)$ induced by $(\mathbb{Z}^2, \kappa^2)$ in the sense of (Example II) above. The fuzzy closures and fuzzy interiors of fuzzy sets are well known by (3.2) in (Example II).

(III-11) (cf. (3.3) in (Example II)) We note that the extended function $f_p : PO(\mathbb{Z}^2, \kappa^2) \to FPO(\mathbb{Z}^2, (\kappa^2)^f)$ of $f : \kappa^2 \to (\kappa^2)^f$ is not surjective. For a singleton $\{y\} \subset \mathbb{Z}^2$, where $y := (2m+1, 2s+1)$ for some integers s and m, we consider the following fuzzy set $\lambda \in I^{\mathbb{Z}^2}$ defined by $\lambda(y) := 1/2, \lambda(x) := 0$ for every point $x \in \mathbb{Z}^2$ with $x \neq y$. Then, it is shown that Int(Cl(λ)) =Int($\chi_{\mathrm{Cl}(\{y\})}$) $= \chi_{\mathrm{Int}(\mathrm{Cl}(\{y\}))} = \chi_{\{y\}} \geq \lambda$ (cf. (3.2) in (Example II) and (III-9)). Namely, λ is fuzzy preopen in $(\mathbb{Z}^2, (\kappa^2)^f)$, i.e., $\lambda \in FPO(\mathbb{Z}^2, (\kappa^2)^f)$. However, we can not find any preopen set V of $(\mathbb{Z}^2, \kappa^2)$ such that $f_p(V) = \lambda$. Indeed, suppose that there exists a preopen set V with $f_p(V) = \lambda$; then we have $\chi_V = \lambda$. We have a contradiction, because $\chi_V(y) \in \{0,1\}$ and $\lambda(y) = 1/2 \notin \{0,1\}$.

(III-12) (cf. (3.5) in (Example II), (III-4)) We note that the extended function $f_s : SO(\mathbb{Z}^2, \kappa^2) \to FSO(\mathbb{Z}^2, (\kappa^2)^f)$ of $f : \kappa^2 \to (\kappa^2)^f$ is not surjective. For a subset $A := \{(2m, 2s), (2m+1, 2s+1)\} \subset \mathbb{Z}^2$, where $s, m \in \mathbb{Z}$, we consider the following fuzzy set $\lambda_A \in I^{\mathbb{Z}^2} : \lambda_A((2m+1, 2s+1)) := 1, \lambda_A((2m, 2s)) := 1/2$ and $\lambda_A(x) := 0$ for every point $x \in \mathbb{Z}^2$ with $x \notin A$. Then, it is shown that Cl(Int(λ_A)) =Cl($\chi_{\{(2m+1,2s+1)\}}$) $= \chi_{\mathrm{Cl}(\{(2m+1,2s+1)\})} \geq \lambda_A$ (cf. (3.2) in (Example II) and (III-9)). Namely, λ_A is fuzzy semi-open in $(\mathbb{Z}^2, (\kappa^2)^f)$, i.e., $\lambda_A \in FSO(\mathbb{Z}^2, (\kappa^2)^f)$. However, we can not find any semi-open set U of $(\mathbb{Z}^2, \kappa^2)$ such that $f_s(U) = \lambda_A$. Indeed, $\chi_U \neq \lambda_A$, because $\chi_U((2m, 2s)) \subset \{0,1\}$ but $\lambda_A((2m, 2s)) = 1/2$.

(III-13) (Proof of Theorem 3.9 (cf. (III-5) above)):

(i) (a) Suppose supp(λ) $\cap (\mathbb{Z}^2)_{\kappa^2} \neq \emptyset$. We put $\mathcal{A} := \{x_{\lambda(x)} \in I^{\mathbb{Z}^2} | x_{\lambda(x)} \in FPO(\mathbb{Z}^2, (\kappa^2)^f), x \in$supp$(\lambda)\}$ and $\mathcal{B} := \{x_{\lambda(x)} \in I^{\mathbb{Z}^2} | x \in$supp$(\lambda) \cap (\mathbb{Z}^2)_{\kappa^2}\}$. We claim that $\mathcal{A} = \mathcal{B}$. Let $x_{\lambda(x)} \in \mathcal{B}$; then $x \in$ supp(λ) $\cap (\mathbb{Z}^2)_{\kappa^2}$. Then, $\{x\} \in \kappa^2$ and so $\{x\} \in PO(\mathbb{Z}^2, \kappa^2)$. By (3.4)(ii) of (Example II) above, $\{x\} \in PO(\mathbb{Z}^2, \kappa^2)$ if and only if $x_{\lambda(x)} \in FPO(\mathbb{Z}^2, (\kappa^2)^f)$. Thus, we have $\mathcal{B} \subset \mathcal{A}$. Conversely, suppose $x_{\lambda(x)} \in \mathcal{A}$; then $\lambda(x) > 0$ and $\{x\} \in PO(\mathbb{Z}^2, \kappa^2)$ (cf. (3.4)(ii) of (Example II) above). Since every preopen singleton $\{x\}$ is open in $(\mathbb{Z}^2, \kappa^2)$, i.e., $x \in (\mathbb{Z}^2)_{\kappa^2}$, we have $x \in$supp$(\lambda) \cap (\mathbb{Z}^2)_{\kappa^2}$; and so $x_{\lambda(x)} \in \mathcal{B}$. Thus, we show $\mathcal{A} \subset \mathcal{B}$. Hence, if supp($\lambda$) $\cap (\mathbb{Z}^2)_{\kappa^2} \neq \emptyset$, we have $\mathcal{A} = \mathcal{B}$ and so $\lambda_{\mathcal{FPO}(\mathbb{Z}^2,(\kappa^2)^f)} = \bigvee \mathcal{A} = \bigvee \mathcal{B} = \bigvee \{x_{\lambda(x)} \in I^{\mathbb{Z}^2} |\ x \in$supp$(\lambda) \cap (\mathbb{Z}^2)_{\kappa^2}\}$. Suppose supp($\lambda$) $\cap (\mathbb{Z}^2)_{\kappa^2} = \emptyset$. Then, for each point $x \in$supp(λ), we have $x_{\lambda(x)} \notin FPO(\mathbb{Z}^2, (\kappa^2)^f)$, because a singleton of $(\mathbb{Z}^2, \kappa^2)$ is open if and ony if it is preopen. Thus, we have supp$(\lambda)^{FPO} = \emptyset$ (cf. Notation II in Section 2). By Remark 2.7(i), it is obtained that $\lambda_{\mathcal{FPO}(\mathbb{Z}^2,(\kappa^2)^f)} = 0_{\mathbb{Z}^2}$.

(b) In general, $\lambda_{\mathcal{FPO}(Y,\tau_Y)}$ is fuzzy preopen in arbitrary fuzzy topological space (Y, τ_Y) (cf. Theorem 2.9(i)).

(ii) We put $E := (\mathbb{Z}^2)_{\mathcal{F}^2} \cup (\mathbb{Z}^2)_{mix}$. **(a)** Suppose supp($\lambda$) $\cap E \neq \emptyset$. We put $\mathcal{A} := \{x_{\lambda(x)} \in I^{\mathbb{Z}^2} | x_{\lambda(x)} \in FND(\mathbb{Z}^2, (\kappa^2)^f)\}$ and $\mathcal{B} := \{x_{\lambda(x)} \in I^{\mathbb{Z}^2} | x \in$supp$(\lambda) \cap E\}$. We claim that $\mathcal{A} = \mathcal{B}$. Let $x_{\lambda(x)} \in \mathcal{B}$; then $x \in$supp$(\lambda) \cap E$. Then, $\{x\}$ is nowhere dense in $(\mathbb{Z}^2, \kappa^2)$, because $x \in E$ and so Int(Cl($\{x\}$)) $= \emptyset$ (cf. (III-9) above). Since $\{x\}$ is nowhere dense in $(\mathbb{Z}^2, \kappa^2)$ if and only if $x_{\lambda(x)} \in FND(\mathbb{Z}^2, (\kappa^2)^f)$ (cf. (3.4)(ii) of (Example II) above), we have the following implication: $\mathcal{B} \subset \mathcal{A}$. By similar argument, it is shown that $\mathcal{A} \subset \mathcal{B}$; and so $\mathcal{B} = \mathcal{A}$. Thus, if supp($\lambda$) $\cap E \neq \emptyset$, then $\lambda_{\mathcal{FND}(\mathbb{Z}^2,(\kappa^2)^f)} = \bigvee \mathcal{A} = \bigvee \{x_{\lambda(x)} \in I^{\mathbb{Z}^2} |\ x \in$supp$(\lambda) \cap E\}$. If supp($\lambda$) $\cap E = \emptyset$, then for each point

$x \in$supp$(\lambda), x \notin E$ and so $x_{\lambda(x)} \notin FND(\mathbb{Z}^2, (\kappa^2)^f)$ (cf. (3.4)(ii) of (Example II) above). Namely, we have supp$(\lambda)^{FND} = \emptyset$. By Remark 2.7(ii), it is shown that $\lambda_{\mathcal{FND}(\mathbb{Z}^2,(\kappa^2)^f)} = 0_{\mathbb{Z}^2}$.

(b) We put $B :=$supp$(\lambda)^{FND}$. We first claim that B is nowhere dense in $(\mathbb{Z}^2, \kappa^2)$. Indeed, using (3.4)(ii) of (Example II) above and the property $(*)$ on singletons in (III-9), we have $B = \{x \in$supp$(\lambda)|\ \{x\}$ is nowhere dense in $(\mathbb{Z}^2, \kappa^2)\} \subset (\mathbb{Z}^2)_{\mathcal{F}^2} \cup (\mathbb{Z}^2)_{mix}$; and so Int(Cl$(B)) \subset$ Int(Int(Cl$((\mathbb{Z}^2)_{\mathcal{F}^2})\cup$ Cl$((\mathbb{Z}^2)_{mix})))$ $\subset$Int[Int(Cl$((\mathbb{Z}^2)_{\mathcal{F}^2}))\cup$ Cl$((\mathbb{Z}^2)_{mix})]$ =Int(Cl$(\emptyset\cup$Cl$((\mathbb{Z}^2)_{mix}))) = \emptyset$, because it is shown that Int(Cl$((\mathbb{Z}^2)_{\mathcal{F}^2})) = \emptyset$ and Int(Cl$((\mathbb{Z}^2)_{mix})) = \emptyset$. Thus, B is nowhere dense in $(\mathbb{Z}^2, \kappa^2)$ and so χ_B is fuzzy nowhere dense in $(\mathbb{Z}^2, (\kappa^2)^f)$. By using Theorem 2.9(ii) in Section 2, it is shown that $\lambda_{\mathcal{FND}(\mathbb{Z}^2,(\kappa^2)^f)}$ is fuzzy nowhere dense in $(\mathbb{Z}^2, (\kappa^2)^f)$.

(iii) The proof follows from Theorem 2.10(iii) and Theorem 3.1 .

An alternative proof of (iii): By (i), (ii) above, Lemma 2.5(i)(ii) and a decomposition of $\mathbb{Z}^2$ (cf. (III-9) above), it is shown that $\lambda = \bigvee\{x_{\lambda(x)} \in I^{\mathbb{Z}^2} | x \in$supp$(\lambda)\}$ $=(\bigvee\{x_{\lambda(x)} \in I^{\mathbb{Z}^2} | x \in$supp$(\lambda)\cap(\mathbb{Z}^2)_{\kappa^2}\})\vee(\bigvee\{x_{\lambda(x)} \in I^{\mathbb{Z}^2} | x \in$supp$(\lambda)\cap((\mathbb{Z}^2)_{\mathcal{F}^2}\cup (\mathbb{Z}^2)_{mix})\}) = (\bigvee\{x_{\lambda(x)} \in I^{\mathbb{Z}^2} | x_{\lambda(x)} \in FPO(\mathbb{Z}^2, (\kappa^2)^f)\})\vee(\bigvee\{x_{\lambda(x)} \in I^{\mathbb{Z}^2} | x_{\lambda(x)} \in FND(\mathbb{Z}^2, (\kappa^2)^f)\})$. Thus, we conclude that $\lambda = \lambda_{\mathcal{FPO}(\mathbb{Z}^2,(\kappa^2)^f)} \vee \lambda_{\mathcal{FND}(\mathbb{Z}^2,(\kappa^2)^f)}$. Theorem 2.10(ii) shows $\lambda_{\mathcal{PO}(\mathbb{Z}^2,(\kappa^2)^f)} \wedge \lambda_{\mathcal{ND}(\mathbb{Z}^2,(\kappa^2)^f)} = 0_{\mathbb{Z}^2}$. □

Remark 3.10. When we choice an alternative and convenient topology σ on $\mathbb{Z}^2$ such that $\sigma \neq \kappa^2$, then using result (Theorem 2.10) of Section 2 and Example II (Corollary 3.7) of Section 3 we can have an alternative decomposition of a given fuzzy set λ on $\mathbb{Z}^2$: $\lambda = \lambda_{\mathcal{FPO}(\mathbb{Z}^2,\sigma^f)} \vee \lambda_{\mathcal{FND}(\mathbb{Z}^2,\sigma^f)}$ with $\lambda_{\mathcal{FPO}(\mathbb{Z}^2,\sigma^f)} \wedge \lambda_{\mathcal{FND}(\mathbb{Z}^2,\sigma^f)} = 0$. Moreover, we choice an alternative and computable fuzzy topology $\tau_{\mathbb{Z}^2}$ on $\mathbb{Z}^2$ such that $\tau_{\mathbb{Z}^2} \neq \sigma^f$ and $\sigma^f \neq (\kappa^2)^f$ and, using result Theorem 2.10 of Section 2, we have an alternative decomposition of a given fuzzy set on $\mathbb{Z}^2$. Thus, as applications, we shall have a lot of decompositions of a given 2-dimensional digital image.

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Communicated by *Hiroaki Ishii*

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