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ON 3-VARIABLE EXTENSION FOR THE INTEGER MEAN

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Abstract. Based on ideas of Lawson-Lim [4] and Jung-Lee-Yamazaki [2], an abstract mean on a metric space was introduced in [5]. In this paper, we discuss a typical example of such a mean on the nonnegative integers and estimate it by the usual mean.

1 Introduction. Since [1] was published, multi-variable geometric operator means have been discussed. Among them, Lawson-Lim [4] introduced abstract means on a metric space. On the other hand,computable geometric operator means was introduced in [2]. In [5], we combined these means and introduced N-means ν on a metric space, particularly as N-variable extension $\nu = \mu^{(N)}$: Let μ is an abstract 2-mean in a metric space X. For a given N-tuple $(x_1, \cdots, x_N)$, let $a_k = a_k^{(0)} = x_k$ and

$$a_k^{(n)} = \mu(a_k^{(n-1)}, a_{k+1}^{(n-1)}) \text{ for } 1 \leqq k \leqq N-1 \quad \text{and} \quad a_N^{(n)} = \mu(a_N^{(n-1)}, a_1^{(n-1)}).$$

If $\lim_{n\to\infty} a_k^{(n)}$ exists for all k and they coincide, say $a^{(\infty)}$, then we put $\nu(x_1, \cdots, x_N) = a^{(\infty)}$, which is the N-variable extension $\mu^{(N)}$ of μ.

Here we give an example of the 3-variable extension $\mu^{(3)}$ for the *integer mean* $\mu : \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{N}_0$

$$\mu(n, m) = \left\lfloor \frac{n+m}{2} \right\rfloor$$

where $\mathbb{N}_0$ is the nonnegative integers and $\lfloor\ \rfloor$ is the Gauss symbol and the metric is the usual one induced by the absolute value. Then, since the total distance

$$d_3(a_1^{(n)}, a_2^{(n)}, a_3^{(n)}) = \max\left\{|a_1^{(n)} - a_2^{(n)}|, |a_2^{(n)} - a_3^{(n)}|, |a_3^{(n)} - a_1^{(n)}|\right\}$$

is monotone decreasing for n, it has the 3-variable extension $\mu^{(3)}$.

By the construction of $\mu^{(3)}$, it is clear that $\mu^{(3)}(a, b, c) \leqq \frac{a+b+c}{3}$. As we show later, the equality holds if and only if $a = b = c$. In this paper, we observe $\mu^{(3)}(a, b, c)$ in detail and estimate it by $\frac{a+b+c}{3}$.

2 Basic value of $\mu^{(3)}$. Let $f(n_1, n_2, n_3) = \mu^{(3)}(n_1, n_2, n_3)$. Then, f is a monotone nondecreasing function. Since this 3-mean is symmetric, we may express $f(n_1, n_2, n_3)$ for the case $n_1 \leqq n_2 \leqq n_3$ and

$$\begin{aligned} f(n_1, n_2, n_3) &= f\left(\left\lfloor \frac{n_1+n_2}{2} \right\rfloor, \left\lfloor \frac{n_2+n_3}{2} \right\rfloor, \left\lfloor \frac{n_3+n_1}{2} \right\rfloor\right) \\ &= f\left(\left\lfloor \frac{n_1+n_2}{2} \right\rfloor, \left\lfloor \frac{n_3+n_1}{2} \right\rfloor, \left\lfloor \frac{n_2+n_3}{2} \right\rfloor\right). \end{aligned}$$

Moreover the following simple lemma allows us that n_1 can be assumed to be 0:

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Lemma 1. $f(n+n_1, n+n_2, n+n_3) = n + f(n_1, n_2, n_3)$.

Proof. The required formula follows from the fact that

$$\mu(n+n_k, n+n_j) = \left\lfloor n + \frac{n_k+n_j}{2} \right\rfloor = n + \left\lfloor \frac{n_k+n_j}{2} \right\rfloor = n + \mu(n_k, n_j)$$

where $j = k \bmod 3 + 1$. □

First we show the case $f(n_1,n_2,n_3) = f(0, 2^m, 2^m)$, which increases at a rate of about 2/3. Here the operation ⫽ means the quotient for integers:

Lemma 2. $f(0, 2^{2k}, 2^{2k}) = \dfrac{2^{2k+1}-2}{3}$ *and* $f(0, 2^{2k+1}, 2^{2k+1}) = \dfrac{2^{2k+2}-1}{3}$ *for* $k \geqq 0$*, or equivalently* $f(0, 2^m, 2^m) = (2^m \times 2) /\!\!/ 3$ *for all* $m \geqq 0$.

Proof. Note that $f(0,1,1) = 0$. For $n = 2^{2k}$ for $k \geqq 1$,

$$\begin{aligned} f(0, 2^{2k}, 2^{2k}) &= f(2^{2k-1}, 2^{2k}, 2^{2k-1}) = 2^{2k-1} + f(0, 2^{2k-1}, 0) = 2^{2k-1} + f(0, 0, 2^{2k-1}) \\ &= 2^{2k-1} + f(0, 2^{2k-2}, 2^{2k-2}) \\ &\quad \cdots = 2^{2k-1} + \cdots + 2 + f(0,1,1) = 2 \times \frac{2^{2k}-1}{3} = \frac{2^{2k+1}-2}{3}. \end{aligned}$$

Thus it also holds for $k = 0$. Lastly for $n = 2^{2k+1}$ for $k \geqq 0$, by $f(0,2,2) = 1$ we have

$$\begin{aligned} f(0, 2^{2k+1}, 2^{2k+1}) &= f(2^{2k}, 2^{2k+1}, 2^{2k}) = f(2^{2k-1} + 2^{2k}, 2^{2k-1} + 2^{2k}, 2^{2k}) \\ &= 2^{2k} + f(0, 2^{2k-1}, 2^{2k-1}) \\ &\quad \cdots = 2^{2k} + \cdots + 4 + f(0,2,2) = 4 \times \frac{2^{2k}-1}{3} + 1 = \frac{2^{2k+2}-1}{3}. \end{aligned}$$

Moreover we have $f(0, 2^m, 2^m) = \frac{2^{m+1}-2+(m \bmod 2)}{3} = (2^m \times 2) /\!\!/ 3$. □

By Lemma 2, $f(0,0,2^m)$ increases at about 1/3 rate:

Corollary 3. *For* $k \geqq 0$*,* $f(0,0,2^{2k}) = \frac{2^{2k}-1}{3}$ *and* $f(0,0,2^{2k+1}) = \frac{2^{2k+1}-2}{3}$*, or equivalently* $f(0,0,2^m) = \frac{2^m-1-(m \bmod 2)}{3} = 2^m /\!\!/ 3$ *for all* $m \geqq 0$.

Proof. For $k \geqq 0$, we have $f(0,0,2^{2k}) = f(0, 2^{2k-1}, 2^{2k-1}) = \frac{2^{2k}-1}{3}$ and $f(0,0,2^{2k+1}) = f(0, 2^{2k}, 2^{2k}) = \frac{2^{2k+1}-2}{3}$. □

Next we observe the special case of the form $f(0,n,n)$. To see this, note that if all 3 variables are odd, then it does not increase in the next step:

Lemma 4. $f(2J, 2K, 2N) = f(2J, 2K, 2N+1)$.

Proof. Since we may assume $J = 0$, we have

$$f(0, 2K, 2N+1) - f(0, 2K, 2N) = f(K, K+N, N) - f(K, K+N, N) = 0.$$ □

Now we observe $f(0,n,n)$:

Proposition 5. *The function $f(0,n,n)$ strictly increases only at*

$$n = 2^{2\ell+1}k - \frac{4^\ell - 1}{3} > 0$$

for $\ell \geqq 0$, $k > 0$. Precisely,

$$f(0,n,n) = \sum_{\ell=0}^{L} \left(n + \frac{4^\ell - 1}{3}\right) /\!\!/ 2^{2\ell+1} = \sum_{\ell=0}^{L} \left(2n + \frac{2(4^\ell - 1)}{3}\right) /\!\!/ 4^{\ell+1}$$

where $L = L(n) = \left\lfloor \log_4 \frac{3n-1}{5} \right\rfloor$.

Proof. Put $n_\ell = 2^{2\ell+1}k - \frac{4^\ell-1}{3}$. At $n = n_0 = 2k$, the function $f(0,n,n)$ is increasing by

$$f(0,2k,2k) = f(k,2k,k) = 1 + f(k-1,2k-1,k-1) = 1 + f(0,2k-1,2k-1).$$

Since n_ℓ is odd for $\ell > 0$, we have

$$2n_{\ell-1} = 2^{2\ell}k - \frac{2(4^{\ell-1}-1)}{3} = \frac{2^{2\ell+1}k - \frac{4(4^{\ell-1}-1)}{3}}{2} = \frac{2^{2\ell+1}k - \frac{4^\ell-1}{3} + 1}{2} = \frac{n_\ell + 1}{2}.$$

Suppose it is increasing at $n = n_\ell$. Then

$$\begin{aligned}
f(0,n_{\ell+1},n_{\ell+1}) &= f\left(\frac{n_{\ell+1}-1}{2}, n_{\ell+1}, \frac{n_{\ell+1}-1}{2}\right) = \frac{n_{\ell+1}-1}{2} + f\left(0,0,\frac{n_{\ell+1}+1}{2}\right) \\
&= \frac{n_{\ell+1}-1}{2} + f(0,0,2n_\ell) = \frac{n_{\ell+1}-1}{2} + f(0,n_\ell,n_\ell) = 1 + \frac{n_{\ell+1}-1}{2} + f(0,n_\ell-1,n_\ell-1) \\
&= 1 + \frac{n_{\ell+1}-1}{2} + f(0,0,2n_\ell-2) = 1 + \frac{n_{\ell+1}-1}{2} + f(0,0,2n_\ell-1) \quad \text{(by Lemma 4)} \\
&= 1 + \frac{n_{\ell+1}-1}{2} + f\left(0,0,\frac{n_{\ell+1}+1}{2}-1\right) = 1 + f\left(\frac{n_{\ell+1}-1}{2}, \frac{n_{\ell+1}-1}{2}, n_{\ell+1}-1\right) \\
&= 1 + f\left(0, n_{\ell+1}-1, n_{\ell+1}-1\right).
\end{aligned}$$

Thus $f(0,n,n)$ is increasing at all $n_\ell = 2^{2\ell+1}k - \frac{4^\ell-1}{3}$. To obtain the upper bound of ℓ, we have $n \geqq 2^{2\ell+1}k - \frac{4^\ell-1}{3}$, so that

$$3n - 1 \geqq 6k \times 4^\ell - 4^\ell = (6k-1)4^\ell \geqq 5 \times 4^\ell.$$

Therefore we have $\frac{3n-1}{5} \geqq 4^\ell$, that is, $L(n) = \left\lfloor \log_4 \frac{3n-1}{5} \right\rfloor$. Since these n_ℓ satisfy that $n_\ell + \frac{4^\ell-1}{3}$ divides $2^{2\ell+1}$, we have

$$f(0,n,n) \geqq \sum_{\ell=0}^{L} \left(n + \frac{4^\ell - 1}{3}\right) /\!\!/ 2^{2\ell+1}.$$

The equality holds since it always holds for $n = 2^m$ by Lemma 2: In fact, for sufficiently large m, we have $L = (\log_2 2^m - 1) /\!\!/ 2 = (m-1) /\!\!/ 2$ and

$$m - 1 - 2L = m - 1 - 2((m-1) /\!\!/ 2) = (m-1) \bmod 2 = (m+1) \bmod 2.$$

Therefore

$$f(0,2^m,2^m) \geqq \sum_{\ell=0}^{L}\left(2^m+\frac{4^\ell-1}{3}\right) /\!\!/ 2^{2\ell+1} = \sum_{\ell=0}^{L} 2^{m-2\ell-1} = 2^{m-2L-1}\sum_{\ell=0}^{L}4^\ell$$
$$= 2^{m-2L-1}\frac{4^{L+1}-1}{3} = \frac{2^{m+1}-2^{m-1-2L}}{3} = \frac{2^{m+1}-2^{(m+1)\bmod 2}}{3} = 2^{m+1} /\!\!/ 3.$$

Thus the equality holds for 2^m for a sufficiently large m, that is, it always holds. □

Since $f(0,0,2n) = f(0,n,n)$ and $f(0,0,2n-1) = f(0,n-1,n-1) = f(0,0,2n-2)$, we have only to observe the former case:

Corollary 6. *The function $f(0,0,n)$ increases only at $n = 2\left(2^{2\ell+1}k - \frac{4^\ell-1}{3}\right) > 0$ for $\ell \geqq 0$, $k > 0$. Precisely*

$$f(0,0,n) = \sum_{\ell=0}^{L}\left(n+\frac{2(4^\ell-1)}{3}\right) /\!\!/ 4^{\ell+1}$$

where $L = L(n) = \left\lfloor \log_4 \frac{3n-2}{10}\right\rfloor$.

Proof. It suffices to obtain $L(n)$. By $n \geqq 2\left(2^{2\ell+1}k - \frac{4^\ell-1}{3}\right)$, we have

$$\frac{3n}{2} \geqq 6k\times 4^\ell - 4^\ell + 1 = (6k-1)4^\ell + 1 \geqq 5\times 4^\ell + 1.$$

It follows from $\frac{3n-2}{2} \geqq 5\times 4^\ell$ that $L(n) = \left\lfloor \log_4 \frac{3n-2}{10}\right\rfloor$. □

Remark 1. Here we consider a function

$$T_\ell(n) = \left(n+\frac{2(4^\ell-1)}{3}\right) /\!\!/ 4^{\ell+1}.$$

It plays a central role for various formulae for f. We easily have a formula: $T_{\ell+1}(4n-2) = T_\ell(n)$. Moreover, since $\frac{2(4^\ell-1)}{3} \bmod 4 = 2$ for $\ell > 0$, we have

$$T_\ell(4m+2) = T_\ell(4m+3) = T_\ell(4m+4) = T_\ell(4m+5)$$

and

$$T_\ell(4m+1) = T_\ell(4m) = T_\ell(4m-1) = T_\ell(4m-2)$$

for $\ell > 0$.

Next we observe $f(0,k,n)$ for small k:

Lemma 7. $f(0,1,n) = f(0,0,n)$.

Proof. Note that

$$f(0,1,4m) = f(0,1,4m+1) = m + f(0,0,m) = f(0,0,4m+1) = f(0,0,4m)$$
$$f(0,1,4m+2) = m + f(0,0,m+1) = f(0,0,4m+2)$$
$$f(0,1,4m+3) = m + f(0,1,m+1), \quad m + f(0,0,m+1) = f(0,0,4m+3).$$

So we have only to verify the case $f(0,1,4m+3) = f(0,0,4m+3)$, which is reduced to the equality $f(0,1,n+1) = f(0,0,n+1)$. By the above observation, we have only to verify the case $n+1 = 4m+3$.

Therefore by this reduction, it suffices to show $f(0,1,k) = f(0,0,k)$ for $k = 0,1,2,3$. Since $f(0,1,3) = 0$, we have $f(0,1,2) = f(0,0,3) = f(0,0,2) = f(0,1,1) = f(0,0,1) = f(0,0,0) = 0$, so that the required equality yields. □

Direct computation shows the following table for $f(0,k,n)$ where $n = 4m - j$:

k	$4m-3$	$4m-2$	$4m-1$	$4m$
2	$m-1+f(0,0,m-1)$	$m+f(0,0,m-1)$	$m+f(0,0,m-1)$	$m+f(0,0,m)$
3	$m-1+f(0,0,m)$	$m+f(0,0,m-1)$	$m+f(0,0,m)$	$m+f(0,0,m)$

Therefore we have the formulae for $f(0,k,n)$ for $k = 2,3$:

In the below, the upper bounds L_ℓ of ℓ are slightly varied, but they are easily obtained as in the above. So we omit the upper bounds for ℓ for the sake of convenience:

Lemma 8. $f(0,2,n) = (n+2)/\!\!/4 + \sum_{\ell \geqq 0} T_\ell(n/\!\!/4)$,

$f(0,3,n) = (n+2)/\!\!/4 + \sum_{\ell \geqq 1} (T_\ell(n+1) + T_\ell(n-1) - T_\ell(n))$.

In fact, in case $n = 4m-3$, then $n-1 = 4(m-1)$, and hence $m-1 = (n-1)/4 = (n-1)/\!\!/4 = n/\!\!/4 = (n+2)/\!\!/4$. Thereby

$$f(0,2,n) = m-1+f(0,0,m-1) = (n+2)/\!\!/4 + f(0,0,n/\!\!/4) = (n+2)/\!\!/4 + \sum_{\ell \geqq 0} T_\ell(n/\!\!/4).$$

By the above remark, we have

$$\begin{aligned} f(0,3,n) = m-1+f(0,0,m) &= (n+2)/\!\!/4 + \sum_{\ell \geqq 0} T_\ell((n-1)/\!\!/4) \\ &= (n+2)/\!\!/4 + \sum_{\ell \geqq 1} T_\ell(n+1). \end{aligned}$$

Since $T_\ell(n+1) = T_\ell(4m-2)$ and $T_\ell(n) = T_\ell(4m-3) = T_\ell(4m-4) = T_\ell(n-1)$ also by the remark, we have

$$\begin{aligned} f(0,3,n) &= (n+2)/\!\!/4 + \sum_{\ell \geqq 1} (T_\ell(n+1) + 0) \\ &= (n+2)/\!\!/4 + \sum_{\ell \geqq 1} (T_\ell(n+1) + T_\ell(n-1) - T_\ell(n)). \end{aligned}$$

3 General formulae for $\mu^{(3)}$. Under modulo 4, the values of $f(0,k,n)$ are classified as the following table:

$k \setminus n$	$4N$	$4N+1$	$4N+2$	$4N+3$
$4K$	$K+N+f(0,K,N)$	$K+N+f(0,K,N)$	$K+N+f(0,K,N+1)$	$K+N+f(0,K,N+1)$
$4K+1$	$K+N+f(0,K,N)$	$K+N+f(0,K,N)$	$K+N+f(0,K,N+1)$	$K+N+f(0,K+1,N+1)$
$4K+2$	$K+N+f(0,K+1,N)$	$K+N+f(0,K+1,N)$	$K+N+1+f(0,K,N)$	$K+N+1+f(0,K,N)$
$4K+3$	$K+N+f(0,K+1,N)$	$K+N+f(0,K+1,N+1)$	$K+N+1+f(0,K,N)$	$K+N+1+f(0,K+1,N+1)$

Summing up, we obtain the reducing formulae for $f(0,k,n)$:

Lemma 9. $f(0,k,n) = k/\!\!/4 + n/\!\!/4 + ((k \bmod 4)/\!\!/2 + (n \bmod 4)/\!\!/2)/\!\!/2$
$+ f(0, k/\!\!/4 + ((k \bmod 4)/\!\!/2 + ((n+2) \bmod 4)/\!\!/2)/\!\!/2 + (k \bmod 2 + n \bmod 4)/\!\!/4,$
$n/\!\!/4 + ((n \bmod 4)/\!\!/2 + ((k+2) \bmod 4)/\!\!/2)/\!\!/2 + (n \bmod 2 + k \bmod 4)/\!\!/4)$.

Combining Lemmas 7–9, we can obtain any value of $f(a,b,c)$ formally. For example, we have the following values:

Example 1.

$$f(0,2^{2j},n) = f(0,4^j,n) = \frac{4^j-1}{3} + \sum_{\ell \geqq 0} T(n,\ell)$$

$$f(0,2^{2j+1},n) = \frac{4^{j+1}-1}{3} + \sum_{\ell \geqq 0} T(n-2\times 4^j,\ell)$$

$$f(0,3\times 4^j,n) = 4^j - 1 + T_\ell(n+2\times 4^j, j) + \sum_{\ell=0}^{j-1} T_\ell(n) + \sum_{\ell \geqq j+1} \Big[T_\ell(n+4^j) + T_\ell(n-4^j) - T_\ell(n)\Big]$$

$$f(0,3\times 2^{2j+1},n) = 2\times 4^j + \sum_{\ell=0}^{j} T_\ell(n) + \sum_{\ell \geqq j+1} \Big[T_\ell(n-4^{j+1}) + T_\ell(n-2\times 4^{j+1}) - T_\ell(n-6\times 4^j)\Big]$$

$$f(0,2^{2j+1}+1,n) = \sum_{m=0}^{j} 4^m + \sum_{\ell \geqq 0} T_\ell(n-2^{2j+1}) + \sum_{\ell \geqq j+1} \Bigg[T_\ell\left(n - \sum_{m=0}^{j} 4^m\right)$$
$$- T_\ell\left(n-1-\sum_{m=0}^{j} 4^m\right) + T_\ell\left(n - \sum_{m=0}^{j} 4^m + 2^{2j+1}\right) - T_\ell\left(n-1-\sum_{m=0}^{j} 4^m + 2^{2j+1}\right)\Bigg]$$

$$f(0,4^j+1,n) = \sum_{m=0}^{j-1} 4^m + \left(n + \sum_{m=0}^{j} 4^m\right) /\!\!/ 4^{j+1} - \left(n-1+\sum_{m=0}^{j} 4^m\right) /\!\!/ 4^{j+1}$$
$$+ \left(n + \sum_{m=0}^{j} 4^m + 4^j\right) /\!\!/ 4^{j+1} - \left(n-1+\sum_{m=0}^{j} 4^m + 4^j\right) /\!\!/ 4^{j+1} + \sum_{\ell \geqq 0} T_\ell(n)$$
$$+ \sum_{\ell \geqq j+1} \left[T_\ell\left(n-1-\sum_{m=0}^{j-1} 4^m\right) - T_\ell\left(n - \sum_{m=0}^{j-1} 4^m\right) + T_\ell\left(n-1-\sum_{m=0}^{j} 4^m\right) - T_\ell\left(n - \sum_{m=0}^{j} 4^m\right)\right]$$

$$f(0,2^{2j+1}-1,n) = \sum_{m=1}^{j} 4^m + \sum_{\ell \geqq 0} T_\ell\left(n+1-2^{2j+1}\right)$$
$$+ \sum_{\ell \geqq j} \left[T_\ell\left(n - \sum_{m=1}^{j} 4^m\right) - T_\ell\left(n-1-\sum_{m=1}^{j} 4^m\right)\right]$$

$$f(0,4^j-1,n) = \sum_{m=1}^{j-1} 4^m + \left(n+1+\sum_{m=0}^{j-1} 4^m\right) /\!\!/ 4^j - \left(n + \sum_{m=0}^{j-1} 4^m\right) /\!\!/ 4^j$$
$$+ \sum_{\ell \geqq 0} T_\ell(n+1) + \sum_{\ell \geqq j} \left[T_\ell\left(n - \sum_{m=0}^{j-1} 4^m\right) - T_\ell\left(n+1-\sum_{m=0}^{j-1} 4^m\right)\right]$$

$$f(0,2^{2j+1}-2,n) = \sum_{m=1}^{j} 4^m + \sum_{\ell \geqq 0} T_\ell\left(n-2^{2j+1}\right)$$
$$+ \sum_{\ell \geqq j} \left[T_\ell\left(n - \sum_{m=1}^{j} 4^m\right) - T_\ell\left(n-2-\sum_{m=1}^{j} 4^m\right)\right]$$

$$f(0,4^j-2,n) = \sum_{m=1}^{j-1} 4^m + \left(n+1+\sum_{m=0}^{j-1} 4^m\right) /\!\!/ 4^j - \left(n-1+\sum_{m=0}^{j-1} 4^m\right) /\!\!/ 4^j$$
$$+ \sum_{\ell \geqq 0} T_\ell(n) + \sum_{\ell \geqq j} \left[T_\ell\left(n-1-\sum_{m=0}^{j-1} 4^m\right) - T_\ell\left(n+1-\sum_{m=0}^{j-1} 4^m\right)\right].$$

4 Estimation. Since [1] was published, multi-variable geometric operator means have been discussed. In general $f(a,b,c) \leqq \frac{a+b+c}{3}$ holds. The equality holds if and only if the variables are equal:

Theorem 10. *The equality* $\mu^{(3)}(a,b,c) \equiv f(a,b,c) = \frac{a+b+c}{3}$ *holds if and only if* $a = b = c$.

Proof. It is clear that the equation holds for $a = b = c$. Conversely suppose the equation holds. Then $f(a,b,c)$ must be equal to $f((a+b)/2,(b+c)/2,(c+a)/2)$, the parities for variables are equal. Since we may assume a is 0, then $\frac{b+c}{3} = f(0,b,c) = f(b/2,(b+c)/2,c/2)$. Thereby $b/2$ and $c/2$ must be even. Such procedure shows that $b/2^k$ and $c/2^k$ are even for all $k \in \mathbb{N}$. If $x = b$ or $x = c$ is 2^ℓ, then $x/2^\ell = 1$, which is odd. Thus b and c must be 0, so that $a = b = c$. □

Next we consider the other cases. The following result is the invariant case that the sum of variables is a constant:

Theorem 11. *For* $m, n \in \mathbb{N}$,

$$f(0,2^m,2^n) = f(0,0,2^m+2^n).$$

Proof. Since we may assume $2 \leqq m < n$ by Lemma 2, the reduction formula shows

$$\begin{aligned} f(0,2^m,2^n) &= 2^{m-2} + 2^{n-2} + f(0,2^{m-2},2^{n-2}) \\ &= \cdots = 2^{m \bmod 2}\frac{4^{m/\!\!/2}-1}{3} + 2^{n-2} + \cdots + 2^{n-2(m/\!\!/2-1)} + f(0,2^{m \bmod 2},2^{n-2(m/\!\!/2-1)}) \\ &= \cdots = 2^{m \bmod 2}\frac{4^{m/\!\!/2}-1}{3} + 2^{n \bmod 2}\frac{4^{n/\!\!/2}-1}{3} + f(0,0,2^{n \bmod 2}) \\ &= \frac{2^m - 2^{m \bmod 2} + 2^n - 2^{n \bmod 2}}{3} + 0 = \frac{2^m + 2^n - 2^{m \bmod 2} - 2^{n \bmod 2}}{3}. \end{aligned}$$

Similar procedure as $f(0,0,2^m+2^n) = 2^{m-2} + 2^{n-2} + f(0,0,2^{m-2}+2^{n-2})$ shows that it is equal to the above. □

Contrastively to Lemma 4, the following formula is a non-constant case:

Theorem 12. *For* $m, n \geqq k + 3 + (k \bmod 2)$ *and* $k \geqq 0$,

$$f(0,2^m,2^n) = f(0,2^m-2^k,2^n+2^k) + 1.$$

Proof. We may assume $m < n$. For $k = 0$, we have

$$\begin{aligned} &f(0,2^m,2^n) - f(0,2^m-1,2^n+1) \\ &\quad = f(2^{m-1},2^{m-1}+2^{n-1},2^{n-1}) - f(2^{m-1}-1,2^{m-1}+2^{n-1},2^{n-1}) \\ &\quad = 1 + f(0,2^{n-1}-2^{m-1},2^{n-1}) - f(0,2^{n-1}-2^{m-1}+1,2^{n-1}+1) \\ &\quad = 1 + f(2^{n-2}-2^{m-2},2^{n-1}-2^{m-2},2^{n-2}) - f(2^{n-2}-2^{m-2},2^{n-1}-2^{m-2}+1,2^{n-2}) \\ &\quad = 1 + f(0,2^{m-2},2^{n-2}) - f(0,2^{m-2},2^{n-2}+1) \\ &\quad = 1 + f(0,2^{m-2},2^{n-2}) - f(0,2^{m-2},2^{n-2}) = 1. \end{aligned}$$

For $k > 0$, putting $K = k - (k \bmod 2)$, we have

$$\begin{aligned}
&f(0, 2^m, 2^n) - f(0, 2^m - 2^k, 2^n + 2^k)\\
&= f(2^{m-1}, 2^{m-1} + 2^{n-1}, 2^{n-1}) - f(2^{m-1} - 2^{k-1}, 2^{m-1} + 2^{n-1}, 2^{n-1} + 2^{k-1})\\
&= f(2^{m-2} + 2^{n-2}, 2^{m-1} + 2^{n-2}, 2^{m-2} + 2^{n-1})\\
&\qquad - f(2^{m-2} + 2^{n-2}, 2^{m-1} + 2^{n-2} - 2^{k-2}, 2^{m-2} + 2^{n-1} + 2^{k-2})\\
&= f(0, 2^{m-2}, 2^{n-2}) - f(0, 2^{m-2} - 2^{k-2}, 2^{n-2} + 2^{k-2})\\
&= \cdots = f(0, 2^{m-K}, 2^{n-K}) - f(0, 2^{m-K} - 2^{k \bmod 2}, 2^{n-K} + 2^{k \bmod 2}).
\end{aligned}$$

Thus it suffices to show that

$$f(0, 2^m, 2^n) - f(0, 2^m - 1, 2^n + 1) = f(0, 2^m, 2^n) - f(0, 2^m - 2, 2^n + 2) = 1$$

In fact, $f(0, 2^m, 2^n) - f(0, 2^m - 1, 2^n + 1) = 1$ has been already shown in the above. Also

$$\begin{aligned}
&f(0, 2^m, 2^n) - f(0, 2^m - 2, 2^n + 2)\\
&= f(2^{m-1}, 2^{m-1} + 2^{n-1}, 2^{n-1}) - f(2^{m-1} - 1, 2^{m-1} + 2^{n-1}, 2^{n-1} + 1)\\
&= f(0, 2^{n-1} - 2^{m-1}, 2^{n-1}) + 1 - f(0, 2^{n-1} - 2^{m-1} + 2, 2^{n-1} + 1)\\
&= f(2^{n-2} - 2^{m-2}, 2^{n-1} - 2^{m-2}, 2^{n-2}) + 1\\
&\qquad - f(2^{n-2} - 2^{m-2} + 1, 2^{n-1} - 2^{m-2} + 1, 2^{n-2})\\
&= f(0, 2^{m-2}, 2^{n-2}) + 1 - f(1, 2^{m-2}, 2^{n-2} + 1)\\
&= f(2^{m-3}, 2^{m-3} + 2^{n-3}, 2^{n-3}) + 1 - f(2^{m-3}, 2^{m-3} + 2^{n-3}, 2^{n-3} + 1)\\
&= f(2^{m-3}, 2^{m-3} + 2^{n-3}, 2^{n-3}) + 1 - f(2^{m-3}, 2^{m-3} + 2^{n-3}, 2^{n-3}) \quad \text{(by Lemma 4)}\\
&= 1.
\end{aligned}$$

□

Remark 2. Let $k = 0$. In case $(m, n) = (0, 0), (3, 2), (1, x), (x, 1)$ for $x \neq 1$, we have

$$f(0, 2^m, 2^n) - f(0, 2^m - 1, 2^n + 1) = 0.$$

Other cases, we also have the above difference is 1.

So we try to estimate $f(a, b, c)$ by $(a + b + c)/3$:

Lemma 13. *For $k \in \mathbb{N}$, let $0 \leqq x, y \leqq 4^k - 1$. Then*

$$f(0, 4^k K + x, 4^k N + y) \geqq \frac{4^k - 1}{4^k} \cdot \frac{4^k(K + N)}{3} - \frac{2}{3} \cdot \frac{(4^k - 1)^2}{4^k} + f(0, K, N)$$

for $K, N \in \mathbb{N}$.

Proof. Since $\frac{x+y}{3} \leqq \frac{2(4^k-1)}{3}$, we have

$$\begin{aligned}
f(0, 4^k K + x, 4^k N + y) &\geqq f(0, 4^k K, 4^k N) = 4^{k-1}(K + N) + f(0, 4^{k-1}K, 4^{k-1}N)\\
&= \cdots = (4^{k-1} + \cdots + 1)(K + N) + f(0, K, N)\\
&= \frac{4^k - 1}{3}(K + N) + f(0, K, N)\\
&= \frac{4^k - 1}{4^k} \cdot \frac{4^k(K + N) + x + y}{3} - \frac{(4^k - 1)(x + y)}{3 \times 4^k} + f(0, K, N)\\
&\geqq \frac{4^k - 1}{4^k} \cdot \frac{4^k(K + N) + x + y}{3} - \frac{2}{3} \cdot \frac{(4^k - 1)^2}{4^k} + f(0, K, N).
\end{aligned}$$

□

Remark 3. If $x = y = 0$ in the above theorem, then

$$f(0, 4^k K, 4^k N) = \frac{4^k - 1}{4^k} \cdot \frac{4^k(K+N)}{3} + f(0, K, N).$$

Corollary 14. $f(0, k, n) \geqq \frac{4^k - 1}{4^k} \cdot \frac{k+n}{3} - \frac{2}{3} \cdot \frac{(4^k - 1)^2}{4^k}.$

Theorem 15. *For nonnegative integers a, b, c and k,*

$$\begin{aligned} f(a, b, c) \ &\geqq \frac{4^k - 1}{4^k} \cdot \frac{a+b+c}{3} + \frac{\min\{a, b, c\}}{4^k} - \frac{2}{3} \cdot \frac{(4^k - 1)^2}{4^k} \\ &\geqq \frac{4^k - 1}{4^k} \cdot \frac{a+b+c}{3} - \frac{2}{3} \cdot \frac{(4^k - 1)^2}{4^k}. \end{aligned}$$

In addition, if $a, b, c \geqq \frac{2(4^k-1)^2}{3}$, then

$$f(a, b, c) \geqq \frac{4^k - 1}{4^k} \cdot \frac{a+b+c}{3}.$$

Proof. Since we may assume $a \leqq b, c$, the above corollary implies

$$\begin{aligned} f(a, b, c) = a + f(0, b-a, c-a) &\geqq a + \frac{4^k - 1}{4^k} \cdot \frac{b+c-2a}{3} - \frac{2}{3} \cdot \frac{(4^k - 1)^2}{4^k} \\ &= \frac{4^k - 1}{4^k} \cdot \frac{a+b+c}{3} + \frac{a}{4^k} - \frac{2}{3} \cdot \frac{(4^k - 1)^2}{4^k} \\ &\geqq \frac{4^k - 1}{4^k} \cdot \frac{a+b+c}{3} - \frac{2}{3} \cdot \frac{(4^k - 1)^2}{4^k}. \end{aligned}$$

If $b, c \geqq a \geqq \frac{2(4^k-1)^2}{3}$, then

$$\frac{a}{4^k} \geqq \frac{2(4^k - 1)^2}{3 \times 4^k},$$

so that we have the last inequality. □

Remark 4. By Remark 3, if $b - a$ and $c - a$ are the multiples of 4^k for $b, c \geqq a$, then

$$f(a, b, c) = \frac{4^k - 1}{4^k} \cdot \frac{a+b+c}{3} + \frac{a}{4^k} + f(0, b-a, c-a).$$

Remark 5. If $a, b, c \geqq \frac{2(4^k-1)^2}{3}$, then

$$f(a, b, c) \geqq \frac{4^k - 1}{4^k} \cdot \frac{a+b+c}{3}.$$

We also pose the case for $k = 1$:

Corollary 16.

$$\begin{aligned} f(a, b, c) &\geqq \frac{3}{4} \cdot \frac{a+b+c}{3} + \frac{\min\{a, b, c\}}{4} - \frac{3}{2} \\ &\geqq \frac{3}{4} \cdot \frac{a+b+c}{3} - \frac{3}{2}. \end{aligned}$$

In addition, if $a, b, c \geqq 12$, then

$$f(a, b, c) \geqq \frac{3}{4} \cdot \frac{a+b+c}{3}.$$

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