PROXIMINALITY IN OPERATOR SPACES

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Abstract. In this article, we study proximinality of some subspaces of operator spaces. Namely, the bounded linear operators and Nuclear operators.

1 Introduction Let X be a normed space, and E be a subspace of X. For $x \in X$, define

$$d(x, E) = \inf\{\|x - e\| : e \in E\}.$$

If this distance is attained for every $x \in X$, we say that E is proximinal. The study of proximinal subspaces is well developed, and many results are available in this direction. We refer the reader to [4] for a comprehensive study of proximinal sets.
For two normed spaces X and Y, we denote by $L(X, Y)$ the space of all bounded linear operators from X into Y.

The study of proximinal subspaces in operator spaces is not as rich as that of normed spaces. However, many results are available in operator spaces like $L^p(I, x)$ and $L(\ell^1, X)$. We refer the reader to [1] for some results on this topic.

Tensor product spaces have played a nice role in the study of proximinal subspaces. For this topic, we refer the reader to [1] and [2].
In this article, we shall study proximinal subspaces in the operator spaces $L(X, Y)$, $N_1(X, \ell^1)$ and $N_p(X, \ell^p)$, where X and Y are any normed spaces.

Among other results, we shall prove that $L(X, G)$ is proximinal in $L(X, Y)$ for any tensorial subspace G of X, $N_1(G, \ell^1)$ is proximinal in $N_1(X.\ell^1)$ and $N_p(G, \ell^p)$ is proximinal in $N_p(X, \ell^p)$ for any closed complemented subspace G of X.

Although our main purpose of this article is to obtain new results on proximinality, many other important results are obtained.

In the sequel, I denotes the unit interval $[0, 1]$, X^* the dual space of the normed space$X$ and $< x^*, x >$ will denote $x^*(x)$, where $x \in X$ and $x^* \in X^*$.

2 Main Results Our study may be divided into two parts; the first part treats the problem of $L(X, Y)$ and the second treats the Nuclear spaces $N_1(X, \ell^1)$ and $N_p(X, \ell^p)$.

2.1 The operator space $L(X, Y)$ For the first part, we study an analogue of the results in [1], where the relation between proximinality of a subspace G in a normed space Y, and that of $L^p(I, G)$ in $L^p(I, Y)$ was discussed. In fact, it is proved, in [1], that G is proximinal in Y if and only if $L^p(I, G)$ is proximinal in $L^p(I, Y)$, provided that G is a tensorial subspace of Y, see definition 1. Then, in [1], it was proved that G is proximinal in Y if and only if

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$L(\ell^1, G)$ is proximinal in $L(\ell^1, Y)$. In fact, some results in [1] were obtained in an attempt to solve the more challenging problem: Given two normed spaces X and Y, is the proximinality of a subspace G of Y equivalent to the proximinality of $L(X, G)$ in $L(X, Y)$?

In the following theorem, we shall use the expression $x^* \otimes y$ as being an operator $x^* \otimes y : X \to Y$ such that $(x^* \otimes y)(x) = x^*(x)\, y$. Here, $x^* \in X^*$ and $y \in Y$. We refer the reader to [2] for more on tensor product spaces.

Theorem 1 *Let X and Y be two normed spaces, and let G be a subspace of Y. If $L(X, G)$ is proximinal in $L(X, Y)$, then G is proximinal in Y.*

Proof. Let $y \in Y$ be arbitrary. We need to find an element $g_0 \in G$ such that

$$\|y - g_0\| \leq \|y - g\|;\ \forall g \in G.$$

For this, let $x_0 \in X$ be such that $\|x_0\| = 1$. By the Hahn-Banach theorem, let $x^* \in X^*$ be such that

$$\|x^*\| = x^*(x_0) = \|x_0\| = 1.$$

Define the bounded linear operator $x^* \otimes y : X \to Y$ by

$$(x^* \otimes y)(x) = x^*(x)\, y,\ x \in X.$$

Since $L(X, G)$ is proximinal in $L(X, Y)$, there is a bounded linear operator $T \in L(X, G)$ such that

$$\|x^* \otimes y - T\| \leq \|x^* \otimes y - S\|;\ \forall S \in L(X, G).$$

In particular, if $g \in G$, then

$$\|x^* \otimes y - T\| \leq \|x^* \otimes y - x^* \otimes g\|.$$

But then,

$$\begin{aligned} \|(x^* \otimes y - T)(x_0)\| &\leq \|x^* \otimes y - T\|\ \|x_0\| \\ &= \|x^* \otimes y - T\| \\ &\leq \|x^* \otimes y - x^* \otimes g\| \\ &\leq \|x^*\|\ \|y - g\| \\ &= \|y - g\|. \end{aligned}$$

Thus, we have shown that $\|x^*(x_0)y - T(x_0)\| \leq \|y - g\|$, for all $g \in G$. But since $x^*(x_0) = 1$ and $T(x_0) \in G$, we infer that $\|y - g_0\| \leq \|y - g\|$ for all $g \in G$, where $g_0 = T(x_0)$. This completes the proof.

So, the converse problem is still open. But, we still can prove that $L(X, G)$ is proximinal in $L(X, Y)$ under certain restrictions. First, a definition.

Definition 1 *Let G be a subspace of the normed space X. Define*

$$\hat{G} = \{x \in X : d(x, G) = \|x\|\}.$$

We say that G is a tensorial subspace of X if $\hat{G}$ is a subspace of X and if $X = G \oplus \hat{G}$, in the sense that every $x \in X$ can be written as $x = g + \hat{g},\ g \in G, \hat{g} \in \hat{G}$.

For example, any closed subspace of an inner product space is tensorial. In fact, if G is a subspace of X, an inner product space, then $\hat{G} = G^{\perp}$.

On the other hand, one can see that if $X = R^2$, with the max norm, and $G = R \subset X$, then $\hat{G}$ is not a subspace of X, hence G is not tensorial.

Observe that, in any normed space X and for every subspace G of X, $G \cap \hat{G} = \{0\}$. Consequently, if $x = g + \hat{g}$ for some $g \in G$ and $\hat{g} \in \hat{G}$, then this representation is unique.

Theorem 2 *If G is a tensorial subspace of X, then G is proximinal in X.*

Proof. Let $x \in X$, and let $g_x \in G$ and $\hat{g}_x \in \hat{G}$ be such that $x = g_x + \hat{g}_x$. Then,

$$\begin{aligned} \|x - g_x\| &= \|\hat{g}_x\| \\ &= d(\hat{g}_x, G) \\ &\leq \|\hat{g}_x - (g - g_x)\|, \ \forall g \in G \\ &= \|x - g\|, \ \forall g \in G. \end{aligned}$$

But then $d(x, G)$ is attained at g_x. Since this is valid for arbitrary $x \in X$, the result follows.

Proposition 1 *If G is a tensorial subspace of G, then G is complemented. That is, the projection of X onto G is a continuous mapping.*

Proof. Let $x \in X$ and let $g_x \in G, \hat{g}_x \in \hat{G}$ be such that $x = g_x + \hat{g}_x$. Then,

$$d(\hat{g}_x, G) = \|\hat{g}_x\| \leq \|\hat{g}_x + g_x\| = \|x\|.$$

Consequently, the projection $P : X \to \hat{G}$ is contractive, and hence $I - P : X \to G$ is continuous. This completes the proof.

Theorem 3 *Let X and Y be two normed spaces, and let G be a tensorial subspace of Y. Then, $L(X, G)$ is proximinal in $L(X, Y)$.*

Proof. Let $T \in L(X, Y)$ and $A = (I - P)T \in L(X, G)$, then for every $F \in L(X, G)$, we have

$$\begin{aligned} \|T - A\| &= \|T - (I - P)T\| \\ &= \|PT\| \\ &= \|(P(T - F)\| \\ &\leq \|T - F\|. \end{aligned}$$

Since F was arbitrary in $L(X, G)$, the result follows.

Now as a matter of notation, let

$$\hat{L}(X, G) = \{T \in L(X, Y) : \|T\| = d(T, L(X, G))\}.$$

Proposition 2 *Let X and Y be normed spaces, and let G be a tensorial subspace of Y. Then,*

$$L(X, \hat{G}) = \hat{L}(X, G).$$

Proof. Let $T \in L(X, \hat{G})$. Then $Tx \in \hat{G}$, and hence $\|Tx\| = d(Tx, G)$. Then, for every $T' \in L(X, G)$, we have

$$\begin{aligned} \|Tx\| &\le \|Tx - T'x\| \\ &\le \|T - T'\| \, \|x\|. \end{aligned}$$

Consequently, $\|T\| \le \|T - T'\|$. Since $T' \in L(X, G)$ was arbitrary, we infer that $T \in \hat{L}(X, G)$. This shows that $L(X, \hat{G}) \subseteq \hat{L}(X, G)$.

For the reverse inclusion, let $T \in \hat{L}(X, G)$. We have seen in Theorem 3 that $T = A + B$ where $A \in L(X, G)$ and $B \in L(X, \hat{G})$. Also, we have just proved that $L(X, \hat{G}) \subseteq \hat{L}(X, G)$, hence $B \in \hat{L}(X, G)$. We assert that $A = 0$, and hence $T \in L(X, \hat{G})$. Since G is tensorial, $L(X, G)$ is proximinal in $L(X, Y)$, and hence

$$L(X, Y) = L(X, G) \oplus \hat{L}(X, G),$$

see [1]. Observe that $T = T + 0$ is the unique representation of T in this direct sum. But also, $T = A + B$ where $A \in L(X, G)$ and $B \in L(X, \hat{G}) \subseteq \hat{L}(X, G)$. Since the decomposition is unique, we infer that $A = 0$. This completes the proof.

2.2 The Nuclear Spaces Now we study proximinality in Nuclear spaces. First, let us recall the definition of these spaces.

Definition 2 *Let X and Y be two normed spaces. We define the Nuclear space $N_1(X, Y)$ as follows:*

$$N_1(X, Y) = \left\{ T : X \to Y : T = \sum_{n=1}^{\infty} x_n^* \otimes y_n, x_n^* \in X^*, y_n \in Y, \sum_{n=1}^{\infty} \|x_n^*\| \, \|y_n\| < \infty \right\}.$$

For $T = \sum_{n=1}^{\infty} x_n^* \otimes y_n \in N_1(X, Y)$, we define the nuclear norm of T to be

$$\|T\|_1 = \inf \sum_{n=1}^{\infty} \|x_n^*\| \, \|y_n\|,$$

where the infimum is taken over all possible representations of T.

We refer the reader to [3] for more on nuclear operators in Banach spaces.

Proposition 3 *If $T \in N_1(X, \ell^1)$, then T can be written as*

$$T = \sum_{n=1}^{\infty} x_n^* \otimes \delta_n, \; x_n^* \in X^*,$$

and $\|T\|_1$ is attained. In fact,

$$\|T\|_1 = \sum_{n=1}^{\infty} \|x_n^*\|.$$

Proof. Let $T = \sum_{n=1}^{\infty} z_n^* \otimes y_n$ be an arbitrary representation of $T \in N_1(X, \ell^1)$. Since $y_n \in \ell^1$, we may write

$$y_n = \sum_{k=1}^{\infty} y_{nk} \delta_k, \; \textit{where} \; \|y_n\| = \sum_{k=1}^{\infty} |y_{nk}|.$$

By the definition of $N_1(X, \ell^1)$, we have

$$\sum_{n=1}^{\infty} \|z_n^*\| \, \|y_n\| < \infty.$$

But then,

$$\begin{aligned} & \sum_{n=1}^{\infty} \|z_n^*\| \sum_{k=1}^{\infty} |y_{nk}| < \infty \\ \Rightarrow \quad & \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \|z_n^*\| \, |y_{nk}| < \infty \\ \Rightarrow \quad & \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \|z_n^*\| \, |y_{nk}| < \infty. \end{aligned}$$

But this last inequality implies that for each $k \geq 1$, we have $\sum_{n=1}^{\infty} \|y_{nk} \; z_n^*\| < \infty$, and hence

$$\sum_{n=1}^{\infty} y_{nk} \; z_n^* \in X^*, \; \forall k \geq 1,$$

because X^* is a complete normed space. Consequently, if $x_k^* = \sum_{n=1}^{\infty} \; y_{nk} \; z_n^*$, then

$$T = \sum_{k=1}^{\infty} x_k^* \otimes \delta_k.$$

Moreover,

$$\begin{aligned} \sum_{n=1}^{\infty} \|z_n^*\| \, \|y_n\| &= \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \|z_n^*\| \, |y_{nk}| \\ &= \sum_{k=1}^{\infty} \left(\sum_{n=1}^{\infty} \|y_{nk} \; z_n^*\| \right) \\ &\geq \sum_{k=1}^{\infty} \left\| \sum_{n=1}^{\infty} y_{nk} \; z_n^* \right\|. \\ &= \sum_{k=1}^{\infty} \|x_k^*\| \end{aligned}$$

This completes the proof of the proposition.

Before proceeding, we emphasize on the use of the notation G^* in the rest of this paper: We know that if G is a subspace of X, then any element $f \in G^*$ maybe extended to an $F \in X^*$. Unfortunately, this extension is not unique in general. However, if G is a complemented closed subspace of X, then any element $g^* \in G^*$ may be extended to X as follows:

$$g^*(x) = g^*(Px), \; P \textit{ is the projection on } G.$$

In the rest of this paper, G^* will refer to the set of these particular extensions, so that G^* becomes a subspace of X^*. In fact, with this convention, G^* is clearly a w^*-closed subspace of X^*.

Theorem 4 *If G is a closed complemented subspace of the normed space X, then $N_1(G,\ell^1)$ is proximinal in $N_1(X,\ell^1)$.*

Proof. Let $T \in N_1(X,\ell^1)$, then

$$T = \sum_{n=1}^{\infty} x_n^* \otimes \delta_n,\ x_n^* \in X^*,\ \textit{and}\ \|T\|_1 = \sum_{n=1}^{\infty} \|x_n^*\|.$$

Since G^* is w^*- closed in X^*, G^* is proximinal in X^*. For each $n \in N$, let $g_n^* \in G^*$ be a best approximant of x_n^* in G^*, and let

$$\hat{T} = \sum_{n=1}^{\infty} g_n^* \otimes \delta_n.$$

Then, clearly $\hat{T} \in N_1(G,\ell^1)$, and, by virtue of proposition 3,

$$\begin{aligned} \|T-\hat{T}\| &= \sum_{n=1}^{\infty} \|x_n^* - g_n^*\| \\ &\le \sum_{n=1}^{\infty} \|x_n^* - w_n^*\|,\ \forall w_n^* \in G^* \\ &= \|T-S\|, \end{aligned}$$

where $S = \sum_{n=1}^{\infty} w_n^* \otimes \delta_n$. But since any operator $S \in N_1(G,\ell^1)$ has the form $S = \sum_{n=1}^{\infty} w_n^* \otimes \delta_n$, for a certain choice of the w_n^*, the result follows.

Now, we move to the study of $N_p(X,\ell^p)$. First, we recall the definition of these spaces.

Definition 3 *Let X be a normed space, and let ℓ^p be the standard sequence spaces, with $p > 1$. We define $N_p(X,\ell^p)$ to be the space of operators $T : X \to \ell^p$ defined by*

$$T = \sum_{n=1}^{\infty} x_n^* \otimes y_n,\ x_n^* \in X^*, y_n \in \ell^p,$$

such that

$$\|(x_n^*)\|_{\pi(p)} := \left(\sum_{n=1}^{\infty} \|x_n^*\|^p\right)^{1/p} < \infty\ \textit{and}\ \|(y_n)\|_{\epsilon(p^*)} := \sup_{y^* \in \ell^{p^*}} \left(\sum_{n=1}^{\infty} |y^*(y_n)|^{p^*}\right)^{1/p^*} < \infty.$$

For $T \in N_p(X,\ell^p)$, we define

$$\|T\| = \inf \|(x_n^*)\|_{\pi(p)} \|(y_n)\|_{\epsilon(p^*)}.$$

We refer the reader to [3] for more on these Nuclear spaces. We now prove an analogue of Proposition 3.

Proposition 4 *Let $T \in N_p(X,\ell^p)$, then*

$$T = \sum_{n=1}^{\infty} z_n^* \otimes \delta_n,\ \text{for some } z_n^* \in X^*,$$

and $\|T\|$ is attained. In fact,

$$\|T\| = \left(\sum_{n=1}^{\infty} \|z_n^*\|^p\right)^{1/p}.$$

Proof. Let $T \in N_p(X, \ell^p)$. We may assume, without loss of generality, that

$$T = \sum_{n=1}^{\infty} \lambda_n \, x_n^* \otimes y_n, \; x_n^* \in X^*, y_n = \sum_{k=1}^{\infty} y_{nk} \delta_k \in \ell^p,$$

where

$$\|x_n^*\| = 1, \; \|y_n\| \leq 1, \; \|(y_n)\|_{\epsilon(p^*)} < 1 \text{ and } \sum_{n=1}^{\infty} |\lambda_n|^p < \infty.$$

Now, let

$$z_k^* = \sum_{n=1}^{\infty} \lambda_n \; y_{nk} \; x_n^* \in X^*.$$

We first show that $z_k^* \in X^*$. Indeed,

$$\begin{aligned} \sum_{n=1}^{\infty} \|\lambda_n x_n^* y_{nk}\| &\leq \sum_{n=1}^{\infty} |\lambda_n| \, |y_{nk}| \\ &\leq \left(\sum_{n=1}^{\infty} |\lambda_n|^p \right)^{1/p} \left(\sum_{n=1}^{\infty} |y_{nk}|^{p^*} \right)^{1/p^*} \\ &= \left(\sum_{n=1}^{\infty} |\lambda_n|^p \right)^{1/p} \left(\sum_{n=1}^{\infty} | < \delta_k, y_n > |^{p^*} \right)^{1/p^*} \\ &< \infty, \end{aligned}$$

where the last inequality follows from the assumption that $\sum_{n=1}^{\infty} |\lambda_n|^p < \infty$ and that

$$\left(\sum_{n=1}^{\infty} | < \delta_k, y_n > |^{p^*} \right)^{1/p^*} \leq \sup_{y^* \in \ell^{p^*}} \left(\sum_{n=1}^{\infty} | < y^*, y_n > |^{p^*} \right)^{1/p^*} \leq 1,$$

by the definition of $N_p(X, \ell^p)$.

Consequently,

$$T = \sum_{k=1}^{\infty} z_k^* \otimes \delta_k.$$

Now, we assert that

$$\|(z_k^*)\|_{\pi(p)} < \infty \text{ and } \|(\delta_k)\|_{\epsilon(p^*)} < \infty.$$

It is clear that $\|(\delta_k)\|_{\epsilon(p^*)} < \infty$. For $\|(z_k^*)\|_{\pi(p)}$, observe that a sequence $(\xi_k) \in \ell^{P^*}$ exists such that

$$\left(\sum_{k=1}^{\infty} |\xi_k|^{p^*} \right)^{1/p^*} \leq 1,$$

and

$$\begin{aligned} \left(\sum_{k=1}^{\infty} \|z_k^*\|^p \right)^{1/p} &= \left| \sum_{k=1}^{\infty} \xi_k \|z_k^*\| \right| \\ &\leq \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} |\xi_k| \, |\lambda_n| \, |y_{nk}| \end{aligned}$$

$$
\begin{aligned}
&\leq \left(\sum_{n=1}^{\infty}|\lambda_n|^p\right)^{1/p} \left(\sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty}|\xi_k|\,|y_{nk}|\right)^{p^*}\right)^{1/p^*} \\
&= \left(\sum_{n=1}^{\infty}|\lambda_n|^p\right)^{1/p} \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\alpha_n|\xi_k|\,|y_{nk}|
\end{aligned}
$$

for some $(\alpha_n) \in \ell^p$ having the property $\|(\alpha_n)\|_p \leq 1$. The last step in our proof is to show that

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\alpha_n|\xi_k|\,|y_{nk}| \leq 1.$$

For this purpose, observe that

$$
\begin{aligned}
\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\alpha_n|\xi_k|\,|y_{nk}| &\leq \sum_{n=1}^{\infty}|\alpha_n|\sum_{k=1}^{\infty}|\xi_k|\,|<\delta_k, y_n>| \\
&\leq \sum_{n=1}^{\infty}|\alpha_n|\left(\sum_{k=1}^{\infty}|<\xi_k\delta_k, y_n>|\right) \\
&\leq \|(\alpha_n)\|_p \sup_{y^*\in\ell^{p^*}}\left(\sum_{n=1}^{\infty}|<y^*, y_n>|^{p^*}\right)^{1/p^*} \\
&\leq 1.
\end{aligned}
$$

Thus, we have shown that

$$\|(z_k^*)\|_{\pi(p)} \leq \|(\lambda_n)\|_p.$$

Since the representation $T = \sum_{k=1}^{\infty} z_k^*\otimes\delta_k$ is unique, with the δ_k's as the second coordinates, for $T \in N_p(X, \ell^p)$, we infer that

$$\|T\| = \|(z_k^*)\|_{\pi(p)} = \left(\sum_{k=1}^{\infty}\|z_k^*\|^p\right)^{1/p}.$$

This completes the proof.

Now having proved this, we present the last main result in this paper, which is an analogue of Theorem 4, and hence we leave the proof to the reader.

Theorem 5 *If G is a closed complemented subspace of the normed space X, then $N_p(G, \ell^p)$ is proximinal in $N_p(X, \ell^p)$.*

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