ON CERTAIN CHARACTERIZATIONS OF INNER PRODUCT SPACES

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Received February 8, 2012; revised April 10, 2012

Abstract. In this note, we present certain characterizations of inner product spaces which are based on geometric norm equalities. In particular, we show that a normed linear space X is an inner product space if and only if for each $x, y \in X$ with $\|x\| = \|y\| = 1$, there exists $t \in (0, 1/2)$ such that $\|(1-t)x + ty\| = \|tx + (1-t)y\|$ holds.

1 Introduction and Preliminaries It is well known that a normed linear space is an inner product space if and only if the space satisfies the parallelogram law. This is due to Jordan and von Neumann [5]. In the book of Amir [1], we have many characterizations of inner product spaces which are based on norm inequalities, various notions of orthogonality in normed linear spaces and so on.

In this note, our aim is to present certain characterizations of inner product spaces in connection with some results seen in the book. Let X be a normed linear space and S_X denotes its unit sphere.

In 1944, Ficken characterized inner product spaces by the following norm equality.

Theorem 1 ([3]). *The following are equivalent:*
(i) *X is an inner product space.*
(ii) *The equality*

$$\|\alpha x + \beta y\| = \|\beta x + \alpha y\|$$

hold for any $x,\ y \in S_X$ and any $\alpha,\ \beta \in \mathbb{R}$.

Lorch showed that the condition (ii) in Theorem 1 can be replaced by a weaker condition, that is, we have following.

Theorem 2 ([7]). *The following are equivalent:*
(i) *X is an inner product space.*
(ii) *There exists a fixed real number $c_0 > 1$ such that*

$$\|c_0 x + y\| = \|x + c_0 y\|$$

hold for any $x,\ y \in S_X$.
(iii) *There exists a fixed real number $t_0 \in (0, 1/2)$ such that*

$$\|(1-t_0)x + t_0 y\| = \|t_0 x + (1-t_0)y\|$$

hold for any $x,\ y \in S_X$.

The following is the main theorem in this note which improves Theorem 2.

Theorem 3. *The following are equivalent:*
(i) *X is an inner product space.*
(ii) *For each $x, y \in S_X$, there exists a real number $c > 1$ such that*

$$\|cx + y\| = \|x + cy\|$$

2010 *Mathematics Subject Classification.* 46C15.
Key words and phrases. inner product space.

holds.

(iii) *For each $x, y \in S_X$, there exists a real number $t \in (0, 1/2)$ such that*

$$\|(1-t)x + ty\| = \|tx + (1-t)y\|$$

holds.

Remark that real numbers c and t are not fixed in Theorem 2. To prove this theorem, we need the following lemmas.

Lemma 1 ([7])**.** *A normed linear space X is an inner product space if and only if it satisfies the following condition:*

(L) *If $x,\ y \in X$ and $\|x\| = \|y\|$, then we have $\|\alpha x + \alpha^{-1} y\| \geq \|x + y\|$ for all $\alpha > 0$.*

In the following lemma, (i) $\Longleftrightarrow$ (ii) is due to Gurarii and Sozonov [4] while Kirk and Smiley [6] proved (i) $\Longleftrightarrow$ (iii). However, we can prove directly that (ii) is equivalent to (iii). For completeness, we prove this lemma.

Lemma 2 ([4, 6])**.** *The following are equivalent:*

(i) *X is an inner product space.*

(ii) *The inequality*

$$\left\| \frac{x+y}{2} \right\| \leq \|(1-t)x + ty\|$$

hold for any $x,\ y \in S_X$ and any $t \in [0, 1]$.

(iii) *The inequality*

$$\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \leq \frac{2\|x-y\|}{\|x\| + \|y\|}$$

hold for any $x,\ y \in X \setminus \{0\}$.

Proof. Suppose that (ii) holds. Take any $x,\ y \in X$ with $\|x\| = \|y\|$ and $\alpha > 0$. Then, by the assumption, we have

$$\begin{aligned} \|\alpha x + \alpha^{-1} y\| &\geq (\alpha + \alpha^{-1}) \left\| \frac{x+y}{2} \right\| \\ &\geq \|x + y\|. \end{aligned}$$

Therefore, by Lemma 1, X is an inner product space. This proves (i) $\Longleftrightarrow$ (ii).

For each $x,\ y \in X \setminus \{0\}$ with $x - y \neq 0$, putting $u = \|x\|^{-1} x$ and $v = -\|y\|^{-1} y$, then

$$\frac{\|x\| + \|y\|}{\|x - y\|} \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| = \frac{\|u + v\|}{\|(1-t)u + tv\|},$$

where $t = \|y\|/(\|x\| + \|y\|)$. On the other hand, if $u,\ v \in S_X$, $u + v \neq 0$ and $t \in (0, 1)$, then we have

$$\frac{\|u + v\|}{\|(1-t)u + tv\|} = \frac{\|x\| + \|y\|}{\|x - y\|} \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\|$$

as $x = (1-t)u$ and $y = -tv$, respectively. From these facts, we have (ii) $\Longleftrightarrow$ (iii). This completes the proof.

2 Proof of Theorem 3 Clearly, (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii). We now suppose that (iii) holds. Take any pair x, y in S_X; we may assume they are linearly independent. We define a subset A of $(0,1/2)$ by

$$A = \{t \in (0,1/2) : \|(1-t)x + ty\| = \|tx + (1-t)y\|\}.$$

By the assumption, $A \neq \emptyset$. Put $t_0 = \sup A$. Then, it follows from continuity of the norm that

$$\|(1-t_0)x + t_0 y\| = \|t_0 x + (1-t_0)y\|.$$

We first show that $t_0 = 1/2$. To show this, suppose contrary that $t_0 < 1/2$. Put $u = (1-t_0)x + t_0 y$ and $v = t_0 x + (1-t_0)y$. Then $\|u\| = \|v\| = c$. By the assumption (iii), for $x_0 = c^{-1}u$ and $y_0 = c^{-1}v$, there exists a real number $s_0 \in (0,1/2)$ such that $\|(1-s_0)x_0 + s_0 y_0\| = \|s_0 x_0 + (1-s_0)y_0\|$. Put $t_1 = (1-s_0)t_0 + s_0(1-t_0)$. Then $t_0 < t_1 < 1/2$ and

$$(1-t_1)x + t_1 y = (1-s_0)u + s_0 v, \quad t_1 x + (1-t_1)y = s_0 u + (1-s_0)v.$$

Therefore we have

$$\|(1-t_1)x + t_1 y\| = \|(1-s_0)u + s_0 v\| = \|s_0 u + (1-s_0)v\| = \|t_1 x + (1-t_1)y\|,$$

that is, $t_1 \in A$. This contradiction implies $t_0 = 1/2$, as desired.

From the fact that $t_0 = 1/2$, we obtain a sequence $\{\lambda_n\}$ in A such that $\lambda_n \nearrow 1/2$. Since the function $t \mapsto \|(1-t)x + ty\|$ is convex, for each $n \in \mathbb{N}$, there is a real number $\mu_n \in [\lambda_n, 1-\lambda_n]$ such that

$$\|(1-\mu_n)x + \mu_n y\| = \min\{\|(1-t)x + ty\| : 0 \leq t \leq 1\}.$$

Since $\mu_n \to 1/2$, for any $t \in [0,1]$, we have

$$\left\|\frac{x+y}{2}\right\| = \lim_{n\to\infty} \|(1-\mu_n)x + \mu_n y\| \leq \|(1-t)x + ty\|.$$

By Lemma 2, X is an inner product space. This completes the proof.

Acknowledgement. The authors would like to thank the referees for their helpful comments and careful reading in this paper.

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