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CONSTRUCTION OF SLOWLY INCREASING FUNCTIONS

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Abstract. We construct a continuous and bijective function $L : (0, \infty) \to (-\infty, \infty)$ which is increasing slower than any nth iterate of logarithmic function. Further, we construct a function which is increasing slower than any nth iterate of L. Using our method, we can construct more and more slowly increasing functions.

1 Introduction In this paper we construct a very slowly increasing function, namely, we construct a continuous and strictly increasing function $L : (0, \infty) \to (-\infty, \infty)$ such that

$$\lim_{r\to 0} L(r) = -\infty, \quad L(1) = 0, \quad \lim_{r\to\infty} L(r) = \infty,$$

and

$$\lim_{r\to 0} \frac{L(r)}{\log^n(1/r)} = \lim_{r\to\infty} \frac{L(r)}{\log^n r} = 0 \quad \text{for each } n \in \mathbb{N} = \{1, 2, \cdots\},$$

where $\log^0 r = r$ and $\log^n r = \log(\log^{n-1} r)$, $n \in \mathbb{N}$. While the logarithmic function has the property $\log r^n = n \log r$, the function $L(r)$ has the following property: There exists a positive constant c such that, for large r,

$$L(r) \le L(\exp(r)) \le cL(r).$$

Further, letting $L^{\langle 1\rangle}(r) = L(r)$, we construct continuous and strictly increasing functions $L^{\langle m\rangle} : (0, \infty) \to (-\infty, \infty)$, $m \ge 2$, such that

$$\lim_{r\to 0} L^{\langle m\rangle}(r) = -\infty, \quad L^{\langle m\rangle}(1) = 0, \quad \lim_{r\to\infty} L^{\langle m\rangle}(r) = \infty$$

and

$$\lim_{r\to 0} \frac{L^{\langle m+1\rangle}(r)}{\log^n L^{\langle m\rangle}(1/r)} = \lim_{r\to\infty} \frac{L^{\langle m+1\rangle}(r)}{\log^n L^{\langle m\rangle}(r)} = 0 \quad \text{for each } m, n \in \mathbb{N}.$$

Moreover, letting $L^{\langle 0,m\rangle}(r) = L^{\langle m\rangle}(r)$, $m \in \mathbb{N}$, we can construct continuous and strictly increasing functions $L^{\langle \ell,m\rangle} : (0, \infty) \to (-\infty, \infty)$, $\ell, m \in \mathbb{N}$, such that

$$\lim_{r\to 0} L^{\langle \ell,m\rangle}(r) = -\infty, \quad L^{\langle \ell,m\rangle}(1) = 0, \quad \lim_{r\to\infty} L^{\langle \ell,m\rangle}(r) = \infty,$$

$$\lim_{r\to 0} \frac{L^{\langle \ell,1\rangle}(r)}{L^{\langle \ell-1,m\rangle}(1/r)} = \lim_{r\to\infty} \frac{L^{\langle \ell,1\rangle}(r)}{L^{\langle \ell-1,m\rangle}(r)} = 0 \quad \text{for each } \ell, m \in \mathbb{N},$$

and

$$\lim_{r\to 0} \frac{L^{\langle \ell,m+1\rangle}(r)}{\log^n L^{\langle \ell,m\rangle}(1/r)} = \lim_{r\to\infty} \frac{L^{\langle \ell,m+1\rangle}(r)}{\log^n L^{\langle \ell,m\rangle}(r)} = 0 \quad \text{for each } \ell, m, n \in \mathbb{N}.$$

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In particular, letting $L^0(r) = r$ and $L^m(r) = L(L^{m-1}(r))$, $m \in \mathbb{N}$, we have

$$\lim_{r\to 0} \frac{L^{\langle 1,1\rangle}(r)}{L^m(1/r)} = \lim_{r\to\infty} \frac{L^{\langle 1,1\rangle}(r)}{L^m(r)} = 0 \quad \text{for each } m \in \mathbb{N},$$

since the relation $L^{\langle m+1\rangle}(r) \le L(L^{\langle m\rangle}(r)) \le cL^{\langle m+1\rangle}(r)$ holds for large r.

Using our method, we can construct more and more slowly increasing functions. Moreover, the inverse functions of them are rapidly increasing as $r \to \infty$ and rapidly decreasing to 0 as $r \to -\infty$. Let E be the inverse function of L. Then

$$\lim_{r\to-\infty} \frac{\exp^n(-r)}{\frac{1}{E(r)}} = \lim_{r\to\infty} \frac{\exp^n(r)}{E(r)} = 0 \quad \text{for each } n \in \mathbb{N},$$

where $\exp^0(r) = r$ and $\exp^n(r) = \exp(\exp^{n-1}(r))$, $n \in \mathbb{N}$.

Several functions are known as rapidly increasing functions, for example, the tetration, the hyperoperation, Ackermann functions, etc., see [1, 5, 6]. The inverse functions of them are slowly increasing. On our functions we can easily check their differentiability. All of our slowly increasing functions are differentiable on $(0, \infty)$ and infinitely differentiable except at 1, and the inverse functions of them are differentiable on $(-\infty, \infty)$ and infinitely differentiable except at 0.

In Sections 2 and 3 we state the definitions and properties of L and $L^{\langle m\rangle}$, $m \in \mathbb{N}$, respectively. Then, based on the idea in Sections 2 and 3, we give the method of construction of slowly increasing functions in Section 4, In the last section we state the definitions and properties of $L^{\langle \ell,m\rangle}$, $\ell, m \in \mathbb{N}$, and more slowly increasing functions. Our idea comes from the study of missing terms of Hardy-Sobolev inequalities [2, 3, 4].

2 Construction of $L(r)$ First we define two sets of functions.

Definition 2.1. Let $\mathcal{L}$ be the set of all continuous, increasing and bijective functions f from $(0, \infty)$ to $(-\infty, \infty)$ satisfying

$$\lim_{r\to 0} f(r) = -\infty, \quad f(1) = 0, \quad \lim_{r\to\infty} f(r) = \infty.$$

For example, the logarithmic function $\log r$ is in $\mathcal{L}$.

Definition 2.2. For $a > 1$, let $\mathcal{F}_a$ be the set of all continuous, increasing and bijective functions from $[a, \infty)$ to itself.

If $f \in \mathcal{F}_a$, then $f(a) = a$ and $\lim_{u\to\infty} f(u) = \infty$. For a function $f \in \mathcal{F}_a$, let $f^0(u) = u$ and $f^k(u) = f(f^{k-1}(u))$, $k \in \mathbb{N}$. Then f^k is also in $\mathcal{F}_a$.

We define a function $F \in \mathcal{F}_a$ as

$$F(u) = F_a(u) = a - \log a + \log u \quad (u \ge a). \tag{2.1}$$

Then the relation

$$(F^k(u))' = \frac{1}{F^{k-1}(u) \cdots F^1(u) F^0(u)} \tag{2.2}$$

holds. That is,

$$F^k(u) = a + \int_a^u \frac{dt}{F^{k-1}(t) \cdots F^1(t) F^0(t)}. \tag{2.3}$$

Let

$$\ell_k(r) = F^k(ar) - a = \int_a^{ar} \frac{dt}{F^{k-1}(t)\cdots F^1(t)F^0(t)} \quad (r \geq 1), \tag{2.4}$$

and let

$$\ell_k(r) = -\ell_k(1/r) = -\int_a^{a/r} \frac{dt}{F^{k-1}(t)\cdots F^1(t)F^0(t)} \quad (0 < r < 1). \tag{2.5}$$

Then $\ell_k \in \mathcal{L}$ and

$$\lim_{r\to 0} \frac{\ell_k(r)}{\log^k(1/r)} = \lim_{r\to\infty} \frac{\ell_k(r)}{\log^k r} = 1 \quad \text{for each } k \in \mathbb{N}. \tag{2.6}$$

To construct the limit function of ℓ_k as $k \to \infty$, we use the integral

$$\int_a^u \frac{dt}{F^{k-1}(t)\cdots F^1(t)F^0(t)} \tag{2.7}$$

with exchanging

$$F^{k-1}(t)\cdots F^1(t)F^0(t) \quad \text{for} \quad \frac{F^{k-1}(t)}{a}\cdots\frac{F^1(t)}{a}\frac{F^0(t)}{a}.$$

Then we can show that the limit exists. This is our main idea.

Definition 2.3. For $a > 1$, let

$$\tilde{F}(u) = \tilde{F}_a(u) = a\prod_{k=0}^{\infty} \frac{F^k(u)}{a} \quad (u \geq a), \tag{2.8}$$

and let

$$\phi(u) = \phi_a(u) = a + \int_a^u \frac{1}{\tilde{F}(t)}\,dt \quad (u \geq a). \tag{2.9}$$

The convergence of the infinite product in (2.8) will be proven later. Note that, if $a = 1$, then the infinite product in (2.8) diverges, see Remark 4.1.

Definition 2.4. For $a > 1$, let

$$L(r) = L_a(r) = \phi(ar) - a = \int_a^{ar} \frac{1}{\tilde{F}(t)}\,dt \quad (r \geq 1), \tag{2.10}$$

and let

$$L(r) = -L(1/r) = -\int_a^{a/r} \frac{1}{\tilde{F}(t)}\,dt \quad (0 < r < 1), \tag{2.11}$$

where $\tilde{F}$ and ϕ are as in (2.8) and (2.9), respectively.

Then we have the following.

Theorem 2.1. *Let $a > 1$.*

(i) *The function $\tilde{F}$ is in $\mathcal{F}_a$, infinitely differentiable and has the following expression:*

$$\tilde{F}(u) = \exp(V(u)), \quad V(u) = \log a + \int_a^u \left(\sum_{k=0}^{\infty} \frac{1}{\prod_{j=0}^{k} F^j(t)} \right) dt. \tag{2.12}$$

Further, $\left(\frac{d}{du}\right)^k \frac{\tilde{F}'}{\tilde{F}}$ *is bounded for each* $k \in \{0\} \cup \mathbb{N}$.

(ii) *The function ϕ is in $\mathcal{F}_a$, infinitely differentiable and concave on $[a, \infty)$.*

(iii) *For each $n \in \mathbb{N}$,*

$$\lim_{u \to \infty} \frac{\phi(u)}{F^n(u)} = 0. \tag{2.13}$$

(iv) *The function L is in $\mathcal{L}$, differentiable on $(0, \infty)$, infinitely differentiable except at 1, and, concave on $[1, \infty)$, Moreover, if $a \geq 2$, then L is concave on $(0, \infty)$.*

(v) *For each $n \in \mathbb{N}$,*

$$\lim_{r \to 0} \frac{L(r)}{\log^n(1/r)} = \lim_{r \to \infty} \frac{L(r)}{\log^n r} = 0. \tag{2.14}$$

(vi) *For $r \geq \exp(a)$,*

$$L(r) \leq L(\exp(r)) \leq (1 + a)L(r).$$

We will prove the theorem above in more general form in Section 4.
By (iv) in Theorem 2.1, L is bijective from $(0, \infty)$ to $(-\infty, \infty)$.

Definition 2.5. Let $E : (-\infty, \infty) \to (0, \infty)$ be the inverse function of L.

Then by Theorem 2.1 we have the following:

Corollary 2.2. *The function E is continuous and strictly increasing and has the following properties:*

(i) $\lim_{r \to -\infty} E(r) = 0, \quad E(0) = 1, \quad \lim_{r \to \infty} E(r) = \infty.$

(ii) *The function E is convex on $[0, \infty)$, differentiable on $(-\infty, \infty)$ and infinitely differentiable except at 0. If $a \geq 2$, then E is convex on $(-\infty, \infty)$.*

(iii) $\lim_{r \to -\infty} \frac{\exp^n(-r)}{\frac{1}{E(r)}} = \lim_{r \to \infty} \frac{\exp^n(r)}{E(r)} = 0$ *for each* $n \in \mathbb{N}$.

(iv) $E(r) \leq \exp(E(r)) \leq E((1 + a)r)$ *for* $r \geq L(\exp(a))$.

Proof. (i), (ii) and (iv) follows from the theorem immediately. Since $-L(s) = L(1/s)$ and $0 \leq L(s) \leq \log^{n+1}(s)$ for large $s > 0$,

$$0 \leq \lim_{r \to -\infty} \frac{\exp^n(-r)}{\frac{1}{E(r)}} = \lim_{s \to \infty} \frac{\exp^n(L(s))}{\frac{1}{E(-L(s))}} \leq \lim_{s \to \infty} \frac{\exp^n(\log^{n+1}(s))}{s} = 0,$$

and

$$0 \leq \lim_{r \to \infty} \frac{\exp^n(r)}{E(r)} = \lim_{s \to \infty} \frac{\exp^n(L(s))}{E(L(s))} \leq \lim_{s \to \infty} \frac{\exp^n(\log^{n+1}(s))}{s} = 0.$$

These show (iii). □

3 Construction of $L^{\langle m\rangle}(r)$ To construct more slowly increasing function, we first give a simple observation. By the relation (2.2) and the definition of ϕ we have

$$(F^k(\phi(u)))' = \frac{1}{F^{k-1}(\phi(u))\cdots F^1(\phi(u))F^0(\phi(u))\tilde{F}(u)}.$$

That is,

$$F^k(\phi(u)) = a + \int_a^u \frac{dt}{F^{k-1}(\phi(t))\cdots F^1(\phi(t))F^0(\phi(t))\tilde{F}(t)}.$$

Then, as the limit of $F^k(\phi(u))$, we let

$$\phi^{\langle 2\rangle}(u) = a + \int_a^u \frac{dt}{\tilde{F}(\phi(t))\tilde{F}(t)} \quad (u \geq a).$$

Similarly, we have

$$(F^k(\phi^{\langle 2\rangle}(u)))' = \frac{1}{F^{k-1}(\phi^{\langle 2\rangle}(u))\cdots F^1(\phi^{\langle 2\rangle}(u))F^0(\phi^{\langle 2\rangle}(u))\tilde{F}(\phi(u))\tilde{F}(u)},$$

and

$$F^k(\phi^{\langle 2\rangle}(u)) = a + \int_a^u \frac{dt}{F^{k-1}(\phi^{\langle 2\rangle}(t))\cdots F^1(\phi^{\langle 2\rangle}(t))F^0(\phi^{\langle 2\rangle}(t))\tilde{F}(\phi(t))\tilde{F}(t)}.$$

So we define $\phi^{\langle m\rangle}$ and $L^{\langle m\rangle}$ as the following:

Definition 3.1. For $a > 1$ and $m \in \mathbb{N}$, let

$$\phi^{\langle m\rangle}(u) = \phi_a^{\langle m\rangle}(u) = a + \int_a^u \frac{dt}{\prod_{j=0}^{m-1} \tilde{F}(\phi^{\langle j\rangle}(t))} \quad (u \geq a), \tag{3.1}$$

where $\phi^{\langle 0\rangle}(u) = u$ and $\tilde{F}$ is as in (2.8).

Note that $\phi^{\langle 1\rangle}$ is the same as ϕ defined by (2.9).

Definition 3.2. For $a > 1$ and $m \in \mathbb{N}$, let

$$L^{\langle m\rangle}(r) = L_a^{\langle m\rangle}(r) = \phi^{\langle m\rangle}(ar) - a = \int_a^{ar} \frac{dt}{\prod_{j=0}^{m-1} \tilde{F}(\phi^{\langle j\rangle}(t))} \quad (r \geq 1),$$

and let

$$L^{\langle m\rangle}(r) = -L^{\langle m\rangle}(1/r) = -\int_a^{a/r} \frac{dt}{\prod_{j=0}^{m-1} \tilde{F}(\phi^{\langle j\rangle}(t))} \quad (0 < r < 1),$$

where $\tilde{F}$ and $\phi^{\langle m\rangle}$ are as in (2.8) and (3.1), respectively.

Proposition 3.1. *The function $\phi^{\langle m\rangle}$ coincides with ϕ^m, $m \in \mathbb{N}$, and there exists a positive constant c such that, for large r,*

$$L^{\langle m+1\rangle}(r) \leq L(L^{\langle m\rangle}(r)) \leq cL^{\langle m+1\rangle}(r). \tag{3.2}$$

Proof. Using the relation $(\phi^{\langle m\rangle}(t))' = \dfrac{1}{\prod_{j=0}^{m-1}\tilde F(\phi^{\langle j\rangle}(t))}$, we have

$$\begin{aligned}\phi^{\langle m+1\rangle}(u)-a &= \int_a^u \frac{dt}{\prod_{j=0}^{m}\tilde F(\phi^{\langle j\rangle}(t))} = \int_a^u \frac{(\phi^{\langle m\rangle}(t))'}{\tilde F(\phi^{\langle m\rangle}(t))}\,dt = \int_a^{\phi^{\langle m\rangle}(u)} \frac{ds}{\tilde F(s)} \\ &= \phi(\phi^{\langle m\rangle}(u)) - a.\end{aligned}$$

This shows that $\phi^{\langle m\rangle} = \phi^m$. Moreover, the equality $\phi^{\langle m+1\rangle}(u) = \phi(\phi^{\langle m\rangle}(u))$ means $L^{\langle m+1\rangle}(r) = L(1+L^{\langle m\rangle}(r)/a)$. By the increasingness and the concavity of L, we have (3.2). □

By Proposition 3.1 and (v) in Theorem 2.1 we have the following:

Corollary 3.2. *For each $m\in\mathbb{N}$, the function $\phi^{\langle m\rangle}$ is in $\mathcal{F}_a$, infinitely differentiable and concave on $[a,\infty)$. The function $L^{\langle m\rangle}$ is in $\mathcal{L}$, differentiable on $(0,\infty)$, infinitely differentiable except at 1, and, concave on $[1,\infty)$. Moreover, if $a\ge a_m$, then $L^{\langle m\rangle}$ is concave on $(0,\infty)$, where a_m is in $[2,2+\sqrt{2})$ and satisfies the equation*

$$\frac{{a_m}^2}{(a_m-1)^2}\left(1-\frac{1}{{a_m}^m}\right)=2.$$

For each $m,n\in\mathbb{N}$,

$$\lim_{r\to 0}\frac{L^{\langle m+1\rangle}(r)}{\log^n L^{\langle m\rangle}(1/r)} = \lim_{r\to\infty}\frac{L^{\langle m+1\rangle}(r)}{\log^n L^{\langle m\rangle}(r)} = 0. \tag{3.3}$$

We will prove the corollary above in Section 4.

Since $L^{\langle m\rangle}\in\mathcal{L}$, $L^{\langle m\rangle}$ is bijective from $(0,\infty)$ to $(-\infty,\infty)$.

Definition 3.3. For $m\in\mathbb{N}$, let $E^{\langle m\rangle}:(-\infty,\infty)\to(0,\infty)$ be the inverse function of $L^{\langle m\rangle}$.

Then by Proposition 3.1 and Corollary 3.2 we have the following:

Corollary 3.3. *For each $m\in\mathbb{N}$, the function $E^{\langle m\rangle}$ is continuous and strictly increasing and has the following properties:*

(i) $\lim_{r\to-\infty} E^{\langle m\rangle}(r)=0, \quad E^{\langle m\rangle}(0)=1, \quad \lim_{r\to\infty} E^{\langle m\rangle}(r)=\infty.$

(ii) *The function $E^{\langle m\rangle}$ is convex on $[0,\infty)$, differentiable on $(-\infty,\infty)$ and infinitely differentiable except at 0. If $a\ge a_m$, then $E^{\langle m\rangle}$ is convex on $(-\infty,\infty)$.*

(iii) $\displaystyle\lim_{r\to-\infty}\frac{E^{\langle m\rangle}(\exp^n(-r))}{\frac{1}{E^{\langle m+1\rangle}(r)}} = \lim_{r\to\infty}\frac{E^{\langle m\rangle}(\exp^n(r))}{E^{\langle m+1\rangle}(r)} = 0$ *for each $n\in\mathbb{N}$.*

(iv) *There exists a positive constant c such that, for large r, $E^{\langle m\rangle}(E(r))\le E^{\langle m+1\rangle}(r)\le E^{\langle m\rangle}(E(cr))$.*

4 Method of construction of slowly increasing functions

To construct the limit function of $L^{\langle m\rangle}$ as $m\to\infty$, we extend Theorem 2.1 to general form. First, we set, for $f\in\mathcal{F}_a$,

$$L_f(r)=\int_a^{ar}\frac{dt}{f(t)}\quad(r\ge 1),\qquad L_f(r)=-\int_a^{a/r}\frac{dt}{f(t)}\quad(0<r<1). \tag{4.1}$$

Theorem 4.1. *For $a > 1$, let f and g be in $\mathcal{F}_a$ and satisfy the relation*

$$f(u) = a + \int_a^u \frac{dt}{g(t)}. \tag{4.2}$$

Assume that g is infinitely differentiable and that $\left(\frac{d}{du}\right)^k \frac{g'}{g}$ is bounded for each $k \in \{0\} \cup \mathbb{N}$. Let

$$h(u) = a \prod_{k=0}^{\infty} \frac{g(f^k(u))}{a}, \quad \varphi(u) = a + \int_a^u \frac{dt}{h(t)} \quad (u \geq a). \tag{4.3}$$

Then we have the following:

(i) *The function h is in $\mathcal{F}_a$, infinitely differentiable and has the following expression:*

$$h(u) = \exp(v(u)), \quad v(u) = \log a + \int_a^u \left(\sum_{k=0}^{\infty} \frac{g'(f^k(t))}{\prod_{j=0}^{k} g(f^j(u))} \right) dt. \tag{4.4}$$

Further, $\left(\frac{d}{du}\right)^k \frac{h'}{h}$ is bounded for each $k \in \{0\} \cup \mathbb{N}$.

(ii) *The function φ is in $\mathcal{F}_a$, infinitely differentiable and concave on $[a, \infty)$.*

(iii) *For each $n \in \mathbb{N}$,*

$$\lim_{u \to \infty} \frac{\varphi(u)}{f^n(u)} = 0. \tag{4.5}$$

(iv) *The function L_h is in $\mathcal{L}$, differentiable on $(0, \infty)$, infinitely differentiable except at 1, and, concave on $[1, \infty)$. Moreover, if $uv'(u) \leq 2$ $(u \geq a)$, then L_h is concave on $(0, \infty)$.*

(v) *Let $g_n(u) = \prod_{j=0}^{n-1} g(f^j(u))$, $n \in \mathbb{N}$. Then L_{g_n} is in $\mathcal{L}$ for each $n \in \mathbb{N}$ and*

$$\lim_{r \to 0} \frac{L_h(r)}{L_{g_n}(r)} = \lim_{r \to \infty} \frac{L_h(r)}{L_{g_n}(r)} = 0. \tag{4.6}$$

(vi) *Let L_g^{-1} be the inverse function of L_g. Then, for $r \geq L_g^{-1}(a)$,*

$$L_h(r) \leq L_h(L_g^{-1}(r)) \leq (1 + a)L_h(r). \tag{4.7}$$

Proof of (i). We first prove that the infinite product in (4.3) converges and that h has the expression (4.4). From the relation (4.2) it follows that

$$(f^k(u))' = f'(f^{k-1}(u))(f^{k-1}(u))' = \frac{(f^{k-1}(u))'}{g(f^{k-1}(u))}.$$

Then we have the relation

$$(f^k(u))' = \frac{1}{\prod_{j=0}^{k-1} g(f^j(u))} \quad (u \geq a), \quad k \in \mathbb{N}. \tag{4.8}$$

Let

$$v_n(u) = \log\left(a\prod_{k=0}^{n}\frac{g(f^k(u))}{a}\right) = \log a + \sum_{k=0}^{n}\log\frac{g(f^k(u))}{a}.$$

Then

$$\begin{aligned} v_n(u) &= v_n(a) + \int_a^u v_n'(t)\,dt \\ &= \log a + \int_a^u \left(\sum_{k=0}^{n}\log\frac{g(f^k(t))}{a}\right)' dt \\ &= \log a + \int_a^u \left(\sum_{k=0}^{n}\frac{g'(f^k(t))}{\prod_{j=0}^{k} g(f^j(t))}\right) dt, \end{aligned}$$

where we use the relation (4.8). Since $g(u) \ge a$ and $0 \le g'(u)/g(u) \le c_g$ for some positive constant c_g,

$$\sum_{k=0}^{n}\frac{g'(f^k(t))}{\prod_{j=0}^{k} g(f^j(t))} \le c_g \sum_{k=0}^{n}\frac{1}{a^k} \quad (t \ge a).$$

Then the sum converges uniformly and the limit function $v(u)$ exists such that

$$v(u) = \lim_{n\to\infty} v_n(u) = \log a + \int_a^u \left(\sum_{k=0}^{\infty}\frac{g'(f^k(t))}{\prod_{j=0}^{k} g(f^j(t))}\right) dt.$$

This shows that v is continuous and strictly increasing and that the infinite product in (4.3) converges to $\exp(v(u))$. That is, $h(u) = \exp(v(u))$ which is also continuous and strictly increasing. Further, h is bijective from $[a,\infty)$ to itself, since

$$h(u) = g(u) \times \sum_{k=1}^{\infty}\frac{g(f^k(u))}{a} \ge g(u) \to \infty \quad \text{as } u \to \infty.$$

Hence $h \in \mathcal{F}_a$.

Moreover, we have

$$\frac{h'(u)}{h(u)} = v'(u) = \sum_{k=0}^{\infty}\frac{g'(f^k(u))}{\prod_{j=0}^{k} g(f^j(u))} \le c_g \sum_{k=0}^{\infty}\frac{1}{a^k} = c_g\frac{a}{a-1}.$$

Similarly, from the boundedness of $\left(\frac{d}{du}\right)^j \frac{g'}{g}$, $0 \le j \le k$, we see that $\left(\frac{d}{du}\right)^{k+1} v$ is bounded. Therefore, h is infinitely differentiable and all derivatives of h'/h is bounded. □

Proof of (ii). Since h is in $\mathcal{F}_a$ and infinitely differentiable, φ is strictly increasing and infinitely differentiable. To prove $\varphi \in \mathcal{F}_a$ we show that $\varphi(u) \to \infty$ as $u \to \infty$. Choose $u_n \in [a,\infty)$ such that $f^n(u_n) = 2a$. Then

$$\begin{aligned} \prod_{k=n}^{\infty}\frac{g(f^k(u_n))}{a} &= \frac{g(f^n(u_n))}{a} \times \frac{g(f^{n+1}(u_n))}{a} \times \frac{g(f^{n+2}(u_n))}{a} \times \cdots \\ &= \frac{g(2a)}{a} \times \frac{g(f(2a))}{a} \times \frac{g(f^2(2a))}{a} \times \cdots \\ &= \frac{h(2a)}{a} = C_a, \end{aligned}$$

which is independent of n, and, for $t \in [a, u_n]$,

$$
\begin{aligned}
h(t) &= a \prod_{k=0}^{n-1} \frac{g(f^k(t))}{a} \prod_{k=n}^{\infty} \frac{g(f^k(t))}{a} \\
&\leq a \prod_{k=0}^{n-1} \frac{g(f^k(t))}{a} \prod_{k=n}^{\infty} \frac{g(f^k(u_n))}{a} \\
&= \frac{\prod_{k=0}^{n-1} g(f^k(t))}{a^{n-1}} \times C_a.
\end{aligned}
$$

Hence, by the relation (4.8),

$$
\varphi(u_n) - a = \int_a^{u_n} \frac{dt}{h(t)} \geq \frac{a^{n-1}}{C_a} \int_a^{u_n} \frac{dt}{\prod_{k=0}^{n-1} g(f^k(t))} = \frac{a^{n-1}}{C_a}(f^n(u_n) - a) = \frac{a^n}{C_a}.
$$

for each $n \geq 1$. Combining this and the strictly increasingness of φ, we have

$$
\lim_{u \to \infty} \varphi(u) = \infty.
$$

From the expression (4.4) it follows that

$$
\varphi'(u) = \frac{1}{h(u)} = \exp(-v(u)) > 0, \quad \varphi''(u) = -v'(u)\exp(-v(u)) < 0.
$$

Hence φ is concave. □

Proof of (iii). For $t \in [a, \infty)$,

$$
h(t) = a \prod_{k=0}^{\infty} \frac{g(f^k(t))}{a} \geq a \prod_{k=0}^{n} \frac{g(f^k(t))}{a} = \frac{1}{a^n} \prod_{k=0}^{n} g(f^k(t)),
$$

and

$$
\varphi(u) - a = \int_a^u \frac{dt}{h(t)} \leq a^n \int_a^u \frac{dt}{\prod_{k=0}^{n} g(f^k(t))} = a^n(f^{n+1}(u) - a).
$$

That is, $0 < \varphi(u)/f^{n+1}(u) \leq 2a^n$ for large u. From the relation (4.2) it follows that

$$
0 < \frac{f(u)}{u} = \frac{a}{u} + \frac{1}{u} \int_a^u \frac{dt}{g(t)} \to 0 \quad \text{as } u \to \infty.
$$

Hence $f^{n+1}(u)/f^n(u) \to 0$ as $u \to \infty$. Therefore, we have (4.5). □

Proof of (iv). Since $L_h(r) = \varphi(ar) - a$ $(r \geq 1)$, the result in (ii) implies $L_h \in \mathcal{L}$, the concavity of L_h on $[1, \infty)$, and infinitely differentiability on $(0,1) \cup (1, \infty)$. Moreover, from

$$
\lim_{r \to 1-0} L_h'(r) = \lim_{r \to 1+0} L_h'(r) = 1,
$$

it follows that L_h is differentiable on $(0, \infty)$.

If $uv'(u) \leq 2$ $(u \geq a)$, then, for $0 < r < 1$,

$$
(L_h(r))' = (-L_h(1/r))' = (-\varphi(a/r) + a)' = \frac{a\varphi'(a/r)}{r^2} = \frac{a\exp(-v(a/r))}{r^2} > 0,
$$

and

$$
(L_h(r))'' = \left(\frac{a\exp(-v(a/r))}{r^2}\right)' = a\exp(-v(a/r))\frac{(a/r)v'(a/r) - 2}{r^3} \leq 0.
$$

Therefore, L_h' is decreasing on $(0, \infty)$. That is, L_h is concave on $(0, \infty)$. □

Proof of (v). From the relation (4.8) it follows that $L_{g_n}(r) = f^n(ar) - a$ $(r \geq 1)$ and $L_{g_n}(r) = -f^n(a/r) + a$ $(0 < r < 1)$. Hence $L_{g_n}(r)$ is in $\mathcal{L}$ for each $n \in \mathbb{N}$. The property (4.6) is a direct consequence of (4.5). □

Proof of (vi). From

$$L_g(r) = \int_a^{ar} \frac{dt}{g(t)} \leq \int_a^{ar} \frac{dt}{a} \leq r$$

it follows that $r \leq L_g^{-1}(r)$ $(r \geq 1)$. Hence the first inequality holds.

Next we show the second inequality. Since $L_g(t) = f(at) - a \leq f(at)$ $(t \geq 1)$, for $L_g(t) \geq a$,

$$g(at)h(L_g(t)) \leq g(at)h(f(at)) = g(f^0(at))a \prod_{k=0}^{\infty} \frac{g(f^k(f(at)))}{a} = ah(at).$$

Observing $L_g^{-1}(r) > 1$ for $r > 0$, we have, for $r \geq L_g^{-1}(a)$,

$$\begin{aligned} L_h(L_g^{-1}(r)) &\leq L_h(L_g^{-1}(ar)) \\ &= \int_a^{aL_g^{-1}(ar)} \frac{1}{h(t)}\,dt \\ &= \int_a^{aL_g^{-1}(a)} \frac{1}{h(t)}\,dt + \int_{aL_g^{-1}(a)}^{aL_g^{-1}(ar)} \frac{1}{h(t)}\,dt \\ &= \int_a^{aL_g^{-1}(a)} \frac{1}{h(t)}\,dt + \int_{L_g^{-1}(a)}^{L_g^{-1}(ar)} \frac{a}{h(at)}\,dt \\ &\leq \int_a^{ar} \frac{1}{h(t)}\,dt + \int_{L_g^{-1}(a)}^{L_g^{-1}(ar)} \frac{a^2}{g(at)h(L_g(t))}\,dt \\ &= (1+a)\int_a^{ar} \frac{1}{h(t)}\,dt = (1+a)L_h(r). \end{aligned}$$

This is the second inequality. □

Proof of Theorem 2.1. In Theorem 4.1, if $f(u) = a - \log a + \log u$ and $g(u) = u$, then we have Theorem 2.1 immediately. Only for the concavity of L on $(0, \infty)$, we need to check that $uV'(u) \leq 2$, where V is as in (2.12). Actually,

$$uV'(u) = u \times \sum_{k=0}^{\infty} \frac{1}{\prod_{j=0}^{k} F^j(u)} \leq u \times \frac{1}{u} \sum_{k=0}^{\infty} \frac{1}{a^k} = \frac{a}{a-1} \leq 2, \quad \text{if } a \geq 2, \tag{4.9}$$

since $F^0(u) = u$ and $F^j(u) \geq a$, $j \in \mathbb{N}$. □

Remark 4.1. In (2.1) if we take $a = 1$, then $F(u) = F_1(u) = 1 + \log u$. In this case $\lim_{k\to\infty} F_1^k(u) = 1$ for all $u \geq 1$, since the graph of $y = 1 + \log x$ is concave and touches the line $y = x$ at the point $(1, 1)$ in the plane. However, the infinite product $\prod_{k=0}^{\infty} F_1^k(u)$ diverges for all $u > 1$. Actually, letting

$$V_n(u) = \log\left(\prod_{k=0}^{n} F_1^k(u)\right) = \sum_{k=0}^{n} \log F_1^k(u),$$

we have

$$V_n(u) = \int_1^u V_n'(t)\,dt = \int_1^u \sum_{k=0}^{n} \left(\log F_1^k(t)\right)'\,dt = \int_1^u \left(\sum_{k=0}^{n} \frac{1}{\prod_{j=0}^{k} F_1^j(t)}\right) dt.$$

If there exists $u > 1$ such that the product $\prod_{j=0}^{k} F_1^j(u)$ converges to some constant $c_u \geq 1$, then it also converges to some constant $c_t \in [1, c_u]$ for $t \in [1, u]$. This implies that the sum in the integral sign diverges for $t \in [1, u]$ and that $V_n(u)$ diverges, which contradicts the convergence of the product.

Proof of Corollary 3.2. By Proposition 3.1 we have $\phi^{\langle m\rangle} = \phi^m$. So the properties of $\phi^{\langle m\rangle}$ and $L^{\langle m\rangle}$ follow from the property of $\phi \in \mathcal{F}_a$ except for the concavity of $L^{\langle m\rangle}$ on $(0, \infty)$. To check the concavity, we note that

$$\frac{(\phi^m)''(u)}{(\phi^m)'(u)} = -\sum_{k=0}^{m-1} \frac{\tilde{F}'(\phi^k(u))}{\prod_{j=0}^{k} \tilde{F}(\phi^j(u))},$$

and

$$\frac{\tilde{F}'(u)}{\tilde{F}(u)} = V'(u) = \sum_{k=0}^{\infty} \frac{1}{\prod_{j=0}^{k} F^j(u)} \leq \frac{1}{u} \sum_{k=0}^{\infty} \frac{1}{a^j} \leq \frac{1}{a-1}. \tag{4.10}$$

Then we can show

$$u \times \sum_{k=0}^{m-1} \frac{\tilde{F}'(\phi^k(u))}{\prod_{j=0}^{k} \tilde{F}(\phi^j(u))} \leq \frac{a^2}{(a-1)^2}\left(1 - \frac{1}{a^m}\right). \tag{4.11}$$

Actually, using (4.9), (4.10) and $\tilde{F}(u) = u \times \prod_{k=1}^{\infty} \frac{F^k(u)}{a} \geq u$, we have

$$\begin{aligned}
&u \times \sum_{k=0}^{m-1} \frac{\tilde{F}'(\phi^k(u))}{\prod_{j=0}^{k} \tilde{F}(\phi^j(u))} \\
&= \frac{u\tilde{F}'(u)}{\tilde{F}(u)} + \frac{u}{\tilde{F}(u)} \frac{\tilde{F}'(\phi(u))}{\tilde{F}(\phi(u))} + \frac{u}{\tilde{F}(u)} \sum_{k=2}^{m-1} \frac{\tilde{F}'(\phi^k(u))}{\tilde{F}(\phi^k(u)) \prod_{j=1}^{k-1} \tilde{F}(\phi^j(u))} \\
&\leq uV'(u) + V'(\phi(u)) + \sum_{k=2}^{m-1} V'(\phi^k(u)) \frac{1}{a^{k-1}} \\
&\leq \frac{a}{a-1} + \frac{1}{a-1} + \sum_{k=2}^{m-1} \frac{1}{a-1} \frac{1}{a^{k-1}} = \frac{a^2}{(a-1)^2}\left(1 - \frac{1}{a^m}\right).
\end{aligned}$$

Then, for $0 < r < 1$,

$$(L^{\langle m\rangle}(r))' = (-L^{\langle m\rangle}(1/r))' = \frac{a(\phi^m)'(a/r)}{r^2} > 0,$$

and

$$\begin{aligned}
(L^{\langle m\rangle}(r))'' &= \left(\frac{a(\phi^m)'(a/r)}{r^2}\right)' \\
&= \frac{a(\phi^m)'(a/r)}{r^3} \left(\frac{a}{r} \sum_{k=0}^{m-1} \frac{\tilde{F}'(\phi^k(a/r))}{\prod_{j=0}^{k} \tilde{F}(\phi^j(a/r))} - 2\right) \\
&\leq \frac{a(\phi^m)'(a/r)}{r^3} \left(\frac{a^2}{(a-1)^2}\left(1 - \frac{1}{a^m}\right) - 2\right) \leq 0,
\end{aligned}$$

if $a \geq a_m$. This shows the concavity on $(0,1)$, and hence the concavity on $(0,\infty)$ because of the differentiability at 1 and the concavity on $(0,1)\cup(1,\infty)$. Finally, the relation (3.3) follows from (2.14) and (3.2). □

5 Construction of $L^{\langle \ell,m\rangle}(r)$ In this section, by using Theorem 4.1, we construct more slowly increasing functions.

Definition 5.1. For $a>1$, let

$$\tilde{F}^{\langle ii\rangle}(u) = a\prod_{m=0}^{\infty}\frac{\tilde{F}(\phi^m(u))}{a} \quad (u \geq a), \tag{5.1}$$

and let

$$\phi^{\langle 1,1\rangle}(u) = \phi_a^{\langle 1,1\rangle}(u) = a + \int_a^u \frac{dt}{\tilde{F}^{\langle ii\rangle}(t)} \quad (u \geq a), \tag{5.2}$$

where $\tilde{F}$ and ϕ are as in (2.8) and (2.9), respectively.

Definition 5.2. For $a>1$, let

$$L^{\langle 1,1\rangle}(r) = L_a^{\langle 1,1\rangle}(r) = \phi^{\langle 1,1\rangle}(ar) - a = \int_a^{ar}\frac{dt}{\tilde{F}^{\langle ii\rangle}(t)} \quad (r \geq 1),$$

and let

$$L^{\langle 1,1\rangle}(r) = -L^{\langle 1,1\rangle}(1/r) = -\int_a^{a/r}\frac{dt}{\tilde{F}^{\langle ii\rangle}(t)} \quad (0<r<1),$$

where $\tilde{F}^{\langle ii\rangle}$ and $\phi^{\langle 1,1\rangle}$ are as in (5.1) and (5.2), respectively.

Then we have the following:

Theorem 5.1. *Let $a>1$.*

(i) *The function $\tilde{F}^{\langle ii\rangle}$ is in $\mathcal{F}_a$, infinitely differentiable, and has the following expression:*

$$\tilde{F}^{\langle ii\rangle}(u) = \exp(V^{\langle ii\rangle}(u)), \quad V^{\langle ii\rangle}(u) = \log a + \int_a^u \left(\sum_{k=0}^{\infty}\frac{\tilde{F}'(\phi^k(t))}{\prod_{j=0}^{k}\tilde{F}(\phi^j(t))}\right) dt.$$

Further, $\left(\frac{d}{du}\right)^k \frac{(\tilde{F}^{\langle ii\rangle})'}{\tilde{F}^{\langle ii\rangle}}$ *is bounded for each $k\in\{0\}\cup\mathbb{N}$.*

(ii) *The function $\phi^{\langle 1,1\rangle}$ is in $\mathcal{F}_a$, infinitely differentiable and concave on $[a,\infty)$.*

(iii) *For each $n\in\mathbb{N}$,*

$$\lim_{u\to\infty}\frac{\phi^{\langle 1,1\rangle}(u)}{\phi^n(u)} = 0.$$

(iv) *The function $L^{\langle 1,1\rangle}$ is in $\mathcal{L}$, differentiable on $(0,\infty)$, infinitely differentiable except at 1, and, concave on $[1,\infty)$. Moreover, if $a\geq 2+\sqrt{2}$, then $L^{\langle 1,1\rangle}$ is concave on $(0,\infty)$.*

(v) *For each* $n \in \mathbb{N}$,

$$\lim_{r\to 0} \frac{L^{\langle 1,1\rangle}(r)}{L^{\langle n\rangle}(r)} = \lim_{r\to\infty} \frac{L^{\langle 1,1\rangle}(r)}{L^{\langle n\rangle}(r)} = 0. \tag{5.3}$$

(vi) *For* $r \geq E(a)$,

$$L^{\langle 1,1\rangle}(r) \leq L^{\langle 1,1\rangle}(E(r)) \leq (1+a)L^{\langle 1,1\rangle}(r).$$

Proof. By the definition (2.9) and Theorem 2.1 the assumptions in Theorem 4.1 hold with $h = \tilde{F}$ and $\varphi = \phi^{\langle 1,1\rangle}$. Therefore, we have the conclusion except for the concavity of $L^{\langle 1,1\rangle}$ on $(0,\infty)$. Using the inequality (4.11), we have

$$u(V^{\langle ii\rangle}(u))' = u \times \sum_{k=0}^{\infty} \frac{\tilde{F}'(\phi^k(u))}{\prod_{j=0}^{k} \tilde{F}(\phi^j(u))} \leq \frac{a^2}{(a-1)^2} \leq 2,$$

if $a \geq 2+\sqrt{2}$. Therefore, we have also the concavity. $\square$

Next, observing

$$(F^k(\phi^{\langle 1,1\rangle}(u)))' = \frac{1}{F^{k-1}(\phi^{\langle 1,1\rangle}(u))\cdots F^1(\phi^{\langle 1,1\rangle}(u))F^0(\phi^{\langle 1,1\rangle}(u))\tilde{F}^{\langle ii\rangle}(u)},$$

we let

$$\phi^{\langle 1,2\rangle}(u) = a + \int_a^u \frac{1}{\tilde{F}(\phi^{\langle 1,1\rangle}(t))\tilde{F}^{\langle ii\rangle}(t)}\,dt.$$

In general, for $m \geq 2$, let

$$\phi^{\langle 1,m\rangle}(u) = \phi_a^{\langle 1,m\rangle}(u) = a + \int_a^u \frac{dt}{\left(\prod_{j=1}^{m-1} \tilde{F}(\phi^{\langle 1,j\rangle}(t))\right)\tilde{F}^{\langle ii\rangle}(t)} \quad (u \geq a).$$

Here, in the same way as Proposition 3.1, we have $\phi^{\langle 1,m+1\rangle}(u) = \phi(\phi^{\langle 1,m\rangle}(u))$. That is,

$$\prod_{j=1}^{m-1} \tilde{F}(\phi^{\langle 1,j\rangle}(t)) = \prod_{j=1}^{m-1} \tilde{F}(\phi^{j-1}(\phi^{\langle 1,1\rangle}(t))) = \prod_{j=0}^{m-2} \tilde{F}(\phi^{j}(\phi^{\langle 1,1\rangle}(t))).$$

Then, we let

$$\phi^{\langle 2,1\rangle}(u) = \phi_a^{\langle 2,1\rangle}(u) = a + \int_a^u \frac{dt}{\tilde{F}^{\langle ii\rangle}(\phi^{\langle 1,1\rangle}(t))\tilde{F}^{\langle ii\rangle}(t)} \quad (u \geq a).$$

Further, in general, we have

$$\phi^{\langle \ell,m+1\rangle}(u) = \phi(\phi^{\langle \ell,m\rangle}(u)), \quad \ell, m \in \mathbb{N}, \tag{5.4}$$

in the same way as Proposition 3.1. So we give the following definition.

Definition 5.3. For $a > 1$ and $\ell \in \mathbb{N}$, let

$$\phi^{\langle \ell,1\rangle}(u) = \phi_a^{\langle \ell,1\rangle}(u) = a + \int_a^u \frac{dt}{\prod_{j=0}^{\ell-1} \tilde{F}^{\langle ii\rangle}(\phi^{\langle j,1\rangle}(t))} \quad (u \geq a),$$

where $\phi^{\langle 0,1\rangle}(u) = u$. For $m \in \mathbb{N}$ with $m \geq 2$,

$$\begin{aligned}\phi^{\langle \ell,m\rangle}(u) &= \phi_a^{\langle \ell,m\rangle}(u)\\ &= a + \int_a^u \frac{dt}{\left(\prod_{k=1}^{m-1} \tilde{F}(\phi^{\langle \ell,k\rangle}(t))\right)\left(\prod_{j=0}^{\ell-1} \tilde{F}^{\langle ii\rangle}(\phi^{\langle j,1\rangle}(t))\right)} \quad (u \geq a).\end{aligned}$$

Definition 5.4. For $a > 1$ and $\ell, m \in \mathbb{N}$, let

$$L^{\langle \ell,m\rangle}(r) = L_a^{\langle \ell,m\rangle}(r) = \phi^{\langle \ell,m\rangle}(ar) - a \quad (r \ge 1),$$

and let

$$L^{\langle \ell,m\rangle}(r) = -L^{\langle \ell,m\rangle}(1/r) \quad (0 < r < 1).$$

Moreover, in the same way as Proposition 3.1 again, we have

$$\phi^{\langle 1,1\rangle}(\phi^{\langle \ell,1\rangle}(u)) = \phi^{\langle \ell+1,1\rangle}(u), \quad \ell \in \mathbb{N}. \tag{5.5}$$

By this property and (5.4) we see that $\phi^{\langle \ell,m\rangle} \in \mathcal{F}_a$ and $L^{\langle \ell,m\rangle} \in \mathcal{L}$ for each $\ell, m \in \mathbb{N}$, and that there exists a positive constant c such that, for large r,

$$\begin{aligned} L^{\langle \ell+1,1\rangle}(r) &\le L^{\langle 1,1\rangle}(L^{\langle \ell,1\rangle}(r)) \le cL^{\langle \ell+1,1\rangle}(r), \\ L^{\langle \ell,m+1\rangle}(r) &\le L(L^{\langle \ell,m\rangle}(r)) \le cL^{\langle \ell,m+1\rangle}(r). \end{aligned}$$

Combining these inequalities and the relations (2.14) and (5.3), we have the following.

Corollary 5.2. *For each $\ell, m \in \mathbb{N}$,*

$$\lim_{r\to 0} \frac{L^{\langle \ell,1\rangle}(r)}{L^{\langle \ell-1,m\rangle}(1/r)} = \lim_{r\to\infty} \frac{L^{\langle \ell,1\rangle}(r)}{L^{\langle \ell-1,m\rangle}(r)} = 0,$$

and, for each $\ell, m, n \in \mathbb{N}$

$$\lim_{r\to 0} \frac{L^{\langle \ell,m+1\rangle}(r)}{\log^n L^{\langle \ell,m\rangle}(1/r)} = \lim_{r\to\infty} \frac{L^{\langle \ell,m+1\rangle}(r)}{\log^n L^{\langle \ell,m\rangle}(r)} = 0.$$

Let $\psi(u) = \phi^{\langle 1,1\rangle}(u)$. Then $\phi^{\langle \ell,1\rangle}(u) = \psi^\ell(u)$ by (5.5). Using this relation, we give the following definition.

Definition 5.5. For $a > 1$, let

$$\tilde{F}^{\langle iii\rangle}(u) = a \prod_{m=0}^{\infty} \frac{\tilde{F}^{\langle ii\rangle}(\psi^m(u))}{a} \quad (u \ge a),$$

and let

$$\phi^{\langle 1,1,1\rangle}(u) = \phi_a^{\langle 1,1,1\rangle}(u) = a + \int_a^u \frac{dt}{\tilde{F}^{\langle iii\rangle}(t)} \quad (u \ge a).$$

Definition 5.6. For $a > 1$, let

$$L^{\langle 1,1,1\rangle}(r) = L_a^{\langle 1,1,1\rangle}(r) = \phi^{\langle 1,1,1\rangle}(ar) - a = \int_a^{ar} \frac{dt}{\tilde{F}^{\langle iii\rangle}(t)} \quad (r \ge 1),$$

and let

$$L^{\langle 1,1,1\rangle}(r) = -L^{\langle 1,1,1\rangle}(1/r) = -\int_a^{a/r} \frac{dt}{\tilde{F}^{\langle iii\rangle}(t)} \quad (0 < r < 1).$$

In this way, we can construct more and more slowly increasing functions such that $L^{\langle k,\ell,m\rangle}$, $L^{\langle 1,1,1,1\rangle}$, $L^{\langle j,k,\ell,m\rangle}$, and so on.

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