GLOBAL EXISTENCE AND DECAY FOR SOME NONLINEAR SECOND ORDER ORDINARY DIFFERENTIAL EQUATIONS OF DEGENERATE TYPE

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Abstract. In this paper we prove the existence and decay of global solutions with small initial data for a nonlinear second order ordinary differential equation of the form $\ddot{x}(t)+\rho(\dot{x}(t))+g(x(t))=f(x,\dot{x},t)$, where $\rho(y)$ behaves as $|y|^r y, r\geq 0$, and $g(x)$ as $|x|^p x, p\geq 0$, in a neighborhood of the origin $(x,y)=(0,0)$, and $f(x,\dot{x},t)$ is a nonlinear pertubation.

1 Introduction. In this paper we are concerned with the initial value problem to second order nonlinear ordinary differential equations of the form

$$\ddot{x}+\rho(\dot{x})+g(x)=f(x,\dot{x},t), 0\geq 0, \tag{1.1}$$

with

$$x(0)=x_0, \dot{x}(0)=x_1, \tag{1.2}$$

where $\rho(y)$ behaves as $|y|^r y, r>0$, and $g(x)$ as $|x|^p x, p>0$, in a neighborhood of the origin $(x,y)=(0,0)$.

For the nonlinear perturbation term $f(x,y,t)$ we assume that as follows:

Hyp.A

(1) $f(x,y,t)$ is continuous in $(x,y,t)\in \mathbf{R}^2\times[0,\infty)$ and satisfies a local Lipshitz condition,

$$|f(x_1,y_1,t)-f(x_2,y_2,t)|\leq L(|x_1-x_2|+|y_1-y_2|) \text{ if } (x_i,y_i)\in B(R), i=1,2,$$

where $R>0$ is a fixed positive number and we set

$$B(R)=\{(x,y)\in\mathbf{R}^2 | |x|^2+|y|^2<R^2\}$$

.

(2)

$$|f(x,y,t)|\leq k(|x|^{\alpha+1}+|y|^{\beta+1}) \text{ if } (x,y)\in B(R)$$

for some $\alpha>0$ and $\beta>0$ with $k>0$.

If both of $\rho(y)$ and $g(x)$ are linear, that is, if $\rho(y)=ay, a>0$, and $g(x)=bx, b>0$, it is easy to show that the problem admits a unique solution $x(t)$ for small data (x_0,x_1), and

$$E(t)\equiv\frac{1}{2}(|\dot{x}(t)|^2+|x(t)|^2)\leq CE(0)e^{-\lambda t}$$

with some $\lambda>0$, where C is a constant . Indeed, it is an easy application of the constant variational formula to semilinear equation. But, such a method is not applicable to the case

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$\rho(y)$ and/or $g(x)$ are nonlinear with $\rho'(0) = 0$ and/or $g'(0) = 0$, that is, $\rho(y)$ and/or $g(x)$ are degenerate at $x = y = 0$.

Further, we note that when f is independent of $\dot{x}$ and t, that is, when $f = f(x)$, the global existence is easily proved for small data (x_0, x_1) by use of the property of the potential function $U(x) = G(x) - F(x)$ where $F'(x) = f(x), G'(x) = g(x)$. But, such a standard argument can not be generally applied if f depends on $\dot{x}$ and/or t.

Our equation (1.1) is a rather standard nonlinear second order ordinary differential equation naturally suggested by the typical weakly nonlinear equation $\ddot{x} + \dot{x} + x = f(x)$. However, no result concerning the global existence is found in standard books(cf. [3],[4],[5] and [6]), and our purpose here is to show the global existence and decay for the problem (1.1) by a new device.

When $\rho(y) = |y|^r y, g(x) = |x|^p x$ and $f \equiv 0$ the problem admits a unique solution $x(t)$ for each $(x_0, x_1) \in \mathbf{R}^2$ and we know

$$E(t) \equiv \frac{1}{2}|\dot{x}(t)|^2 + \frac{1}{p+2}|x(t)|^{p+2} \leq C_0(1+t)^{-\mu} \tag{1.3}$$

with $\mu = (p+2)/(pr+p+r)$ where C_0 is a constant depending on $E(0)$ (see [8]). It is easy to see that the decay rate is optimal if $p = 0$. For the special case $\rho(y) = y$, $g(x) = x^3$ and $f \equiv 0$ the asymptotic behaviour of solutions is precisely investigated by Ball and Carr [2]. By a result of [2] we see that the decay rate in (1.3) is also optimal if $r = 0$ and $p = 2$. In the present paper we refine the argument in [8] and show the global existence of small amplitude solutions satisfying the property (1.3) for the problem (1.1)-(1.2). For generality we consider in fact the equation of the form

$$\ddot{x} + \rho(x, \dot{x}, t) + g(x) = f(x, \dot{x}, t), t \geq 0, \tag{1.1$'$}$$

where $\rho(x, y, t)$ is a function which behaves as $|y|^r y$ in a neighbourhod of the origin $(x, y) = (0, 0)$.

2 Statements of main results.

First we state the precise assumptions on $\rho(x, y, t)$ and $g(x)$.

Hyp.B $\rho(x, y, t)$ is locally Lipshitz continuous on $B(R), R > 0$, for each t , continuous on $B(R) \times [0, \infty)$ and satisfies

$$k_0|y|^{r+2} \leq \rho(x, y, t)y \leq k_1(|y|^2 + |y|^{r+2}), \tag{2.1}$$

for some $r \geq 0$ and some constants $k_0, k_1 > 0$ independent of t.

Hyp.C $g(x)$ is a Lipshitz continuous function on $[-R, R], R > 0$, satisfying

$$g(0) = 0 \text{ and } k_2|x|^{p+2} \leq G(x) \leq k_3 g(x)x \leq k_4(|x|^2 + |x|^{p+2}), \tag{2.2}$$

for some $p \geq 0$ with some positive constants $k_i, i = 2, 3, 4$.

It is easy to see that the problem admits a unique local in time solution $x(t)$ for each $(x_0, x_1) \in B(R)$. Our result reads as follows:

Theorem 2.1 *Assume that*

$$2\alpha > 3p + 2r(p+2) \textit{ and } \beta > 2(pr+p+r)/(p+2).$$

Then under assumptions Hyp.A,Hyp.B and Hyp.C, there exists $\delta > 0$ such that if $(x_0, x_1) \in B(\delta)$, the problem (1.1)'-(1.2) admits a unique global solution $x(t)$ on $[0, \infty)$, satisfying

$$E(t) \equiv \frac{1}{2}|\dot{x}(t)|^2 + G(x(t)) \leq C_0(1+t)^{-\mu} \tag{2.3}$$

with $\mu = (p+2)/(pr+p+r)$ where $G(x) = \int_0^x g(\xi)d\xi$ and C_0 denotes a constant depending on $|x_0| + |x_1|$. (When $p = r = 0$ (2.3) is replaced by an exponential decay estimate.)

For illustration we give three nontrivial examples.

Example1. Consider

$$\ddot{x} + \dot{x}^3 + x^3 = x^8\dot{x}^2. \tag{2.4}$$

By Young's inequality we see

$$|x^8y^2| \leq \frac{16}{25}(|x|^{25/2} + |y|^{50/9}).$$

In this case we can take $p = r = 2, \alpha = 23/2 > 11$ and $\beta = 41/9 > 4$, and we see that the required conditions are satisfied. Thus , if $|x_0| + |x_1|$ is sufficiently small the above problem admits a unique solution on $[0, \infty)$. The solutions satisfy

$$E(t) = \frac{1}{2}|\dot{x}(t)|^2 + \frac{1}{4}|x(t)|^4 \leq C_0(1+t)^{-1/2}.$$

Example 2. Consider

$$\ddot{x}(t) + (1 - |x|^2)\dot{x} + |x|x = x^3(t)\text{sin}t. \tag{2.6}$$

Since

$$|x^3\text{sin}t| \leq |x|^3$$

and

$$\frac{3}{4}|y|^2 \leq |(1-|x|^2)y^2| \leq |y|^2 \text{ if } |x| \leq 1/2$$

we can take $r = 0, p = 1, \alpha = 2 > 3/2$ and $\beta =$ arbitrary $> 2/3$. Thus , if $|x_0| + |x_1|$ is sufficiently small the problem admits a unique solution on $[0, \infty)$. The solutions satisfy

$$E(t) = \frac{1}{2}|\dot{x}(t)|^2 + \frac{1}{3}|x(t)|^3 \leq C_0(1+t)^{-3}.$$

Example 3. Consider

$$\ddot{x}(t) + (1 - |x|)|\dot{x}|\dot{x} + x = x^2\dot{x}. \tag{2.7}$$

Since

$$\frac{1}{2}|y|^3 \leq |(1-|x|)|y|y^2| \leq |y|^3 \text{ if } |x| \leq 1/2$$

and

$$|x|^2|y| \leq \frac{4}{7}(|x|^{7/2} + |y|^{7/3})$$

we can take $r = 1, p = 0, \alpha = 5/2 > 2$ and $\beta = 4/3 > 1$. Thus , if $|x_0| + |x_1|$ is sufficiently small the problem admits a unique solution on $[0, \infty)$. The solutions satisfy

$$E(t) = \frac{1}{2}(|\dot{x}(t)|^2 + |x(t)|^2) \leq C_0(1+t)^{-2}.$$

3 Estimation for $0 \le t \le 1$**.** Let $(x_0, x_1) \in B(R)$. We can assume that the solution $x(t)$ exists on $[0, T)$ for some $T > 0$. In this section we show that T can be taken as $T > 1$ if $|x_0| + |x_1|$ is appropriately small.

Proposition 3.1 *There exists* $\delta_0 > 0$ *such that if* $0 < E(0) \le \delta_0$*, then*

$$E(t) < K < 2E(0) \text{ if } 0 \le t < T \text{ and } 0 \le t \le 1 \tag{3.1}$$

where $E(t)$ *is defined in (2.3) and* $K > 0$ *is a constant independent of* T*.*

Proof.

Suppose that $E(t) \le 2E(0)$ and $(x(t), \dot{x}(t)) \in B(R)$ on $[0, \hat{T}]$ for some $\hat{T}$ such that $\hat{T} < T$ and $\hat{T} \le 1$. Then, multiplying the equation by $\dot{x}(t)$ we see

$$\begin{aligned} E(t) &\le E(0) + \int_0^t |f(x, \dot{x}, t)\dot{x}| ds \\ &\le E(0) + C \int_0^t (|x(s)|^{\alpha+1} |\dot{x}(s)| + |\dot{x}(s)|^{\beta+2}) ds \\ &\le E(0) + C \sup_{0 \le s \le \hat{T}} (E^{(\alpha+1)/(p+2)+1/2}(s) + E^{(\beta+2)/2}(s)) \\ &\le E(0) + C \left((2E(0))^{(2\alpha-p)/2(p+2)} + (2E(0))^{\beta/2} \right) \sup_{0 \le s \le \hat{T}} E(s), 0 \le t \le \hat{T}, \end{aligned} \tag{3.2}$$

where C denotes constants independent of $x(t)$ which may change from line to line.

Assume that

$$C \left((2E(0))^{(2\alpha-p)/2(p+2)} + (2E(0))^{\beta/2} \right) < 1/2$$

which is satisfied if $E(0) \le \delta_0$ with a small $\delta_0 > 0$(note that $2\alpha > p$ and $\beta > 0$). Then we have from (3.2) that if $E(0) \ne 0$,

$$\sup_{0 \le s \le \hat{T}} E(s) < K < 2E(0) \tag{3.3}$$

for some $K > 0$ independent of $\hat{T}$ and T. Since δ_0 is small the estimate (3.3) implies that if $0 < E(0) < \delta_0$, then $x(t)$ exists and $E(t)$ never reaches the value K on the interval $[0, T)$ or $[0, 1]$. Consequently, the assertion (3.1) holds. **q.e.d.**

The estimate (3.1) is independent of T and hence it shows that the solution $x(t)$ exists in fact beyond $t = 1$, that is, we can take $T > 1$. Further, by the estimate (3.3) we see that if $0 < E(0) \le \delta_0$, then

$$E(t) < 2E(0) \text{ for } 0 \le t \le \tilde{T}. \tag{3.1$'$}$$

for some $\tilde{T}, 1 < \tilde{T} < T$.

4 A difference inequality. We assume $E(0) < \delta_0$ as in Proposition 3.1. We derive a difference inequality for $E(t)$ on $[0, T)$ by use of the similar arguments as in [7, 8] which will be useful to show the boundedness and decay of $E(t)$.

Multiplying the equation by $\dot{x}(t)$ and integrating we have for $0 \le t < T - 1$,

$$\int_t^{t+1} \rho(x(s), \dot{x}(s), s)\dot{x}(s) ds$$

$$= E(t) - E(t+1) + \int_t^{t+1} F(s)\dot{x}(s)ds \equiv D(t)^2, \tag{4.1}$$

where we set $F(t) = f(x(t), \dot{x}(t), t)$.

By the assumption on ρ ,

$$\int_t^{t+1} |\dot{x}(s)|^{r+2} ds \le CD(t)^2 \tag{4.2}$$

and hence,

$$\int_t^{t+1} |\dot{x}(s)|^2 ds \le CD(t)^{4/(r+2)}. \tag{4.3}$$

From (4.3) there exist $t_1 \in [t, t+1/4], t_2 \in [t+3/4, t+1]$ such that

$$|\dot{x}(t_i)| \le CD(t)^{2/(r+2)}.$$

(Note that C may change from line to line.)

Multiplying the equation by $x(t)$ and integrating on $[t_1, t_2]$ we have

$$\int_{t_1}^{t_2} g(x(s))x(s)ds = \dot{x}(t_1)x(t_1) - \dot{x}(t_2)x(t_2)$$

$$+ \int_{t_1}^{t_2} |\dot{x}(s)|^2 ds + \int_{t_1}^{t_2} (F(s)x(s) - \rho(x(s), \dot{x}(s), s)x(s))ds$$

$$\le C\left(D(t)^{2/(r+2)} \sup_{t\le s\le t+1} |x(s)| + D(t)^{4/(r+2)} + D(t)^{2(r+1)/(r+2)} \sup_{t\le s\le t+1} |x(s)|\right)$$

$$+C\int_t^{t+1} |F(s)||x(s)|ds. \tag{4.4}$$

It follows from (4.3),(4.4) and the assumption (2.2) that

$$\int_{t_1}^{t_2} E(s)ds \le C\left(D(t)^{2/(r+2)} + D(t)^{2(r+1)/(r+2)}\right) \sup_{t\le s\le t+1} E(s)^{1/(p+2)}$$

$$+CD(t)^{4/(r+2)} + +C\int_t^{t+1} |F(s)||x(s)|ds \equiv A(t)^2 \tag{4.5}$$

and there exists $t^* \in [t+1/4, t+3/4]$ such that

$$E(t^*) \le 2A(t)^2.$$

Returning to the energy identity as (4.1), that is,

$$E(\hat{t}) + \int_{t^*}^{\hat{t}} \rho(x(s), \dot{x}(s), s)\dot{x}(s)ds = E(t^*) + \int_{t^*}^{\hat{t}} F(x(s))\dot{x}(s)ds, t \le \hat{t} \le t+1,$$

we have

$$\sup_{t\le s\le t+1} E(s) \le E(t^*)$$

$$+ \int_t^{t+1} \rho(x(s), \dot{x}(s), s))\dot{x}(s)ds + \int_t^{t+1} |F(s)\dot{x}(s)|ds$$

$$\le 2A(t)^2 + D(t)^2 + \int_t^{t+1} |F(s)\dot{x}(s)|ds. \tag{4.6}$$

Recalling the definition of $A(t)^2$ and applying Young's inequality we arrive at the following difference inequality.

Proposition 4.1 *We have*

$$\sup_{t\le s\le t+1} E(s) \le C\Big(D(t)^{2(p+2)/(p+1)(r+2)} + D(t)^{2(r+1)(p+2)/(p+1)(r+2)}$$

$$+D(t)^{4/(r+2)} + D(t)^2\Big) + \int_t^{t+1} |F(s)|(|\dot{x}(s)| + |x(s)|)ds, 0 \le t < T-1, \tag{4.7}$$

where $D(t)^2$ *is defined by (4.1).*

5 Boundedness of $E(t)$ on $[0,\infty)$. We shall derive the boundedness of $E(t)$ on $[0,\infty)$ from the difference inequality (4.7). We assume for a moment that

$$E(0) < \delta_0 \text{ and } E(t) \le 2E(0) \text{ for } 0 \le t \le \tilde{T}, 1 < \tilde{T} < T. \tag{5.1}$$

By (3.1)' we see that (5.1) is certainly valid for some $\tilde{T} > 1$. We shall show that (5.1) is valid in fact for all $\tilde{T}, \tilde{T} < T$. For this we use an idea similar to the one used in [9](see also Amerio and Prouse [1]).

We suppose that $E(t) \le E(t+1)$ for some $0 \le t \le \tilde{T} - 1$. Then it follows from (4.7) that

$$\sup_{t\le s\le t+1} E(s)$$

$$\le C\{\Big((\int_t^{t+1} |F(s)\dot{x}(s)|ds\Big)^{(p+2)/(p+1)(r+2)} + \Big(\int_t^{t+1} |F(s)\dot{x}(s)|ds\Big)^{(r+1)(p+2)/(p+1)(r+2)}$$

$$+\Big(\int_t^{t+1} |F(s)\dot{x}(s)|ds\Big)^{2/(r+2)} + \Big(\int_t^{t+1} |F(s)|(|\dot{x}(s)| + |x(s)|)ds\Big)\}. \tag{5.2}$$

Here, we see by **Hyp A**,(2),

$$\int_t^{t+1} |F(s)\dot{x}(s)|ds \le C\int_t^{t+1} (|x(s)|^{\alpha+1} + |\dot{x}(s)|^{\beta+1})|\dot{x}(s)|ds$$

$$\le C(\sup_{t\le s\le t+1} E(s)^{(2\alpha+p+4)/2(p+2)} + \sup_{t\le s\le t+1} E(s)^{(\beta+2)/2}). \tag{5.3}$$

Similarly,

$$\int_t^{t+1} |F(s)||x(s)|ds \le C\int_t^{t+1} (|x(s)|^{\alpha+1} + |\dot{x}(s)|^{\beta+1})|x(s)|ds$$

$$\le C(\sup_{t\le s\le t+1} E(s)^{(\alpha+2)/(p+2)} + \sup_{t\le s\le t+1} E(s)^{((\beta+1)(p+2)+2)/2(p+2)}). \tag{5.4}$$

We have from (5.1), (5.2),(5.3) and (5.4) that

$$\sup_{t\le s\le t+1} E(s) \le C\{\sup_{t\le s\le t+1} E(s)^{(2\alpha+p+4)/2(p+1)(r+2)} + \sup_{t\le s\le t+1} E(s)^{(\beta+2)(p+2)/2(p+1)(r+2)}$$

$$+ \sup_{t\le s\le t+1} E(s)^{(2\alpha+p+4)(r+1)/2(p+1)(r+2)} + \sup_{t\le s\le t+1} E(s)^{(\beta+2)(p+2)(r+1)/2(p+1)(r+2)}$$

$$+ \sup_{t\le s\le t+1} E(s)^{(2\alpha+p+4)/(p+2)(r+2)} + \sup_{t\le s\le t+1} E(s)^{(\beta+2)/(r+2)}$$

$$+ \sup_{t\le s\le t+1} E(s)^{(2\alpha+p+4)/2(p+2)} + \sup_{t\le s\le t+1} E(s)^{(\beta+2)/2}\}$$

$$+C(\sup_{t\le s\le t+1} E(s)^{(\alpha+2)/(p+2)} + \sup_{t\le s\le t+1} E(s)^{((\beta+1)(p+2)+2)/2(p+2)}).$$

$$\le Q_1(E(0)) \sup_{t\le s\le t+1} E(s), 0 \le t \le \tilde{T}-1, \tag{5.5}$$

where we set

$$Q_1(E(0)) = C\sum_{i=1}^{5}(E(0)^{\mu_i} + E(0)^{\nu_i}) \tag{5.6}$$

with

$$\mu_1 = \frac{2\alpha - 3p - 2(p+1)r}{2(p+1)(r+2)}, \mu_2 = \frac{(2\alpha+p+4)r + 2\alpha - 3p - 2(p+1)r}{2(p+1)(r+2)},$$

$$\mu_3 = \frac{2\alpha - p - pr - 2r}{(p+2)(r+2)}, \mu_4 = \frac{2\alpha - p}{2(p+2)}, \mu_5 = \frac{\alpha - p}{p+2}$$

and

$$\nu_1 = \frac{(p+2)\beta - 2(pr+p+r)}{2(p+1)(r+2)}, \nu_2 = \frac{(\beta+2)r + (p+2)\beta - 2(pr+p+r)}{2(p+1)(r+2)},$$

$$\nu_3 = \frac{\beta - r}{r+2}, \nu_4 = \frac{\beta}{2}, \nu_5 = \frac{\beta(p+2)-p}{2(p+2)}.$$

By the assumptions $2\alpha > 3p+2(p+1)r$ and $(p+2)\beta > 2(pr+p+r)$ we see that $\mu_i > 0$ and $\nu_i > 0$ for all $i = 1, \cdots, 5$, and hence, $Q_1(0) = 0$. Therefore there exists $0 < \delta_1 \le \delta_0$ such that if $E(0) < \delta_1$, then $Q_1(E(0)) < 1$. Thus, we obtain from (5.5) that $E(s) = 0, t \le s \le t+1$. Note that (5.5) is derived under the assumption that $E(t) \le E(t+1)$. Consequently, we see that $E(t+1) \le E(t)$ for all $0 \le t \le \tilde{T}-1$ if $E(0) < \delta_1$, and this fact implies further that

$$E(t) \equiv 0 \text{ or } E(t) \le \sup_{0\le s\le 1} E(s) < 2E(0), 0 \le t \le \tilde{T}. \tag{5.7}$$

Note that the estimate (5.7) is proved under the assumption (5.1) and further (5.7) is more strict or better than (5.1). From this observation we can conclude that the solution in fact exists on $[0,\infty)$ and satisfies the estimate (5.7) for all $t \ge 0$.

6 Completion of the proof of Theorem. We shall show the decay estimate of $E(t)$ and complete the proof of Theorem 2.1. We return to the difference inequality (4.7), which is now valid for all $t \ge 0$. Since $E(t+1) \le E(t)$ for all $t \ge 0$ we have from (4.7) and (5.4) that

$$\sup_{t\le s\le t+1} E(s) \le C\Big(D_0(t)^{2(p+2)/(p+1)(r+2)} + D_0(t)^{2(p+2)(r+1)/(p+1)(r+2)}$$

$$+D_0(t)^{4/(r+2)} + D_0(t)^2\Big) + CQ_1(E(0))\sup_{t\le s\le t+1} E(s) \tag{6.1}$$

where we set

$$D_0(t)^2 = E(t) - E(t+1).$$

Now we make the further assumption which is essentially the same as $Q_1(E(0)) < 1$,

$$CQ_1(E(0)) \le \frac{1}{2}. \tag{6.2}$$

The condition (6.2) is realized if

$$E(0) \le \delta_2 \tag{6.3}$$

for some small $\delta_2, 0 < \delta_2 < \delta_1 \leq \delta_0$. Then we have from (6.1) that

$$\sup_{t \leq s \leq t+1} E(s)$$

$$\leq C\left(D_0(t)^{2(p+2)/(p+1)(r+2)} + D_0(t)^{2(p+2)(r+1)/(p+1)(r+2)} + D_0(t)^{4/(r+2)} + D_0(t)^2\right)$$

$$\leq C_0 D_0^{2(p+2)/(p+1)(r+2)}, \tag{6.4}$$

where C_0 denotes constants depending on $E(0)$. It follows from (6.4) that

$$\sup_{t \leq s \leq t+1} E(s)^{1+(p+r+pr)/(p+2)} \leq C_0(E(t) - E(t+1)), 0 \leq t < \infty. \tag{6.5}$$

Applying the lemma below we obtain the decay estimate

$$E(t) \leq C_0(1+t)^{-(p+2)/(p+r+pr)}.$$

When $p = r = 0$ we have the usual exponential decay $E(t) \leq C_0 e^{-\lambda t}$ with some $\lambda > 0$. The proof of Theorem3.1 is complete.

Lemma 6.1 *([7]) Let $\phi(t)$ be a positive function on $[0, T), T > 1$, satisfying $\phi(t+1) \leq \phi(t)$ and*

$$\sup_{t \leq s \leq t+1} \phi(s)^{1+\gamma} \leq k\left(\phi(t) - \phi(t+1)\right), 0 \leq t \leq T-1$$

with $k > 0, \gamma > 0$. Then

$$\phi(t) \leq \left((\sup_{0 \leq s \leq 1} \phi(s))^{-\gamma} + \frac{\gamma}{k}(t-1)^+\right)^{-1/\gamma}$$

$$\leq C_0(1+t)^{-1/\gamma}, 0 \leq t < T,$$

where C_0 is a constant depending on $\sup_{0 \leq s \leq 1} \phi(s)$ and k. (When $\gamma = 0$ we have $\phi(t) \leq C_0 e^{-\lambda t}$ with some $\lambda > 0$.)

Remark. The proof of the above lemma is given in [7] under the assumption that $\phi(t)$ is nonincreasing in t. But it is valid if $\phi(t+1) \leq \phi(t)$ for each $t \geq 0$.

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