DE MORGAN ALGEBRAS: NEW PERSPECTIVES AND APPLICATIONS

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Dedicated affectionately to my wife Nalinaxi

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Abstract. It is well known that Boolean algebras can be defined using only the implication and the constant 0. It is, then, natural to ask whether De Morgan algebras can also be characterized using only a binary operation (implication) $\to$ and a constant 0. In this paper, we give an affirmative answer to this question by showing that the variety of De Morgan algebras is term-equivalent to a variety of type $\{\to, 0\}$. As a natural consequence, we describe Kleene algebras also as a variety using only $\to$ and 0. (The afore-mentioned result for Boolean algebras is also deduced.) As a second consequence, we give a simplification of an axiom system of Bernstein (along with a new proof for his system of axioms). We also describe De Morgan algebras in terms of a NAND operation $|$ and the constant 0. Motivated by the the afore-mentioned results, we define, and initiate, the investigation of a new variety $\mathbf{I}$ of algebras, called "Implication zroupoids" (I-zroupoids, for short) and show that $\mathbf{I}$ satisfies the identity $x''' \to y \approx x' \to y$, where $x' := x \to 0$. Furthermore, we introduce several important subvarieties of $\mathbf{I}$ and establish some relationships among them; in particular, we give several characterizations of the subvariety defined by $x'' = x$. The paper ends with some open problems for further research.

1. Introduction

It is well known that Boolean algebras can be defined using only the implication and the constant 0. It is, then, natural to ask whether De Morgan algebras can also be characterized using only a binary operation (implication) $\to$ and a constant 0. In this paper, we give an affirmative answer to this question by showing that the variety of De Morgan algebras is term-equivalent to a variety of type $\{\to, 0\}$. As a natural consequence, we describe a variety, using only an implication $\to$ and a constant 0, that is term-equivalent to the variety of Kleene algebras. (The afore-mentioned result for Boolean algebras is also deduced.) As a second consequence, we give a simplification of an axiom system of Bernstein (along with a new proof for his system of axioms). We also describe De Morgan algebras and Kleene algebras in terms of a NAND operation $|$ and the constant 0. Motivated by the the afore-mentioned results, we define, and initiate, the investigation of a new variety of algebras, called "Implication zroupoids" (I-zroupoids, for short) and show that the variety of I-zroupoids satisfies the identity: $x'''' \approx x''$, where $x' := x \to 0$. In fact, the stronger identity: $x''' \to y \approx x' \to y$ is shown to hold. Furthermore, we introduce several important subvarieties of $\mathbf{I}$ and establish some relationships among them; in particular, we give several

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characterizations of the subvariety defined by $x'' = x$. The paper ends with some open problems for further research.

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2. Preliminaries

Definition 2.1. An algebra $\mathbf{A} = \langle A, \vee, \wedge, ^c, 0, 1\rangle$ is a *De Morgan algebra* if the following conditions hold in $\mathbf{A}$:

(1) $\langle A, \vee, \wedge, 0, 1\rangle$ is a distributive lattice with $0, 1$

(2) $x^{cc} \approx x$

(3) $(x \vee y)^c \approx x^c \wedge y^c$

(4) $(x \wedge y)^c \approx x^c \vee y^c$.

It is well known that $\vee$ and 1 can be defined in terms of $\wedge$ and 0 in the above definition. Thus we have the following equivalent definition.

Definition 2.2. An algebra $\mathbf{A} = \langle A, \wedge, ^c, 0\rangle$ is a *De Morgan algebra* if $\mathbf{A}$ satisfies the following conditions, where we define $x \vee y = (x^c \wedge y^c)^c$ and $1 = 0^c$:

(d1) $\langle A, \vee, \wedge, 0, 1\rangle$ is a distributive lattice with $0, 1$

(d2) $x^{cc} \approx x$.

DM denotes the variety of De Morgan algebras.

The following definitions are also well known.

Definition 2.3. A De Morgan algebra $\mathbf{A}$ is a *Kleene algebra* if $\mathbf{A}$ satisfies:

(K) $x \wedge x^c \leq y \vee y^c$.

Let **KL** denote the variety of Kleene algebras.

Definition 2.4. A De Morgan algebra $\mathbf{A}$ is a *Boolean algebra* if $\mathbf{A}$ satisfies:

(B) $x \wedge x^c \approx 0$.

Let **BA** denote the variety of Boolean algebras.

For n a positive integer, an *n-base* for a variety $\mathbf{V}$ is an independent set Σ of n identities in the language of $\mathbf{V}$ such that Mod $\Sigma = \mathbf{V}$.

3. De Morgan Algebras: From the usual perspective

In this section we present axiomatizations of De Morgan algebras which will be useful in the next section. The following theorem simplifies Definition 2.2.

Theorem 3.1. *Let $\mathbf{A} = \langle A, \wedge, ', 0\rangle$ be an algebra. Then the following are equivalent:*

(1) $\mathbf{A} \in \mathbf{DM}$ (in the sense of Definition 2.2).

(2) $\mathbf{A}$ *satisfies the following five axioms, where we define $x \vee y = (x' \wedge y')'$:*

(a) $x \wedge (x \vee y) \approx x$

(b) $x \wedge 0 \approx 0$

(c) $x \vee (y \wedge z) \approx (x \vee y) \wedge (x \vee z)$

(d) $x'' \approx x$

(e) $x \vee y \approx y \vee x$.

We note here that the dual of the above theorem is also true. To prove this theorem it suffices to prove that (2) implies (1), which will be accomplished using the following two lemmas.

Lemma 3.2. *Let $\mathbf{A} = \langle A, \wedge, ', 0\rangle$ be an algebra satisfying* (a), (b) and (d) of condition (2) of Theorem 3.1, and let x, $y \in \mathbf{A}$. Then $\mathbf{A}$ satisfies:

(i) $(x \vee y)' = x' \wedge y'$

(ii) $(x \wedge y)' = x' \vee y'$

(iii) $x \vee (x \wedge y) = x$

(iv) $x \vee 0' = 0'$

(v) $x \wedge 0' = x$

(vi) $x \vee 0 = x$

(vii) $x \wedge x = x$

(viii) $x \vee x = x$

Proof. (i)-(iii) easily follow from the definition of $\vee$, and the hypothesis (d) and (a). For (iv) use (d), (ii) and (b), while (v) follows from (iv) and (a), and (vi) is immediate from (v) and (d). (vii) follows since $x \wedge x = x \wedge (x \vee 0) = x$ by (vi) and (a). (viii) follows immediately from (d) and (vii). □

Lemma 3.3. *Let* $\mathbf{A} = \langle A, \wedge, ', 0\rangle$ *be an algebra satisfying the condition* (2) of Theorem 3.1, and let x, y, z denote arbitrary elements of $\mathbf{A}$. Then

(i) $x \wedge y = y \wedge x$

(ii) $x \vee (x \vee y) = x \vee y$

(iii) $(x \vee y) \vee z = (x \vee y) \vee (y \vee z)$

(iv) $(x \vee y) \vee z = x \vee (y \vee z)$

(v) $(x \wedge y) \wedge z = x \wedge (y \wedge z)$.

Proof. It is clear that (i) follows from (e), (d), and Lemma 3.2(i). Using the hypothesis (a), (e), (i), and Lemma 3.2(iii), we have

$x \vee (x \vee y) = [x \wedge (x \vee y)] \vee (x \vee y) = (x \vee y) \vee [(x \vee y) \wedge x] = x \vee y$, proving (ii).

For (iii),

$(x \vee y) \vee z$

$= (x \vee y) \vee [(y \vee z) \wedge z]$ by (a), (i), and (e)

$= [(x \vee y) \vee (y \vee z)] \wedge [(x \vee y) \vee z]$ by (c)

$= [(x \vee y) \vee (y \vee z)] \wedge [(x \vee y) \vee \{(x \vee y) \vee z\}$ by (ii)

$= (x \vee y) \vee [z \vee \{y \wedge (x \vee y)\}]$ by (c) and (e)

$= (x \vee y) \vee (y \vee z)$ by (a) and (e), proving (iii).

To prove (iv), we have

$(x \vee y) \vee z$

$= (x \vee y) \vee (y \vee z)$ by (iii)

$= (z \vee y) \vee (y \vee x)$ by (e)

$= (z \vee y) \vee x$ by (iii)

$= x \vee (y \vee z)$ by (e).

It is clear that (v) follows from (d), (iv), Lemma 3.2(ii), and definition of $\vee$. □

From the preceding lemmas we conclude (in view of of Definition 2.2) that $\mathbf{A} \in \mathbf{DM}$. The other implication being trivial, the proof of Theorem 3.1 is now complete.

Even the commutative axiom (e) in Theorem 3.1(2) can be made redundant by slightly modifying the distributive law (c).

Theorem 3.4. *The following axioms, in the language* $\{\wedge, ', 0\}$, *form a base for the variety of De Morgan algebras, where* $x \vee y := (x' \wedge y')'$:

(a) $x \wedge (x \vee y) \approx x$

(b) $x \wedge 0 \approx 0$

(c′) $x \vee (y \wedge z) \approx (x \vee y) \wedge (z \vee x)$

(d) $x'' \approx x$.

Since (a), (b), (c') and (d) of Theorem 3.4 clearly hold in the 4-element subdirectly irreducible De Morgan algebra, they hold in every De Morgan algebra. So, we only need to prove the converse. Thus, to complete the proof of Theorem 3.4, it suffices to show (c) and (e) of Theorem 3.1 hold. The following lemma aids us in achieving this goal.

Lemma 3.5. *Let* $\mathbf{A} = \langle A, \wedge, ', 0\rangle$ *be an algebra satisfying the axioms of* Theorem 3.4. *Let* x, y, z *denote arbitrary elements of* $\mathbf{A}$. *Then*

(i) $(x \wedge y) \vee (0' \wedge x) = x$

(ii) $0' \wedge x = x$

(iii) $x \vee y = y \vee x$.

Proof. First, observe that Lemma 3.2 holds in $\mathbf{A}$.

For (i),

$(x \wedge y) \vee (0' \wedge x)$

$= [(x \wedge y)' \wedge (0' \wedge x)']'$

$= [(x' \vee y') \wedge (0 \vee x')]'$ by Lemma 3.2(ii) and (d)

$= [x' \vee (y' \wedge 0)]'$ by (c′)

$= (x' \vee 0)'$ by (b)

$= x''$ by Lemma 3.2(vi)

$= x$ by (d).

Next, we prove (ii):

$0' \wedge x$

$= 0' \wedge [x \vee (x \wedge y)]$ by Lemma 3.2(iii)

$= [(x \wedge y) \vee 0'] \wedge [x \vee (x \wedge y)]$ by Lemma 3.2(iv)

$= (x \wedge y) \vee (0' \wedge x)$ by (c')

$= x$ by (i).

Finally,

$x \vee y$

$= x \vee (0' \wedge y)$ by (ii)

$= (x \vee 0') \wedge (y \vee x)$ by (c')

$= 0' \wedge (y \vee x)$ by Lemma 3.2(iv)

$= y \vee x$ by (ii), proving (iii). $\square$

Thus (e) holds in $\mathbf{A}$. Since (c) is immediate from (c$'$) and (e), we conclude from Theorem 3.1 that $\mathbf{A}$ is a De Morgan algebra, which completes the proof of Theorem 3.4.

4. De Morgan Algebras: From a New Perspective

We shall now cast De Morgan algebras as a variety of type $\{\to, 0\}$, where $\to$ is binary, and 0 is a constant symbol. For this we need to make the following definition.

Definition 4.1. An algebra $\mathbf{A} = \langle A, \to, 0\rangle$ is a $\mathbf{DM}^{\to}$-*algebra* if $\mathbf{A}$ satisfies the following axioms, where $x' = x \to 0$:

(I) $(x \to y) \to z \approx [(z' \to x) \to (y \to z)']'$

(J) $(x \to y) \to x \approx x$.

Let $\mathbf{DM}^{\to}$ denote the variety of $\mathbf{DM}^{\to}$-algebras.

The following theorem shows that the variety of De Morgan algebras can be characterized in the language $\{\to, 0\}$, as claimed in the Introduction.

Theorem 4.2. *The variety* $\mathbf{DM}$ *is term-equivalent to the variety* $\mathbf{DM}^{\to}$*. More precisely,*

(a) For $\mathbf{A} \in \mathbf{DM}$, let $\mathbf{A}^{\to}$ be the algebra $\langle A, \to, 0\rangle$ where $\to$ is defined by $x \to y = (x \wedge y^c)^c$. Then $\mathbf{A}^{\to} \in \mathbf{DM}^{\to}$.

(b) For $\mathbf{A} \in \mathbf{DM}^{\to}$, let $\mathbf{A}^*$ be the algebra $\langle A, \wedge, ^c, 0\rangle$ such that $x \wedge y := (x \to y')'$, where $x' := x \to 0$, and $x^c := x'$. Then $\mathbf{A}^* \in \mathbf{DM}$.

(c) If $\mathbf{A} \in \mathbf{DM}$, then $(\mathbf{A}^{\to})^* = \mathbf{A}$.

(d) If $\mathbf{A} \in \mathbf{DM}^{\to}$, then $(\mathbf{A}^*)^{\to} = \mathbf{A}$.

Proof. (a): Let $\mathbf{A} = \langle A, \wedge, ^c, 0\rangle$ be a De Morgan algebra (in the sense of Definition 2.2), and let $\mathbf{A}^{\rightarrow}$ be the algebra as in (a) of the above theorem. Let x, y, z denote arbitrary elements of $\mathbf{A}^{\rightarrow}$. First, note that $x' = x \rightarrow 0 = (x \wedge 0^c)^c = x^c$. Now, $(x \rightarrow y) \rightarrow z = [(x \wedge y^c)^c \wedge z^c]^c = z \vee (x \wedge y^c) = (z \vee x) \wedge (y^c \vee z) = [(z' \rightarrow x) \rightarrow (y \rightarrow z)']'$, which proves (I). Next, $(x \rightarrow y) \rightarrow x = [(x \wedge y^c)^c \wedge x^c]^c = x \vee (x \wedge y^c) = x$, which prove (J). Thus $\mathbf{A}^{\rightarrow} \in \mathbf{DM}^{\rightarrow}$, proving Theorem 4.2(a). □

The proof of Theorem 4.2(b) will be obtained from the following lemma.

Lemma 4.3. *Let* $\mathbf{A} \in \mathbf{DM}^{\rightarrow}$ *and let* $\mathbf{A}^*$ *be as in* (b) (of the preceding theorem). *Let* $x, y, z \in \mathbf{A}^*$. *Let* $x \vee y = (x^c \wedge y^c)^c$. (Note $x^c = x'$.) *Then*

(i) $x^{cc} = x$

(ii) $x^c \rightarrow y = y^c \rightarrow x$

(iii) $x \wedge (x \vee y) = x$

(iv) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$

(v) $x \wedge 0 = 0$.

Proof. $x^{cc} = (x \rightarrow 0) \rightarrow 0$

$= [\{(x \rightarrow 0) \rightarrow x\} \rightarrow \{(0 \rightarrow x) \rightarrow 0\}] \rightarrow 0$ by (J)

$= (x \rightarrow 0) \rightarrow x$ by (I)

$= x$ by (J), proving (i).

Next, $x^c \rightarrow y$

$= (x^c \rightarrow y)^{cc}$ by (i)

$= [(x^c \rightarrow y) \rightarrow 0] \rightarrow 0$

$= [(x' \rightarrow y) \rightarrow \{(0 \rightarrow x) \rightarrow 0\}] \rightarrow 0$ by (J)

$= (y \rightarrow 0) \rightarrow x$ by (I)

$= y^c \rightarrow x$, proving (ii).

(iii) is immediate from (i), (ii) and (J).

To prove (iv),

$x \vee (y \wedge z)$

$= x' \rightarrow (y \rightarrow z')'$, in view of (i) and definitions of $\vee$, $\wedge$ and ^{c}

$= (y \rightarrow z') \rightarrow x$ by (i) and (ii)

$= [(x' \to y) \to (z' \to x)']'$ by (I)

$= (x \vee y) \wedge (z \vee x)$ in view of (i) and definitions of $\vee$ and $\wedge$.

Using (i), (ii) and (J), we have $x \wedge 0 = (x \to 0')' = (x'' \to 0')' = (0 \to x') \to 0 = 0$, proving (v). $\square$

In view of the above Lemma, $\mathbf{A}^*$ satisfies the axioms of Theorem 3.4, from which we can infer that $\mathbf{A}^* \in \mathbf{DM}$, thus proving Theorem 4.2(b). The proofs of (c) and (d) of Theorem 4.2 are left to the reader.

Definition 4.4. A $\mathbf{DM}^{\to}$-algebra $\mathbf{A} = \langle A, \to, 0 \rangle$ is a $\mathbf{KL}^{\to}$-*algebra* if $\mathbf{A}$ satisfies the following axiom:

(K1) $(x \to x)' \to (y \to y)' \approx x \to x$

or, equivalently,

(K2) $(y \to y) \to (x \to x) \approx x \to x$.

Let $\mathbf{KL}^{\to}$ denote the variety of $\mathbf{KL}^{\to}$-algebras.

Corollary 4.5. *The variety* $\mathbf{KL}$ *is term-equivalent to the variety* $\mathbf{KL}^{\to}$.

To prove Corollary 4.5, it is sufficient, in view of Theorem 4.2, to prove that (K) and (K1) (or (K2)) are equivalent, which is left to the reader to verify.

Definition 4.6. A $\mathbf{DM}^{\to}$-algebra $\mathbf{A} = \langle A, \to, 0 \rangle$ is a $\mathbf{BA}^{\to}$-*algebra* if $\mathbf{A}$ satisfies the following axiom:

(B1) $x \to x \approx 0'$.

Let $\mathbf{BA}^{\to}$ denote the variety of $\mathbf{BA}^{\to}$-algebras.

The following corollary is also immediate from Theorem 4.2.

Corollary 4.7. *The variety* $\mathbf{BA}$ *is term-equivalent to the variety* $\mathbf{BA}^{\to}$.

We denote by $\mathcal{DM}$ the category whose objects are De Morgan algebras and whose morphisms are $\{\wedge, ', 0\}$-homomorphisms. We also let $\mathcal{DM}^{\to}$ denote the category, whose objects are $\mathbf{DM}^{\to}$-algebras and whose morphisms are $\{\to, 0\}$-homomorphisms. The categories $\mathcal{KL}$, $\mathcal{KL}^{\to}$, $\mathcal{BA}$ and $\mathcal{BA}^{\to}$ are similarly defined. The following remark is now pretty clear.

Remark 4.8. The category $\mathcal{DM}$ (respectively, $\mathcal{KL}$, $\mathcal{BA}$) is equivalent to the category $\mathcal{D}M^{\to}$ (respectively, $\mathcal{KL}^{\to}$, $\mathcal{BA}^{\to}$).

There is a large number of varieties of algebras arising from nonclassical logics, such as Łukasiewicz algebras, Nelson algebras and so on, which have De Morgan algebras or Kleene algebras as reducts. We wish to note here that Theorem 4.2 and its consequences can be applied to give new axiomatizations for those varieties by suitably expanding the lanuage $\{\to, 0\}$.

5. A Simplified version of Bernstein's axiom system for Boolean algebras

In this section we give a slightly different axiomatization for De Morgan algebras, still in the language $\{\to, 0\}$, and use it to give a simplification of Bernstein's axiom system (see [3]) for Boolean algebras.

Theorem 5.1. *The following axioms, in the language $\{\to, 0\}$, form a 2-base for the variety of De Morgan algebras, where $x' = x \to 0$:*

(I): $(x \to y) \to z \approx [(z' \to x) \to (y \to z)']'$

(L): $(0 \to x) \to y \approx y.$

In light of Theorem 3.4(b), it suffices to show that (J) and (L) are equivalent in the presence of (I). So, first suppose (I) and (J) hold in $\mathbf{A}$. Then
$(0 \to x) \to y$
$= [(y' \to 0) \to (x \to y)']'$ by (I)
$= [(x \to y) \to y']'$ by Lemma 4.3(ii) and (i)
$= [(y' \to x') \to y']'$ by Lemma 4.3(ii) and (i)
$= y''$ by (J)
$= y$ by Lemma 4.3(i).
Thus (L) holds in $\mathbf{A}$, proving the first half of the theorem. For the other half, we need the following lemma.

Lemma 5.2. *let $\mathbf{A} = \langle A, \to, 0\rangle$ be an algebra satisfying* (I) *and* (L). Let x, y, z be arbitrary elements of $\mathbf{A}$. Then

(i) $x''' = x'$

(ii) $x' \to y = y' \to x$

(iii) $x'' = x$

(iv) $[x \to (x' \to y)']' = [y \to (0 \to z)] \to x$

(v) $(x \to y) \to x = x.$

Proof. To prove (i),

$x' = (0' \to x) \to 0$ by (L)

$= [(0' \to 0') \to (x \to 0)']'$ by (I)

$= (0' \to x'')'$ by (L)

$= x'''$ by (L).

Next,

$y' \to x$

$= [(x' \to y) \to (0 \to x)']'$ by (I)

$= [\{(x \to 0) \to y\} \to 0]'$ by (L)

$= [(x \to 0) \to y]''$

$= [(y' \to x) \to (0 \to y)']'''$ by (I)

$= x' \to y$ by (i) and (I), proving (ii).

(iii) is immediate from (ii) and (L), and (iv) follows from (ii), (iii), (L) and (I).

Finally,

$(x \to y) \to x$

$= [(x \to y) \to x]''$ by (iii)

$= [x' \to (x'' \to y)']''$ by (ii) and (iii)

$= [\{y \to (0 \to z)\} \to x']'$ by (iv)

$= [\{(0 \to z)' \to y'\} \to x']'$ by (ii) and (iii)

$= [\{0 \to (y \to 0)] \to x']'$ by (L)

$= x$ by (L) and (iii). □

Thus (J) holds in $\mathbf{A}$ and the proof of Theorem 5.1 is now complete.

Next we present a base for Boolean algebras, which is a simplification of a well known axiom system due to Bernstein([3]).

Theorem 5.3. *The following axioms form a 2-base for the variety of Boolean algebras, where $x' := x \to 0$:*

(J) $(x \to y) \to x \approx x$

(M) $(y \to y) \to ((x \to y) \to z) \approx [(z' \to x) \to (y \to z)']'$.

Since (J) and (M) hold in the 2-element Boolean algebra, they hold in every Boolean algebra. To prove the converse, we let $\mathbf{B} = \langle B, \wedge, ', 0\rangle$ be an algebra satisfying (J) and (M). We need to show that (B1) and (I) hold in $\mathbf{B}$, which will be accomplished in the following lemma.

Lemma 5.4. *Let $x, y, z \in \mathbf{B}$. Then*

(i) $[x \to (y \to x)']' = (y \to y) \to x$

(ii) $(x \to x)' = 0$

(iii) $x'' = x$

(iv) $0 \to 0 = x \to x$. *Hence* (B1) *holds in* **B**

(v) $(0 \to 0) \to x = x$

(vi) $(x \to y) \to z = [(z' \to x) \to (y \to z)']'$. *Hence,* (I) *holds in* **B**.

Proof. We have

$(y \to y) \to x$

$= (y \to y) \to [(x \to y) \to x]$ by (J)

$= [(x' \to x) \to (y \to x)']'$ by (M)

$= [x \to (y \to x)']'$ by (J), proving (i).

Next, for (ii),

$(x \to x) \to 0$

$= [0 \to (x \to 0)'] \to 0$ by (i)

$= 0$ by (J).

Next,

x''

$= (x \to 0)'$

$= [x \to (x \to x)']'$ by (ii)

$= (x \to x) \to x$ by (i)

$= x$ by (J), proving (iii).

For (iv),

$0 \to 0$

$= (x \to x)' \to 0$ by (ii)

$= x \to x$ by (iii).

To prove (v), $(0 \to 0) \to x = (x \to x) \to x = x$, using (iv) and (J).

Finally,

$[(z' \to x) \to (y \to z)']'$

$= (x \to x) \to [(x \to y) \to z]$ by (M)

$= (0 \to 0) \to [(x \to y) \to z]$ by (iv)

$= (x \to y) \to z$ by (v), proving (vi). □

Thus, (I) and (B1) hold in **B** by the preceding lemma, whence Theorem 5.3 is proved, in view of Theorem 4.7.

In 1934, Bernstein([3]) gave an axiom system using only implication; but it was not equational since one of the axioms was existential. It is clear from his proof that the existential statement can be eliminated if we expand the language to include a constant 0, which leads us to the following

Corollary 5.5 (Bernstein [3] modified). *The following (equational) axioms form a 2-base for the variety of Boolean algebras in the language $\{\to, 0\}$, where $x' = x \to 0$:*

(B2) $(u \to u) \to ((x \to y) \to z) \approx [(z' \to x) \to (y \to z)']'$

(J) $(x \to y) \to x \approx x$.

Proof. Let **B** ba an algebra satisfying (B2) and (J). Since (M) (see Theorem 5.3), a special case of (B2), holds in **B**, it follows from Theorem 5.3 that **B** is a Boolean algebra. The converse holds since (B2) and (J) are true in the 2-element Boolean algebra. □

Perhaps, it may be remarked that the base given in Theorem 5.3 contains 3 variables, whereas Bernstein's base (Corollary 5.5) has 4 variables.

We conclude this section by mentioning yet another axiomatization for Boolean algebras (without giving the proof here).

Theorem 5.6. *The following axioms form a base for the variety of Boolean algebras, where $x' = x \to 0$:*

(I) $(x \to y) \to z \approx [(z' \to x) \to (y \to z)']'$

(P) $(x \to x) \to y \approx y$.

6. De Morgan Algebras: From Yet Another Perspective

It is also well known that Boolean algebras can be defined in terms of the NAND operation. In this section we investigate a similar question for De Morgan algebras.

Definition 6.1. An algebra $\mathbf{A} = \langle A, |, 0\rangle$ is a **NAND**-*algebra* if **A** satisfies the following axioms, where we set $x^c := x|x$:

(N1) $0|x \approx 0^c$

(N2) $x^c|(x|y) \approx x$

(N3) $[x|(y^c|z^c)]^c \approx (y|x)|(x|z)$.

Let **NAND** denote the variety of **NAND**-algebras.

We now give a characterization of De Morgan algebras using the operation $|$ and the constant 0.

Theorem 6.2. *The variety* $\mathbf{DM}^{\rightarrow}$ *is term-equivalent to the variety* **NAND**. *More precisely,*

(a) For $\mathbf{A} \in \mathbf{DM}^{\rightarrow}$, let $\mathbf{A}^*$ be the algebra $\langle A, |, 0\rangle$ where $|$ is defined by $x|y := x \rightarrow y'$. Then $\mathbf{A}^* \in \mathbf{NAND}$ such that $x^c = x'$.

(b) For $\mathbf{A} \in \mathbf{NAND}$, let $\mathbf{A}^+$ be the algebra $\langle A, \rightarrow, 0\rangle$ where $\rightarrow$ is defined by $x \rightarrow y := x|y^c$. Then $\mathbf{A}^+ \in \mathbf{DM}^{\rightarrow}$ such that $x' = x^c$.

(c) If $\mathbf{A} \in \mathbf{DM}^{\rightarrow}$, then $(\mathbf{A}^*)^+ = \mathbf{A}$.

(d) If $\mathbf{A} \in \mathbf{NAND}$, then $(\mathbf{A}^+)^* = \mathbf{A}$.

To prove (a), we need the following lemma.

Lemma 6.3. *Let* $\mathbf{A} \in \mathbf{DM}^{\rightarrow}$ *and* $\mathbf{A}^*$ *be as in* (a) (of the above theorem). *Let* x, y, z *be elements of* A.

(i) $0|x = 0'$

(ii) $x'|(x|y) = x$

(iii) $[x \rightarrow (y \rightarrow x)']' = x$

(iv) $(0 \rightarrow x) \rightarrow y = y$

(v) $[(x \rightarrow y) \rightarrow z]' = 0' \rightarrow [(z' \rightarrow x) \rightarrow (y \rightarrow z)']$

(vi) $[x|(y'|z')]' = (y|x)|(x|z)$

(vii) $x' = x|x$.

Proof. First, we note that lemma 4.3 holds in $\mathbf{A}$. Using Lemma 4.3(i) and (J), we have $0|x = 0 \rightarrow x' = [(0 \rightarrow x') \rightarrow 0] \rightarrow 0 = 0 \rightarrow 0 = 0'$, proving (i). Observe that (ii) follows from (i) and (ii) of Lemma 4.3 and (J). For (iii), we have, using (J) and (I), that $x = (x \rightarrow y) \rightarrow x = [(x' \rightarrow x) \rightarrow (y \rightarrow x)']' = [x \rightarrow (y \rightarrow x)']'$. The proof of (iv) already occurs in the first half of the proof of Theorem 5.1.

To prove (v),

$[(x \to y) \to z] \to 0$

$= [(x \to y) \to z]'' \to 0$ by Lemma 4.3(i)

$= 0' \to [(x \to y) \to z]'$ by Lemma 4.3(ii)

$= 0' \to [(z' \to x) \to (y \to z)']''$ by Axiom (I)

$= 0' \to [(z' \to x) \to (y \to z)']$ by Lemma 4.3(i).

For (vi), we have

$[x|(y'|z')]'$

$= [x \to (y' \to z'')']'$

$= [x \to (y' \to z)']'$ by Lemma 4.3(i)

$= [(z' \to y) \to x']'$ by (i) and (ii) of Lemma 4.3

$= 0' \to [(x'' \to z') \to (y \to x')']$ by (v)

$= (x \to z') \to (y \to x')'$ by (iv) and Lemma 4.3(i)

$= (y \to x') \to (x \to z')'$ by (i) and (ii) of Lemma 4.3

$= (y|x)|(x|z)$.

Finally, using (J), Lemma 4.3(i) we have $x' = (x' \to 0) \to x' = x'' \to x' = x \to x' = x|x$, proving (vii).

□

The proof of Theorem 6.2(a) is now complete in view of Lemma 6.3. To prove Theorem 6.2(b) we need the following lemma.

Lemma 6.4. *Let* $\mathbf{A} \in \mathbf{NAND}$. *Let* $\mathbf{A}^+$ *be as in* (b)(of the above theorem). *Let* $x, y, z \in \mathbf{A}^+$. *Then*

(i) $x^{cc} = x$

(ii) $(x|y)|(y|x) = (y|x)^c$

(iii) $x|y = y|x$

(iv) $[x|(y^c|z^c)]^c = (x|y)|(x|z)$

(v) $[(x|y)|(y|z)]^c = y|(x^c|z^c)$

(vi) $x^c = x \to 0$

(vii) $(x \to y) \to z = [(z' \to x) \to (y \to z)']'$

(viii) $(x \to y) \to x = x$.

Proof. (i) is immediate from (N2). Next,

$(x|y)|(y|x)$

$= [y|(x^c|x^c)]^c$ by Axiom (N3)

$= (y|x^{cc})^c$

$= (y|x)^c$ by (i), proving (ii).

To prove (iii),

$x|y$

$= (x|y)^{cc}$ by (i)

$= [(x|y)|(y|x)]|[(y|x)|(x|y)]$ by (ii)

$= (y|x)^c|[(y|x)|(x|y)]$ by (ii)

$= y|x$ by Axiom (N2).

Next,

$[x|(y^c|z^c)]^c$

$= (y|x)|(x|z)$ by Axiom (N3)

$= (x|y)|(x|z)$ by (iii), proving (iv).

To prove (v),

$y|(x^c|z^c)$

$= \{y|(x^c|z^c)\}^c|[\{y|(x^c|z^c)\}|\{y|(x^c|z^c)\}]$ by (N2)

$= [(x|y)|(y|z)]|[\{y|(x^c|z^c)\}|\{y|(x^c|z^c)\}]$ by (N3)

$= [(x|y)|(y|z)]|[(y|x)|(y|z)]$ by (iv)

$= [(x|y)|(y|z)]^c$ by (iii).

Next,

$x \to 0$

$= x|0^c$

$= x|(0|x^c)$ by Axiom (N1)

$= x^{cc}|(x^c|0)$ by (i) and (iii)

$= x^c$ by Axiom (N2), proving (vi).

Next,

$[(z' \to x) \to (y \to z)']'$

$= [(z^c|x^c)|(y|z^c)^{cc}]^c$ by (vi) and definition of $\to$

$= [(z^c|x^c)|(y|z^c)]^c$ by (i)

$= [(x^c|z^c)|(z^c|y)]^c$ by (iii)

$= z^c|(x^{cc}|y^c)$ by (v)

$= (x|y^c)|z^c$ by (iii) and (i)

$= (x \to y) \to z$ by definition of $\to$, proving (vii).

To prove (viii),

$(x \to y) \to x$

$= (x|y^c)|x^c$

$= x^c|(x|y^c)$ by (iii)

$= x$ by Axiom (N2).

□

The proof of Theorem 6.2(b) is complete in view of (vi), (vii), and (viii) of Lemma 6.4. It, then, follows from Theorem 4.2 that the variety **NAND** is also term-equivalent to the variety of De Morgan algebras.

As consequences of Theorem 6.2, we can now define Kleene algebras (and Boolean algebras) using $|$ and 0.

Definition 6.5. A **NAND**-algebra $\mathbf{A} = \langle A, |, 0\rangle$ is a **KAND**-*algebra* if $\mathbf{A}$ satisfies the following axiom, where $x^c = x|x$:

(Q) $(x|x^c)^c|(y|y^c) \approx x|x^c$.

Let **KAND** denote the variety of **KAND**-algebras.

The following theorem is immediate from Theorem 6.2.

Theorem 6.6. *The variety* $\mathbf{KL}^{\to}$ *is term-equivalent to the variety* **KAND**.

Definition 6.7. A **NAND**-algebra $\mathbf{A} = \langle A, |, 0\rangle$ is a **BAND**-*algebra* if $\mathbf{A}$ satisfies the following axiom:

(R) $x|x^c \approx 0^c$.

Let **BAND** denote the variety of **BAND**-algebras.

The following theorem is also immediate from Theorem 6.2.

Theorem 6.8. *The variety* $\mathbf{BA}^{\to}$ *is term-equivalent to the variety* **BAND**.

We conclude that the varieties **KAND** and **BAND** are term-equivalent to Kleene algebras and Boolean algebras respectively.

7. Implication Zroupoids

The observation that the axiom (I) has played a significant role in the earlier sections has led us to investigate it in its own right. In this section we define, and initiate the investigation of, a new equational class of algebras, called "implication zroupoids", which generalize De Morgan algebras.

Definition 7.1. A *groupoid with zero* (*zroupoid*, for short) is an algebra $\mathbf{A} = \langle A, \to, 0, \rangle$, where $\to$ is a binary operation and 0 is a constant. A zroupoid $\mathbf{A} = \langle A, \to, 0, \rangle$ is an *implication zroupoid* (I-*zroupoid*, for short) if the following identities hold in $\mathbf{A}$, where $x' := x \to 0$:

(I) $(x \to y) \to z \approx [(z' \to x) \to (y \to z)']'$

(Z) $0'' \approx 0$.

The variety of I-zroupoids is denoted by $\mathbf{I}$.

It follows from earlier sections that the varieties $\mathbf{BA}^{\to}$(of Boolean algebras), $\mathbf{KL}^{\to}$ (of Kleene algebras) and $\mathbf{DM}^{\to}$ (of De Morgan algebras) are important examples (up to term-equivalence) of subvarieties of $\mathbf{I}$. Examples of I-zroupoids also arise from an entirely different and unexpected source: right distributive groupoids. Recall that a groupoid $\mathbf{G} = \langle G, \cdot \rangle$ is *right distributive* (r-distributive, for short) if $\mathbf{G} \models (x \cdot y) \cdot z \approx (x \cdot z) \cdot (y \cdot z)$.

Definition 7.2. Let $\mathbf{A}$ be an I-zroupoid. We say that $\mathbf{A}$ is *strong right distributive* if the following condition holds in $\mathbf{A}$:

(SRD) $(x \to y) \to z \approx (z \to x) \to (y \to z)$.

$\mathbf{A}$ is *right distributive* if $\mathbf{A}$ satisfies:

(RD) $(x \to y) \to z \approx (x \to z) \to (y \to z)$.

$\mathbf{A}$ is *commutative* if the following condition holds in $\mathbf{A}$:

(C) $x \to y \approx y \to x$.

$\mathbf{A}$ is an I-*zroupoid with right identity* if $\mathbf{A}$ satisfies: $x \to 0 \approx x$ (that is, $x' \approx x$).

$\mathbf{A}$ is an I-*zroupoid with left identity* if $\mathbf{A}$ satisfies: $0 \to x \approx x$.

$\mathbf{A}$ is an I-*zroupoid with identity* if $\mathbf{A}$ satisfies: $x \to 0 \approx x$ and $0 \to x \approx x$.

$\mathbf{A}$ has *trivial multiplication* if $\mathbf{A} \models x \to y \approx 0$, and such a zroupoid is called a *zero zroupoid.* Let $\mathbf{SRD}$ and $\mathbf{RD}$ denote the subvarieties of $\mathbf{I}$ consisting of strong right distributive, right distributive I-zroupoids respectively. $\mathbf{SLD}$ and $\mathbf{LD}$ are defined dually. Also, we let $\mathbf{C}$, $\mathbf{Z}$, $\mathbf{I_{1,0}}$, $\mathbf{I_{lid}}$ and $\mathbf{I_{id}}$ denote the subvarieties of $\mathbf{I}$ consisiting respectively of commutative zroupoids, zero zroupoids, I-zroupoids with right identity, I-zroupoids with left identity and I-zroupoids with identity. **A** is *strong distributive* if $\mathbf{A} \in \mathbf{SLD} \cap \mathbf{SRD}$ and is *distributive* if $\mathbf{A} \in \mathbf{LD} \cap \mathbf{RD}$. We let **D** and **SD** denote the varieties of distributive, and strong distributive I-zroupoids respectively.

One can also define a groupoid to be *strong right distributive*, *strong left distributive*, or *strong distributive* if it satisfies (SRD), or (SLD) or both, respectively. (These concepts for groupoids seem to be new).

There are exactly 3 two-element I-zroupoids as shown in Figure 1. It immediately follows that these are simple and the varieties they generate are atoms in the lattice of subvarieties of **I**. Observe that $\mathbf{2} \in \mathbf{BA}^{\rightarrow}$, $\mathbf{2_z} \in \mathbf{Z}$ and $\mathbf{2_{id}} \in \mathbf{I_{id}} \cap \mathbf{C}$.

$\mathbf{2}$:

$\rightarrow$	0	1
0	1	1
1	0	1

$\mathbf{2_z}$:

$\rightarrow$	0	1
0	0	0
1	0	0

$\mathbf{2_{id}}$:

$\rightarrow$	0	1
0	0	1
1	1	1

Figure 1

Another class of examples of I-zroupoids is obtained as follows: For a set X, let $Su(X)$ denote the set of all subsets of X. Then $\langle Su(X), \cup, \varnothing \rangle$ is a commutative I-zroupoid with identity, if we interpret the $\rightarrow$ as $\cup$ and the constant 0 as $\varnothing$. This class of examples is further generalized in [14].

Remark 7.3. If **A** be an I-zroupoid with right identity, then **A** is strong right distributive.

Let $\mathbf{A} = \langle A, \rightarrow \rangle$ be a strong right distributive groupoid. Let us define an expansion of **A** to be the the algebra $\mathbf{A^e} = \langle A^e, \rightarrow, 0 \rangle$, where $A^e = A \cup \{0\}$, with 0 not in A, where $\rightarrow$ is extended to A^e by setting $x \rightarrow 0 = x$, for $x \in A^e$. Then it is clear that $\mathbf{A^e}$ is an I-zroupoid with right identity.

The following lemmas will be used in proving our main result of this section that the identity $x''' \rightarrow y \approx x' \rightarrow y$ holds in every I-zroupoid.

Lemma 7.4. *Let* $\mathbf{A} \in \mathbf{I}$ *and let* $x, y, z \in \mathbf{A}$. *Then*

(1) $[(0' \rightarrow x) \rightarrow y'']' = (x \rightarrow y)'$

(2) $(0' \rightarrow x)'' = x''$

(3) $[x'' \rightarrow \{y \rightarrow (0' \rightarrow x)\}']' = (0 \rightarrow y) \rightarrow (0' \rightarrow x)$

(4) $[x \to (0' \to y)]' = (x \to y)'$

(5) $(0 \to x) \to (0' \to y) = (0 \to x) \to y$.

Proof. (1) is immediate from (I), and (2) follows easily from (I) and (Z). For (3), we have, using (I) and (2), that $(0 \to y) \to (0' \to x) = [(0' \to x)'' \to \{y \to (0' \to x)\}']' = [x'' \to \{y \to (0' \to x)\}']'$. Next, $[x \to (0' \to y)]' = [(0' \to x) \to (0' \to y)'']' = (0' \to x) \to y'' = (x \to y)'$, in view of (I), (2) and (1), which proves (4). Using (I), (4) and (3), $(0 \to x) \to y = [y'' \to (x \to y)']' = [y'' \to \{x \to (0' \to y)\}']' = (0 \to x) \to (0' \to y)$, which proves (5). □

Lemma 7.5. *Let $x, y, z \in \mathbf{A} \in \mathbf{I}$. Then*

(a) $[(0' \to x) \to y]' = (x \to y)'$

(b) $(x \to y'')' = (x \to y)'$

(c) $(x \to y) \to z = [x \to (0' \to y)] \to z$

(d) $(x \to y) \to z = [(0' \to x) \to y] \to z$

(e) $(x \to y)' = [(0' \to x) \to y]'$

(f) $[x \to (0' \to y)']' = (x \to y')'$.

Proof. $[(0' \to x) \to y]' = [\{(0 \to 0) \to (0' \to x)\} \to y'']' = [\{(0 \to 0) \to x\} \to y'']' = (x \to y)'$, in view of (I), and (5) and (1) of the preceding lemma, thus proving (a). For (b), $(x \to y'')' = [(0' \to x) \to y'']' = (x \to y)'$ by (a) and Lemma 7.4(1). (c) follows from (I) and (a). Next, using (I) and (c), we have $[(0' \to x) \to y] \to z = [\{z' \to (0' \to x)\} \to (y \to z)']' = [(z' \to x) \to (y \to z)']' = (x \to y) \to z$, proving (d), which, together with (I), implies (e), as is easily seen. (f) can be proved using (I), Lemma 7.4(2) and (b). □

Theorem 7.6. *Let $x, y \in \mathbf{A} \in \mathbf{I}$. Then $x''' \to y = x' \to y$.*

Proof. $x' \to y = (0' \to x)' \to y = (0' \to x'')' \to y = x''' \to y$ in view of (d), (b) of the preceding Lemma. □

The following corollary is now immediate.

Corollary 7.7. *Let* $\mathbf{A}$ *be an* I*-zroupoid. Then* $x'''' = x''$*.*

8. Subvarieties of I

In this section we wish to introduce several important subvarieties of $\mathbf{I}$ and establish some relationships among them.

In view of Corollary 7.7, one is naturally led to ask if the identity $x''' \approx x'$ also holds in $\mathbf{I}$. The answer, however, turns to be negative, since the identity $x''' = x'$ fails in the 3-element I-zroupoid whose $\rightarrow$-table is given in Figure 2.

$\rightarrow$	0	a	1
0	0	0	0
a	1	0	0
1	0	0	0

Figure 2

Definition 8.1. Let $\mathbf{I_{3,1}}$, $\mathbf{I_{2,0}}$ and $\mathbf{I_{1,0}}$ denote, respectively, the subvarieties of $\mathbf{I}$ satisfying $x''' \approx x'$, $x'' \approx x$, and $x' \approx x$.

Let **TII** denote the subvariety of $\mathbf{I}$ satisfying the identity:

(TII) $0' \rightarrow (x \rightarrow y) \approx x \rightarrow y$. (**TII** is abbreviated from: " **T**ruth **I**mplies **I**mplication".)

Our first goal in this section is to show that $\mathbf{TII} \subset \mathbf{I_{3,1}}$. To that end we need the following lemma.

Lemma 8.2. *Let* $\mathbf{A} \in \mathbf{TII}$*. Let* $x, y, z \in \mathbf{A}$*. Then*

(a) $(0' \rightarrow x)' = x'''$

(b) $[(x \rightarrow y) \rightarrow z]'' = (x \rightarrow y) \rightarrow z$

(c) $0' \rightarrow x = x''$.

Proof. (a) follows immediately from Lemma 7.5(b) and (TII). To prove (b),

$[(x \rightarrow y) \rightarrow z]''$

$= [(z' \rightarrow x) \rightarrow (y \rightarrow z)']'''$ by (I)

$= 0' \rightarrow [(z' \rightarrow x) \rightarrow (y \rightarrow z)]'$ by (a)

$= [(z' \to x) \to (y \to z)]'$ by (TII)

$= (x \to y) \to z$ by (I).

Finally, using (b), (a), and Corollary 7.7, we have $0' \to x = (0' \to x)'' = x'''' = x''$, proving (c). □

Theorem 8.3. $\mathbf{TII} \subset \mathbf{I_{3,1}}$.

Proof. Let $x \in \mathbf{A} \in \mathbf{TII}$. Then $x' = 0' \to x'$ by (TII), from which we get $x' = x'''$ by (c) of the preceding lemma, implying $\mathbf{A} \subseteq \mathbf{I_{3,1}}$. The example in Figure 3 (with 0 as the constant) shows that the inclusion is proper, since $0' \to (a \to a) = 0$ and $a \to a = 1$. □

$\to$	0	a	1
0	0	0	0
a	0	1	0
1	0	0	0

Figure 3

Definition 8.4. Let $\mathbf{A} \in \mathbf{I}$. $\mathbf{A}$ is *strong contrapositive* if $\mathbf{A}$ satisfies the identity:

(SCP) $x \to y \approx y' \to x'$.

$\mathbf{A}$ is *contrapositive* if $\mathbf{A}$ satisfies the identity:

(CP) $x \to y' \approx y \to x'$.

$\mathbf{A}$ is *weak contrapositive* if $\mathbf{A}$ satisfies the identity:

(WCP) $x' \to y \approx y' \to x$.

Let **SCP**, **CP** and **WCP** denote the subvarieties of **I** consisting of strong contrapositive, contrapositive and weak contrapositive I-zroupoids respectively.

It is clear that $\mathbf{DM}^{\to} \subset \mathbf{SCP}$ in view of (i) and (ii) of Lemma 4.3. Next, we wish to show that $\mathbf{SCP} \subset \mathbf{CP} \subset \mathbf{WCP}$.

Lemma 8.5. $\mathbf{SCP} \subset \mathbf{CP}$.

Proof. Let $x, y \in \mathbf{A} \in \mathbf{SCP}$. Using (SCP) and (Z), we have,
$y \to x' = x'' \to (0' \to y') = x'' \to y''' = y'' \to x' = x \to y'$. The example in Figure 3 shows that the inclusion is proper, since $a' \to a' = 0$, while $a \to a = 1$. □

Lemma 8.6. *Let $x, y, z \in \mathbf{A} \in \mathbf{CP}$. Then,*

(i) $(x'' \to y) \to z = (x \to y) \to z$
(ii) $(x \to y) \to z' = [(z \to x) \to (z \to y')']'$
(iii) $(x \to y) \to z'' = (x \to y) \to z.$

Proof. To prove (i), we have
$(x'' \to y) \to z$
$= [(y \to z) \to (z' \to x'')']'$ by (I) and (CP)
$= [(y \to z) \to (z' \to x)']'$ by Lemma 7.5 (b)
$= [(z' \to x) \to (y \to z)']'$ by (CP)
$= (x \to y) \to z$ by (I).
For (ii), we have
$(x \to y) \to z'$
$= [(z'' \to x) \to (y \to z')']'$ by (I)
$= [(z'' \to x) \to (z \to y')']'$ by (CP)
$= [(z \to x) \to (z \to y')']'$ by (i).
To prove (iii),
$(x \to y) \to z$
$= [(z' \to x) \to (y \to z)']'$ by (I)
$= [(z' \to x) \to (y \to z'')']'$ by Lemma 7.5 (b)
$= [(z' \to x) \to (z' \to y')']'$ by (CP)
$= (x \to y) \to z''$ by (ii). □

Theorem 8.7. $\mathbf{SCP} \subset \mathbf{CP} \subset \mathbf{WCP}$.

Proof. First half is already proved in Lemma 8.5. By (iii) of the preceding Lemma and (CP), we get

$y' \to x = y' \to x'' = x' \to y'' = x' \to y$. That the inclusion in the second half is proper is shown by the example in Figure 2 (with $x = 0$ and $y = a$); thus proving the second half. $\square$

It may be noted here that the algebra given in Figure 2 is also in **WCP**, hence it follows that $\mathbf{WCP} \not\subseteq \mathbf{I_{3,1}}$. Next, we will show that **SCP** is also a (proper) subvariety of **TII**, for which we need the following

Lemma 8.8. *Let* $x, y, z \in \mathbf{A} \in \mathbf{SCP}$. *Then*
$0' \to \{(x \to y) \to z\} = (x \to y) \to z$.

Proof. Using (I) and (SCP) we have $(x \to y) \to z = [(z' \to x) \to (y \to z)']' = 0' \to [(z' \to x) \to (y \to z)']' = 0' \to [(x \to y) \to z]$. $\square$

Theorem 8.9. $\mathbf{SCP} \subset \mathbf{TII}$.

Proof. Let $x, y \in \mathbf{A} \in \mathbf{SCP}$. Then, using the preceding lemma and (SCP), we have $x \to y = x'' \to y'' = (x' \to 0) \to y'' = 0' \to (x'' \to y'') = 0' \to (x \to y)$, implying that $\mathbf{A} \in \mathbf{TII}$. The algebra in Figure 4 (with $x = a$ and $y = a$) is an example to show that the inclusion is proper. $\square$

We would like to note here that the algebra given in Figure 3 is in **CP** also, hence $\mathbf{CP} \not\subseteq \mathbf{TII}$. However, we have the following

Theorem 8.10. $\mathbf{CP} \subset \mathbf{I_{3,1}}$.

Proof. Using (Z), (CP), and Lemma 8.6(iv), we have $x' = x \to 0'' = 0' \to x' = 0' \to x''' = x'' \to 0'' = x'''$. The algebra given in Figure 4 (with $x = a$ and $y = 1$) shows that the inclusion is proper. $\square$

$\to$	0	a	1
0	0	a	1
a	1	a	1
1	a	a	1

Figure 4

Observe, however, that $\mathbf{WCP} \not\subseteq \mathbf{I_{3,1}}$, which can be verified using the algebra given in Figure 4.

Lemma 8.11. *Let* $\mathbf{A} \in \mathbf{I_{2,0}}$. *Let* $a, b, c \in \mathbf{A}$. *Then*

(a) $a''' = a'$

(b) $(a \to b) \to c'' = (a \to b) \to c$.

Proof. (a) is immediate from Lemma 7.5(b). To prove (b), we have

$(a \to b) \to c''$

$= [(c''' \to a) \to (b \to c'')']'$ by (I)

$= [(c' \to a) \to (b \to c)'$ by Lemma 7.5(b) and (a)

$= (a \to b) \to c$. □

Lemma 8.12. *Let* $\mathbf{A} \in \mathbf{I}$. *Then T.F.A.E.:*

(a) $\mathbf{A} \models 0' \to x \approx x$.

(b) $\mathbf{A} \in \mathbf{I_{2,0}}$.

Proof. Suppose (a) holds; then, from (a) of the preceding lemma, we have $x'' = 0' \to x'' = 0' \to x = x$, proving (b). It is clear from Lemma 7.4(2) that (b) implies (a); thus (a) and (b) are equivalent. □

It is now clear that $\mathbf{I_{2,0}} \subset \mathbf{TII}$, as the algebra $\mathbf{2_z}$ is in $\mathbf{TII}$ but not in $\mathbf{I_{2,0}}$.

Lemma 8.13. *Let* $\mathbf{A} \in \mathbf{I_{2,0}}$. *Let* $x, y, z \in \mathbf{A}$. *Then*

(a) $(y \to z) \to x' = [(x \to y) \to (z \to x')']'$

(b) $(0 \to y) \to x = [x \to (y \to x)']'$

(c) $x \to 0' = (0 \to x) \to 0'$

(d) $0 \to 0' = 0'$

(e) $x' \to 0' = 0 \to x$

(f) $0 \to x' = x \to 0'$.

Proof. $(y \to z) \to x' = [(x'' \to y) \to (z \to x')']' = [(x \to y) \to (z \to x')']'$, (b) is immediate from (I) and the identity $0' \to x \approx x$, in view of the preceding lemma. Now, using (a), the identity $0' \to x \approx x$, and the hypothesis, we get $x \to 0' = (x \to 0')'' = [0' \to (x \to 0')']' = (0 \to x) \to 0'$, proving (c). $0 \to 0' = (0 \to 0) \to 0' = 0'$ by (c) and the identity $0' \to x \approx x$, proving (d). $x' \to 0' = [(0 \to x) \to (0 \to 0')']' = [(0 \to x) \to 0'']' = [(0 \to x) \to 0]' = (0 \to x)'' = 0 \to x$, by (a), (d), and the hypothesis, proving (e), from which (f) is immediate. □

Lemma 8.14. *Let* $\mathbf{A}$ *be an* I-*zroupoid such that* $\mathbf{A} \models x' \to x \approx x$. *Let* $x, y, z \in \mathbf{A}$. *Then*

(i) $(x \to y) \to (0' \to z)' = (x \to y) \to z'$

(ii) $(0' \to x)' = x'$

(iii) $(x \to y) \to (0' \to z) = (x \to y) \to z$

(iv) $0' \to x = x$.

Proof.

$$\begin{aligned}(x \to y) \to (0' \to z)' &= [\{(0' \to z)'' \to x\} \to \{y \to (0' \to z)'\}']' \quad \text{by (I)}\\ &= [(z'' \to x) \to \{y \to (0' \to z)'\}']' \quad \text{by Lemma 7.4(2)}\\ &= [(z'' \to x) \to (y \to z')']' \quad \text{by Lemma 7.5(f)}\\ &= (x \to y) \to z' \quad \text{by (I),}\end{aligned}$$

proving (i).

$$\begin{aligned}(0' \to x)' &= (0' \to x)'' \to (0' \to x)' \quad \text{by hypothesis}\\ &= x'' \to (0' \to x)' \quad \text{by Lemma 7.4(2)}\\ &= x'' \to x \quad \text{by (i)}\\ &= x \quad \text{by hypothesis,}\end{aligned}$$

proving (ii). To prove (iii), we have

$$\begin{aligned}(x \to y) \to (0' \to z) &= [\{(0' \to z)' \to x\}' \to \{y \to (0' \to z)\}']' \quad \text{by (I)}\\ &= [(z' \to x)' \to (y \to z)']' \quad \text{by (ii) and Lemma 7.4(4)}\\ &= (x \to y) \to z \quad \text{by (I),}\end{aligned}$$

Finally,

$$\begin{aligned}0' \to x &= (0' \to x)' \to (0' \to x) \quad \text{by hypothesis}\\ &= x' \to (0' \to x) \quad \text{by (ii)}\\ &= x' \to x \quad \text{by (iii)}\\ &= x \quad \text{by hypothesis,}\end{aligned}$$

which proves (iv). $\square$

The following theorem gives several characterizations of the variety $\mathbf{I_{2,0}}$.

Theorem 8.15. *Let* $\mathbf{A}$ *be a* I*-zroupoid. Then T.F.A.E.:*

(a) $\mathbf{A} \models 0' \to x \approx x$

(b) $\mathbf{A} \in \mathbf{I_{2,0}}$

(c) $\mathbf{A} \models (x \to x')' \approx x$

(d) $\mathbf{A} \models x' \to x \approx x$.

Proof. The equivalence of (a) and (b) is already proved in Lemma 8.12. Let $x \in \mathbf{A}$. Assume (b) holds. Then, by Lemma 8.12, $0' \to x = x$. So, using (b), and Lemma 8.13(d), we have $(x \to x')' = (x'' \to x')' = [(x' \to 0) \to (0' \to x)']' = (0 \to 0') \to x = 0' \to x = x$, thus proving (b) implies (c). Next, suppose (c) holds. Then $x = (x \to x')' = [(0' \to x) \to x''']' = (x'' \to x''')' = x''$ by (I) and Lemma 8.2, proving (b); thus (b) and (c) are equivalent. Finally, we show that (a) and (d) are equivalent. We note that (d) implies (a) by (iv) of the preceding lemma. If (a) holds in $\mathbf{A}$ then (b) and (c) also hold in $\mathbf{A}$ (as proved earlier), from which it is easy to prove (d), thus completing the proof. □

We now introduce another important subvariety of $\mathbf{I}$.

Definition 8.16. An I-zroupoid $\mathbf{A} \in \mathbf{I}$ is called a *left associative 3-potent* if $\mathbf{A}$ satisfies the identity:

(LAP) $(x \to x) \to x \approx x$.

Let **LAP** be the (equational) class of all left associative 3-potent I-zroupoids.

Our next goal is to show that **LAP** is, in fact, a (proper) subvariety of $\mathbf{I_{2,0}}$.

Lemma 8.17. *Let* $x, y, z \in \mathbf{A} \in \mathbf{I}$. *Then*

(i) $[x \to (0' \to y)]' = (x \to y)'$

(ii) $(0 \to x) \to (0' \to y) = (0 \to x) \to y$

(iii) $0' \to (0' \to x) = 0' \to x$

(iv) $[(0' \to x) \to y]' = (x \to y)'$

(v) $[x \to (0' \to y)] \to z = (x \to y) \to z$

(vi) $[(0' \to x) \to y] \to z = (x \to y) \to z$

(vii) $[x \to \{y \to (0' \to z)\}]' \to z = [x \to (y \to z)]'$

(viii) $(x \to y) \to [z \to (0' \to u)] = (x \to y) \to (z \to u)$.

Proof. (i) is proved using (I) and Lemma 7.4(2). To prove (ii), use (I), Lemma 7.4(2) and (i). (iii) is a particular case of (ii), and (iv) can be proved using (ii), (I) and (iii). (v) follows from (I) and (iv), and (vi) from (I) and (v). For (vii), use (I) and (i). To prove (viii),

$$\begin{aligned}(x \to y) \to (z \to u) &= [\{(z \to u)' \to x\} \to \{y \to (z \to u)\}']' \quad \text{by (I)}\\ &= [\{(z \to u)' \to x\} \to \{y \to (z \to (0' \to u)\}']' \quad \text{by (vii)}\\ &= [\{(x \to (0' \to y))' \to z] \to \{u \to (x \to (0' \to y))\}']' \text{ by (i)}\\ &= (x \to y) \to [z \to (0' \to u)] \quad \text{by (I).}\end{aligned}$$

□

We now turn to the variety **LAP**.

Lemma 8.18. *Let* $x, y, z \in \mathbf{A} \in \mathbf{LAP}$. *Then* $x \to (0' \to y) = x \to y$.

Proof. By (LAP) and Lemma 8.17(v) we obtain that $x \to (0' \to y) = [\{x \to (0' \to y)\} \to \{x \to (0' \to y)\}] \to \{x \to (0' \to y)\} = [(x \to y) \to \{x \to (0' \to y)\}] \to \{x \to (0' \to y)\}$. Hence, by Lemma 8.17(viii) and Lemma 8.17(viii) and (LAI), we have $x \to (0' \to y) = [(x \to y) \to (x \to y)] \to \{x \to (0' \to y)\} = [(x \to y) \to (x \to y)] \to (x \to y) = x \to y$. □

We are ready to prove the theorem that we had set out to prove.

Theorem 8.19. $\mathbf{LAP} \subset \mathbf{I_{2,0}}$.

Proof.

$$\begin{aligned}0' \to x &= [(0' \to x) \to (0' \to x)] \to (0' \to x) \quad \text{by (LAP)}\\ &= [x \to (0' \to x)] \to (0' \to x) \quad \text{by Lemma 8.17(vi)}\\ &= (x \to x) \to (0' \to x) \quad \text{by Lemma 8.17(v)}\\ &= (x \to x) \to x \quad \text{by the preceding lemma}\\ &= x \quad \text{by (LAP),}\end{aligned}$$

thus proving that $\mathbf{LAP} \subseteq \mathbf{I_{2,0}}$. The example in Figure 5 shows that the inclusion is proper (take $x = a$). $\square$

$\to$	0	a	b	1
0	0	a	b	1
a	1	b	b	1
b	b	b	b	b
1	a	a	b	b

Figure 5

Definition 8.20. An I-zroupoid $\mathbf{A} \in \mathbf{I}$ is called *idempotent* if $\mathbf{A}$ satisfies the identity:

(IDMP) $x \to x \approx x$.

Let **IDMP** denote the subvariety of $\mathbf{I}$ consisting of idempotent I-zroupoids.

We now hope to show that $\mathbf{I_{1,0}} \subset \mathbf{IDMP} \subset \mathbf{LAP}$. But, first we need the following

Lemma 8.21. *Let $x, y, z \in \mathbf{A} \in \mathbf{I_{1,0}}$. Then*

(i) $(x \to y) \to (z \to x) = (y \to z) \to x$

(ii) $(0 \to x) \to y = x \to y$

(iii) $x \to (y \to x) = y \to x$

(iv) $(x \to y) \to (0 \to x) = y \to x$

(v) $0 \to x = x$

(vi) $(x \to y) \to x = y \to x$

(vii) $(x \to y) \to x = x \to (y \to x)$.

Proof. (i) is immediate from (I) and the hypothesis. For (ii), using the hypothesis and (i), we get $(0 \to x) \to y = (0 \to x) \to y' = (x \to y)' = x \to y$. Use the hypothesis, (i) and (ii) to get $x \to (y \to x) = x' \to (y \to x) = (0 \to y) \to x = y \to x$, proving (iii), while (iv) is immediate from (i) and the hypothesis. Using the hypothesis and (ii), we have $0 \to x = (0 \to x)' = x' = x$, thus proving (v). For (vi), $(x \to y) \to x = (x \to y) \to (0 \to x) = y \to x$, using (vi) and (iv), while (vii) is immediate from (iii) and (vi). $\square$

Theorem 8.22. $\mathbf{I_{1,0}} \subset \mathbf{IDMP} \subset \mathbf{LAP}$.

Proof. Let $x \in \mathbf{A} \in \mathbf{I_{1,0}}$. Now, using (v) and (iii) of the preceding lemma, we have $x \to x = x \to (0 \to x) = 0 \to x = x$, thus $\mathbf{I_{1,0}} \subseteq \mathbf{IDMP}$; furthermore, this inclusion is proper as the example in Figure 4 shows. The proof of the the second half is trivial, since the Boolean algebra $\mathbf{2}$ is in $\mathbf{LAP}$ but not in $\mathbf{IDMP}$. □

The following corollary summarizes the relationships among the subvarieties of $\mathbf{I}$ introduced so far.

Corollary 8.23. *We have*

(a) $\mathbf{BA}^{\to} \subset \mathbf{KA}^{\to} \subset \mathbf{DM}^{\to} \subset \mathbf{LAP} \subset \mathbf{I_{2,0}} \subset \mathbf{TII} \subset \mathbf{I_{3,1}} \subset \mathbf{I}$

(b) $\mathbf{I_{1,0}} \subset \mathbf{IDMP} \subset \mathbf{LAP}$

(c) $\mathbf{BA}^{\to} \subset \mathbf{KA}^{\to} \subset \mathbf{DM}^{\to} \subset \mathbf{SCP} \subset \mathbf{CP} \subset \mathbf{WCP} \subset \mathbf{I}$

(d) $\mathbf{Z} \subset \mathbf{C}$.

These investigations on (the lattice of) subvarieties of $\mathbf{I}$ and other related algebras are continued in [14] and [15]. It is shown, among other things, in [14] that $\mathbf{SRD} \subset \mathbf{RD}$ and $\mathbf{C} \subset \mathbf{CP} \cap \mathbf{SD}$.

9. Concluding Remarks

We will conclude this paper with some open problems to stimulate further research.

PROBLEM 1: Find a 1-base for De Morgan algebras in the language $\{\to, 0\}$.
PROBLEM 2: Find a 1-base for De Morgan algebras in the language $\{|, 0\}$
PROBLEM 3: Is their a characterization of Stone algebras in the language $\{\to, 0\}$?
PROBLEM 4: Is their a characterization of Stone algebras in the language $\{|, 0\}$.
PROBLEM 5: Is the lattice of subvarieties of I-zroupoids distributive?
PROBLEM 6: Investigate the lattice of subvarieties of $\mathbf{I}$.
PROBLEM 7: Investigate the strong distributive groupoids.

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