QUASI-COMMUTATIVE WEAK BCC-ALGEBRAS

Bushra Karamdin* and Janus Thomys**

Received February 2, 2012; revised February 9, 2012

Abstract. We describe weak BCC-algebras (called also BZ-algebras) in which the condition $(xy)z = (xz)y$ is satisfied only in the case when elements x, y belong to the same branch. We also characterize quasi-commutative weak BCC-algebras various types.

1 Introduction BCK-algebras which are a generalization of the notion of algebra of sets with the set subtraction as the only fundamental non-nullary operation, and on the other hand, the notion of implication algebra (cf. [17]) were defined by Imai and Iséki in [15]. The class of all BCK-algebras does not form a variety. To prove this fact Y.Komori introduced in [18] the new class of algebras called BCC-algebras. In view of strongly connections with a BIK$^+$-logic, BCC-algebras also are called BIK$^+$-algebras (cf. [22] or [23]). Nowadays, the mathematicians especially from China, Japan and Korea, have been studying various generalizations of BCC-algebras such as, for example, B-algebras, difference algebras, implication algebras, GB-algebras, Hilbert algebras, d-algebras and many others. All these algebras have one distinguished element and satisfy some common identities playing a crucial role in these algebras.

One of very important identities is the identity $(xy)z = (xz)y$. It holds in BCK-algebras and in some generalizations of BCK-algebras, but not in BCC-algebras. BCC-algebras satisfying this identity are BCK-algebras (cf. [6] or [7]). Therefore, it makes sense to consider such BCC-algebras and some their generalizations for which this identity is satisfied only by elements belonging to some subsets. Such study has been initiated by W.A. Dudek in [9].

On the other hand, many mathematicians investigate BCI-algebras in which some basic properties are restricted to some subset called branches. For example, branchwise commutative BCI-algebras were described in [2], branchwise implicative and branchwise positive implicative BCI-algebras in [3] and [4]. But, as it was observed many years ago, results obtained for BCI-algebras can not be transferred to weak BCC-algebras.

Below we begin the study of weak BCC-algebras in which the condition $(xy)z = (xz)y$ is satisfied only in the case when elements x, y belong to the same branch.

2 Basic definitions and facts The BCC-operation will be denoted by juxtaposition. Dots will be used only to avoid repetitions of brackets. For example, the formula $((xy)(zy))(xz) = 0$ will be written in the abbreviated form as $(xy \cdot zy) \cdot xz = 0$.

Definition 2.1. A *weak BCC-algebra* is a system $(G; \cdot, 0)$ of type $(2, 0)$ satisfying the following axioms:

(i) $(xy \cdot zy) \cdot xz = 0$,

(ii) $xx = 0$,

2010 *Mathematics Subject Classification.* 03G25, 06F35.

Key words and phrases. weak BCC-algebra, BCC-algebra, quasi-commutative BCC-algebra, branch.

(*iii*) $x0 = x$,

(*iv*) $xy = yx = 0 \Longrightarrow x = y$.

A weak BCC-algebra satisfying the identity

(*v*) $0x = 0$

is called a *BCC-algebra.* A BCC-algebra with the condition

(*vi*) $(x \cdot xy)y = 0$

is called a *BCK-algebra.*

One can prove (see [6]) that *a BCC-algebra is a BCK-algebra if and only if it satisfies the identity*

(*vii*) $xy \cdot z = xz \cdot y$.

An algebra $(G; \cdot, 0)$ of type $(2, 0)$ satisfying the axioms (*i*), (*ii*), (*iii*), (*iv*) and (*vi*) is called a *BCI-algebra.* A BCI-algebra satisfies also (*vii*). A weak BCC-algebra is a BCI-algebra if and only if it satisfies (*vii*).

A BCC-algebra which is not BCK-algebra is called *proper.* Similarly, a weak BCC-algebra which is not a BCC-algebra is called *proper* if it is not a BCI-algebra. A proper BCC-algebra has at least four elements (see [7]). Direct computation shows that there exist 45 distinct proper BCC-algebras of order four. Each of these BCC-algebras is isomorphic to one of eight proper BCC-algebras mentioned in [7]. One can prove (see [6]) that for every natural $n \geqslant 4$ there exists at least one proper BCC-algebra containing n elements. Proper weak BCC-algebras also have at least four elements (see [8]). But there are only two non-isomorphic weak BCC-algebras of order four:

*	0	1	2	3
0	0	0	2	2
1	1	0	2	2
2	2	2	0	0
3	3	3	1	0

Table 2.1.

*	0	1	2	3
0	0	0	2	2
1	1	0	3	3
2	2	2	0	0
3	3	3	1	0

Table 2.2.

They are proper, because in both cases $(3 * 2) * 1 \neq (3 * 1) * 2$.

The methods of construction of weak BCC-algebras proposed in [8] show that for every $n \geqslant 4$ there exist at least two non-isomorphic proper weak BCC-algebras of order n.

Any weak BCC-algebra can be considered as a partially ordered set. In any weak BCC-algebra we can define a natural partial order $\leqslant$ putting

$$x \leqslant y \Longleftrightarrow xy = 0. \tag{1}$$

This means that a weak BCC-algebra can be considered as a partially ordered set with some additional properties.

Proposition 2.2. *An algebra $(G; \cdot, 0)$ of type $(2, 0)$ with a relation $\leqslant$ defined by* (1) *is a weak BCC-algebra if and only if for all $x, y, z \in G$ the following conditions are satisfied:*

(*i′*) $xy \cdot zy \leqslant xz$,

(*ii′*) $x \leqslant x$,

(iii') $x0 = x$,

(iv') $x \leqslant y$ *and* $y \leqslant x$ *imply* $x = y$. □

Since two non-isomorphic weak BCC-algebras may have the same partial order, they cannot be investigated as partially ordered sets only. For example, weak BCC-algebras defined by Tables 2.1 and 2.2 have the same partial order but they are not isomorphic.

¿From (i') it follows that in weak BCC-algebras implications

$$x \leqslant y \Longrightarrow xz \leqslant yz \tag{2}$$

$$x \leqslant y \Longrightarrow zy \leqslant zx \tag{3}$$

are satisfied by all $x, y, z \in G$.

In the investigations of algebras connected with various types of logics an important role plays the so-called *Dudek's map* φ defined as $\varphi(x) = 0x$. The main properties of this map in the case of weak BCC-algebras are collected in the following theorem proved in [12].

Theorem 2.3. *Let G be a weak BCC-algebra. Then*

(1) $\varphi^2(x) \leqslant x$,

(2) $x \leqslant y \Longrightarrow \varphi(x) = \varphi(y)$,

(3) $\varphi^3(x) = \varphi(x)$,

(4) $\varphi^2(xy) = \varphi^2(x)\varphi^2(y)$,

(5) $\varphi^2(xy) = \varphi(yx)$,

(6) $\varphi(x)(yx) = \varphi(y)$

for all $x, y \in G$. □

The set

$$B(a) = \{x \in G : a \leqslant x\},$$

where $a \in G$ is fixed, is called a *branch* of G initiated by a. A branch $B(a)$ is *proper* if $B(b) = B(a)$ for every $b \leqslant a$. The set of initial elements of all proper branches of a weak BCC-algebra G is denoted by $I(G)$. Elements of $I(G)$ are called *initial.* A branch containing only initial element is called *trivial.*

Theorem 2.4. $I(G) = \{a \in G : \varphi^2(a) = a\}$. □

The proof of this theorem is given in [10]. Comparing this result with Theorem 2.3 (4) we obtain

Corollary 2.5. *$I(G)$ is a subalgebra of G.* □

Corollary 2.6. *$I(G) = \varphi(G)$ for any weak BCC-algebra G.*

Proof. Indeed, if $x \in \varphi(G)$, then $x = \varphi(y)$ for some $y \in G$. Thus, by Theorem 2.3, $\varphi^2(x) = \varphi^3(y) = \varphi(y) = x$. Hence $\varphi^2(x) = x$, i.e., $x \in I(G)$. So, $\varphi(G) \subset I(G)$.

Conversely, for $x \in I(G)$ we have $x = \varphi^2(x) = \varphi(\varphi(x)) = \varphi(y)$, where $y = \varphi(x) \in G$. Thus $I(G) \subset \varphi(G)$, which completes the proof. □

Corollary 2.7. *An element a of a weak BCC-algebra G is its initial element if and only if there exists an element $x \in G$ such that $a = \varphi(x)$.* □

This means that *the first row of the multiplication table determining a weak BCC-algebra contains only initial elements.*

According to Corollary 2.7 each element satisfying the condition $\varphi(a) = a$ is initial, but this condition is not characteristic for initial elements, i.e., there are initial elements for which $\varphi(a) \neq a$.

Example 2.8. By computer we can check that the following table defines a weak BCC-algebra.

$*$	0	a	b	c	d	e
0	0	0	0	d	c	d
a	a	0	a	d	c	d
b	b	b	0	d	c	d
c	c	c	c	0	d	0
d	d	d	d	c	0	c
e	e	c	e	a	d	0

Table 2.3.

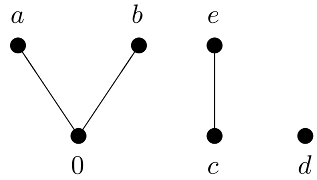

Diagram 2.3.

This weak BCC-algebra has three initial elements: 0, c, d. But $\varphi(c) \neq c$ and $\varphi(d) \neq d$. □

Corollary 2.9. *$\varphi(a) = a$ if and only if $\varphi(x) \leqslant x$ for every $x \in B(a)$.*

Proof. Let $\varphi(a) = a$ for some $a \in G$. Then $\varphi^2(a) = a$, so $a \in I(G)$. Hence for every $x \in B(a)$ we have $a \leqslant x$. From this, applying Theorem 2.3, we obtain $\varphi(x) = \varphi(a) = a \leqslant x$.

Conversely, $\varphi(x) \leqslant x$ for every $x \in B(a)$ means that also $\varphi(a) \leqslant a$. Since a is a minimal element in $B(a)$, the last implies $\varphi(a) = a$. □

The branch initiated by 0, i.e., the set

$$B(0) = \{x \in G : 0 \leqslant x\}$$

is called a *BCC-part* of a weak BCC-algebra G.

One can show (cf. [10]) that $B(0)$ is the greatest BCC-algebra contained in a weak BCC-algebra G.

3 Congruences and ideals

In many algebras congruences are uniquely determined by some subsets. For example, congruences of groups are determined by normal subgroups, congruences of rings – by ideals.

In weak BCC-algebras the situation is more complicated. Indeed, as it was observed many years ago (cf. for example [14] or [19]) the kernel

$$\rho(0) = \{x \in G \,:\, x\rho 0\}$$

of a congruence ρ on a BCK-algebra G has the following property: $y \in \rho(0), xy \in \rho(0)$ imply $x \in \rho(0)$. Moreover, if A is an ideal of BCK-algebra G, then A determines some congruence of G, but there are congruences which are not determined by such subsets (cf. [21]).

According to [14] and [17], we say that a subset A of a BCK-algebra G is an *ideal* of G if

(1) $0 \in A$,

(2) $y \in A$ and $xy \in A$ imply $x \in A$.

Such defined ideal is an ideal in the sense of ordered sets. The relation

$$x\theta y \iff xy,\, yx \in A \tag{4}$$

is a congruence on a BCK-algebra G. Unfortunately it is not true for weak BCC-algebras (cf. [11]). In connection with this fact, W. A. Dudek and X. H. Zhang introduced in [11] the new concept of ideals. Now, in the literature these new ideals are called *BCC-ideals*, old ideals are called *ideals* or *BCK-ideals*.

Definition 3.1. A non-empty subset A of a weak BCC-algebra G is called a *BCC-ideal* if
(1) $0 \in A$,
(2) $y \in A$ and $xy \cdot z \in A$ imply $xz \in A$.

By putting $z = 0$ we can see that a BCC-ideal is a BCK-ideal. In a BCK-algebra any BCK-ideal is a BCC-ideal, but there are BCC-algebras with BCC-ideals which are not BCK-ideals (cf. [11]).

Proposition 3.2. *$B(0)$ is a BCC-ideal of each weak BCC-algebra.*

Proof. Obviously $0 \in B(0)$. Let $xy \cdot z,\, y \in B(0)$. Then $0 \leqslant xy \cdot z$ and $0 \leqslant y$. From the last inequality, by (2) and (3) we obtain $xy \cdot z \leqslant xz$, which implies $0 \leqslant xz$ and consequently $xz \in B(0)$. □

Each BCC-ideal of a BCC-algebra G is a kernel of some congruence on G, and conversely, each BCC-ideal of G determines some congruence on G. Similarly to BCK-algebras in infinite BCC-algebras there are congruences which are not determined by BCC-ideals. In finite BCC-algebras all congruences are determined by BCC-ideals (cf. [11]).

For a congruence θ an equivalence class containing an element x is denoted by C_x^θ. The quotient algebra $G/\theta = \{C_x^\theta : x \in G\}$ satisfies all axioms of a weak BCC-algebra except (iv). This axiom is satisfied only in some cases.

Definition 3.3. The congruence θ defined on a weak BCC-algebra G is called *regular* if and only if $C_x^\theta \cdot C_y^\theta = C_y^\theta \cdot C_x^\theta = C_0^\theta$ implies $C_x^\theta = C_y^\theta$.

Regular congruences are characterized by BCC-ideals.

Proposition 3.4. *A congruence of a weak BCC-algebra is regular if and only if it is defined by some BCC-ideal.*

Proof. The proof of this proposition is identical with the proof given in [11] for BCC-algebras. □

Example 3.5. The relation $\sim$ defined on a weak BCC-algebra G by

$$x \sim y \iff \varphi(x) = \varphi(y)$$

is an equivalence on G. Moreover, if $x \sim y$ and $u \sim v$, then $\varphi(x) = \varphi(y)$, $\varphi(u) = \varphi(v)$. Hence, by Theorem 2.3, we obtain

$$\varphi(ux) = \varphi^2(xu) = \varphi^2(x)\varphi^2(u) = \varphi^2(y)\varphi^2(v) = \varphi^2(yv) = \varphi(yv),$$

which implies $ux \sim vy$. Thus $\sim$ is a congruence. It is clear that the corresponding quotient algebra $G/{\sim} = \{C_x : x \in G\}$ satisfies the first three conditions of Definition 2.1. Moreover, if $C_x \cdot C_y = C_y \cdot C_x = C_0$ for some $C_x, C_y \in G/\sim$, then $\varphi(xy) = \varphi(yx) = \varphi(0) = 0$. This by Theorem 2.3 implies

$$\varphi^2(y)\varphi^2(x) = \varphi^2(yx) = \varphi(xy) = 0 = \varphi(yx) = \varphi^2(xy) = \varphi^2(x)\varphi^2(y).$$

Therefore $\varphi^2(x) = \varphi^2(y)$, and consequently $\varphi(x) = \varphi^3(x) = \varphi^3(y) = \varphi(y)$. Thus, $C_x = C_y$. Hence $G/\sim$ is a weak BCC-algebra and $\sim$ is a regular congruence. □

Proposition 3.6. *The congruence $\sim$ coincides with the congruence induced by $B(0)$.*

Proof. Indeed, if $x \sim y$, then $\varphi(x) = \varphi(y)$ and, by Theorem 2.3,

$$\varphi(xy) = \varphi^2(yx) = \varphi^2(y)\varphi^2(x) = 0,$$

i.e., $0 \leqslant xy$. Hence $xy \in B(0)$. Similarly, $yx \in B(0)$. Thus $x\theta y$, where θ is defined by (4) with $A = B(0)$.

Conversely, let $x\theta y$, where θ is defined by (4) with $A = B(0)$. Then $xy, yx \in B(0)$ and consequently $\varphi(xy) = \varphi(xy) = 0$. Thus

$$0 = \varphi^2(xy) = \varphi^2(x)\varphi^2(y).$$

Analogously, $0 = \varphi^2(x)\varphi^2(y)$. This implies $\varphi^2(x) = \varphi^2(y)$. Therefore

$$\varphi(x) = \varphi^3(x) = \varphi^3(y) = \varphi(y),$$

which proves $x \sim y$. □

Proposition 3.7. *The class C_x coincides with the branch containing x.*

Proof. Let $x \in G$ and $y \in C_x$. Then by Corollary 2.6 $\varphi(y) = \varphi(x) = a \in I(G)$ and so by Theorems 2.4 and 2.3, we obtain $a = \varphi^2(a) = \varphi^2(y) \leqslant y$, which implies $y \in B(a)$. Thus $C_x \subset B(a)$.

Now let $z \in B(a)$. Then $a \leqslant z$ and, by Theorem 2.4, $\varphi(a) = \varphi(z)$. Thus

$$\varphi(z) = \varphi(a) = \varphi^3(a) = \varphi(\varphi^2(a)) = \varphi(\varphi^2(y)) = \varphi^3(y) = \varphi(y)$$

for any $y \in C_x$. Hence $z \in C_x$, i.e., $B(a) \subset C_x$. Consequently, $C_x = B(a)$ for $a = \varphi(x)$. □

Corollary 3.8. *Branches of a weak BCC-algebra coincide with the equivalence classes of a congruence induced by its BCC-part $B(0)$, i.e., $B(a) = C_a$ for any $a \in I(G)$.* □

Corollary 3.9. *Let G be a weak BCC-algebra and $a, b \in I(G)$. Then*

$$B(a)B(b) = B(ab).$$ □

As a simple consequence of the above results we obtain the following characterization of elements belonging to the same branch. This characterization was firstly presented in [10] with another proof.

Corollary 3.10. *Elements $x, y \in G$ are in the same branch if and only if $xy \in B(0)$.*

Proof. If $x, y \in B(a)$, then $x, y \in C_a$, so $xy, yx \in B(0)$. Conversely, if $xy \in B(0)$, then, by (i'), we have $0 = 0 \cdot xy = yy \cdot xy \leqslant yx$, which means that $yx \in B(0)$. Thus $x, y \in C_a$ for some $a \in I(G)$. Corollary 3.8 completes the proof. □

Corollary 3.11. *Comparable elements are in the same branch.* □

Proposition 3.12. *If $x, y \in B(a)$, then also $x \cdot xy$ and $y \cdot yx$ are in $B(a)$.*

Proof. Let $x, y \in B(a)$. Then $xy, yx \in B(0)$. Thus $0 \leqslant xy$ and $0 \leqslant yx$. From this, using (3), we obtain $x \cdot xy \leqslant x$ and $y \cdot yx \leqslant y$. Corollary 3.11 completes the proof. □

Proposition 3.13. *Let G be a weak BCC-algebra. The sum of all branches $B(a)$ of G such that $a \in A \subset I(G)$ is a subalgebra of G if and only if A is a subalgebra of G.*

Proof. Let S be the sum of all branches $B(a)$ of G such that $a \in A$. Obviously $a \in S$. If S is a subalgebra of G, then $0 \in B(a)$ for some $a \in A$. Since $0 \in B(a)$ only in the case when $a = 0$, we obtain $0 \in A$. Now let $a, b \in A$. Then $a, b \in S$, and consequently $ab \in S \cap I(G) = A$ (Corollary 2.5). Hence A is a subalgebra of G.

Conversely, if A is a subalgebra of G, then $0 \in A \subset S$. Moreover, for any $x, y \in S$ there are $a, b \in A$ such that $x \in B(a)$ and $y \in B(b)$. Thus $xy \in B(a)B(b) = B(ab)$. But $ab \in A$, so $B(ab) \subset S$. Hence $xy \in S$. □

4 Group-like weak BCC-algebras One of important classes of weak BCC-algebras is the class of the so-called *group-like weak BCC-algebras* called also *anti-grouped BZ-algebras* [24]. It is a subclass of group-like BCI-algebras described in [5] and [20].

Definition 4.1. A weak BCC-algebra is *group-like* if all its branches are trivial.

This means that a group-like weak BCC-algebra contains only incomparable elements. From results proved in [5] it follows that such BCC-algebras are strongly connected with groups (see also [24]). The connection between group-like weak BCC-algebras and groups is given in the theorem presented below.

Theorem 4.2. *A weak BCC-algebra $(G; \cdot, 0)$ is group-like if and only if $(G; *, e)$, where $e = 0$ and $x * y = x \cdot 0y$, is a group. Moreover, in this case $xy = x * y^{-1}$.* □

It is not difficult to see that if in the above theorem a group $(G; *, e)$ is abelian then the corresponding weak BCC-algebra is a BCI-algebra. Thus, a group-like weak BCC-algebra is proper if and only if it is induced by a non-abelian group.

The conditions under which a weak BCC-algebra is group-like are found in [10]. These conditions are presented below.

Theorem 4.3. *A weak BCC-algebra G is group-like if and only if at least one of the following conditions is satisfied:*

(1) $\varphi^2(x) = x$ *for all* $x \in G$,

(2) $\varphi(xy) = yx$ *for all* $x, y \in G$,

(3) $xy \cdot zy = xz$ *for all* $x, y, z \in G$,

(4) $\operatorname{Ker}\varphi = \{0\}$,

(5) $xy = zy$ *implies* $x = z$ *for all* $x, y, z \in G$,

(6) $xy = 0$ *implies* $x = y$ *for all* $x, y \in G$. □

As a consequence of Theorems 2.4 and 4.3 we obtain

Corollary 4.4. *A weak BCC-algebra G is group-like if and only if $G = I(G)$, or equivalently, if and only if $G = \varphi(G)$.* □

Corollary 4.5. *$\varphi(G)$ is a maximal group-like subalgebra of each weak BCC-algebra G.*

Proof. By Corollaries 2.5 and 2.6 $\varphi(G) = I(G)$ is a subalgebra. By Corollary 4.4 it is group-like. To prove it is maximal, let us consider an arbitrary group-like subalgebra A of G. Then, by Theorem 4.3, for any $x \in A$ we have $x = \varphi^2(x)$, i.e., $x = \varphi(\varphi(x))$ which means that $x \in \varphi(G)$. Thus $A \subset \varphi(G)$ for any group-like subalgebra A of G. Hence $\varphi(G)$ is a maximal group-like subalgebra of G. □

As a simple consequence of Theorem 4.2 we obtain

Corollary 4.6. *ρ is a congruence of a group-like weak BCC-algebra if and only if it is a congruence of the corresponding group.* □

5 Solid weak BCC-algebras

Definition 5.1. A weak BCC-algebra $(G; \cdot, 0)$ is called *solid*, if for all x and y belonging to the same branch the identity

(vii) $\quad xy \cdot z = xz \cdot y$

is satisfied. If this identity is satisfied also in the case when y, z are in the same branch, then we say such a weak BCC-algebra is *super solid*.

All BCI-algebras and all BCK-algebras are solid weak BCC-algebras. A solid weak BCC-algebra containing only one branch is a BCK-algebra. But there are solid weak BCC-algebras which are not BCI-algebras. For example, a proper weak BCC-algebra defined by Table 2.1 is solid but it is not super solid. A weak BCC-algebra defined by Table 2.2 is not solid because in this algebra we have $(3 * 2) * 3 \neq (3 * 3) * 2$.

Theorem 5.2. *In solid weak BCC-algebras the map φ is a homomorphism.*

Proof. Indeed,

$$\begin{aligned}\varphi(x)\varphi(y) &= 0x \cdot 0y = ((xy \cdot xy)x) \cdot 0y = ((xy \cdot x) \cdot xy) \cdot 0y \\ &= ((xx \cdot y) \cdot xy) \cdot 0y = (0y \cdot xy) \cdot 0y = (0y \cdot 0y) \cdot xy \\ &= 0 \cdot xy = \varphi(xy)\end{aligned}$$

for all $x, y \in G$. □

Lemma 5.3. *In any solid weak BCC-algebra*

$$ax = ab$$

for all $a, b \in I(G)$ and $x \in B(b)$.

Proof. Let $a, b \in I(G)$. Then for any $x \in B(b)$ we have $b \leqslant x$, which, by (3), implies $ax \leqslant ab$. Since $I(G)$ is a subalgebra of G (Corollary 2.5), hence $ab \in I(G)$. This means that ab is a minimal element of G. Thus $ax = ab$. □

Lemma 5.4. *If in a solid weak BCC-algebra $ax = ab$ holds for some $a, x \in G$ and $b \in I(G)$, then $x \in B(b)$.*

Proof. If $ax = ab$ holds for some $a, x \in G$ and $b \in I(G)$, then, according to (i), we have

$$0 = (ab \cdot xb) \cdot ax = (ab \cdot ax) \cdot xb = 0 \cdot xb.$$

Thus $0 \leqslant xb$. This, by Corollary 3.10, means that x and b are in the same branch. □

Corollary 5.5. *Elements x, y of a solid weak BCC-algebra G are in the same branch if and only if $ax = ay$ for some $a \in I(G)$.*

Proof. If elements x, y belong to the branch $B(b)$, where $b \in I(G)$, then from Lemma 5.3 it follows $ax = ab = ay$ for all $a \in I(G)$.

Conversely, if $ax = ay$ for some $a \in I(G)$, then

$$0 = (ax \cdot yx) \cdot ay = (ax \cdot ay) \cdot yx = 0 \cdot yx.$$

Thus $yx \in B(0)$. Corollary 3.10 completes the proof. □

Definition 5.6. For $x, y \in G$ and non-negative integers n we define

$$xy^0 = x, \quad xy^{n+1} = (xy^n)y.$$

Lemma 5.7. *In solid weak BCC-algebras we have*

$$0 \cdot 0x^n = 0 \cdot (0x)^n$$

for every $x \in G$ and every natural n.

Proof. For $n = 1$ this identity is obvious. If it is valid for $n = k$, then for $n = k + 1$, using Theorem 5.2, we obtain

$$0 \cdot 0x^{k+1} = 0 \cdot (0x^k \cdot x) = (0 \cdot 0x^k) \cdot 0x = (0 \cdot (0x)^k) \cdot 0x = 0 \cdot (0x)^{k+1},$$

which completes the proof. □

Lemma 5.8. ([9], Lemma 2). *In a solid weak BCC-algebra*

$$x(x \cdot xy) = xy$$

for x, y belonging to the same branch. □

We present some generalizations of the above result.

Proposition 5.9. *In a solid weak BCC-algebra*

$$x(x \cdot xy)^2 = xy^2$$

for x, y belonging to the same branch.

Proof. Indeed, using Lemma 5.8, we obtain
$x(x \cdot xy)^2 = x(x \cdot xy) \cdot (x \cdot xy) = xy \cdot (x \cdot xy) = x(x \cdot xy) \cdot y = xy \cdot y = xy^2$. □

Theorem 5.10. *In a super solid weak BCC-algebra*

$$x(x \cdot xy)^n = xy^n$$

for all natural n and x, y belonging to the same branch.

Proof. For $n = 1$ this theorem coincides with Lemma 5.8, for $n = 2$ with Proposition 5.9.

For $n \geqslant 3$, by Lemma 5.8, we have

$$\begin{aligned} x(x \cdot xy)^n &= x(x \cdot xy) \cdot (x \cdot xy)^{n-1} = xy \cdot (x \cdot xy)^{n-1} \\ &= (xy \cdot (x \cdot xy)) \cdot (x \cdot xy)^{n-2} = (x(x \cdot xy))y \cdot (x \cdot xy)^{n-2} \\ &= (xy \cdot y) \cdot (x \cdot xy)^{n-2} = ((xy \cdot y) \cdot (x \cdot xy)) \cdot (x \cdot xy)^{n-3}. \end{aligned}$$

Since, by the assumption, x, y belong to the same branch $B(a)$, then, by Proposition 3.12, also $x \cdot xy \in B(a)$. Thus

$$\begin{aligned}((xy \cdot y) \cdot (x \cdot xy)) \cdot (x \cdot xy)^{n-3} &= (xy \cdot (x \cdot xy))y \cdot (x \cdot xy)^{n-3} \\ &= (x(x \cdot xy) \cdot y)y \cdot (x \cdot xy)^{n-3} \\ &= (xy \cdot y)y \cdot (x \cdot xy)^{n-3} \\ &= xy^3 \cdot (x \cdot xy)^{n-3} \\ &\cdots\cdots\cdots\cdots \\ &= xy^{n-1} \cdot (x \cdot xy) \\ &= \ldots = xy^n.\end{aligned}$$

This completes the proof. □

Theorem 5.11. *Any (solid) weak BCC-algebra can be extended to a (solid) weak BCC-algebra containing one element more.*

Proof. Let $(G; \cdot, 0)$ be a (solid) weak BCC-algebra and let $\theta \notin G$. Then the set $G' = G \cup \{\theta\}$ with the operation

$$x \star y = \begin{cases} xy & \text{for } x, y \in G, \\ x & \text{for } x \in G,\ y = \theta, \\ 0y & \text{for } x = \theta,\ y \in G - \{0\}, \\ \theta & \text{for } x = \theta,\ y = 0, \\ 0 & \text{for } x = y = \theta \end{cases}$$

is a (solid) weak BCC-algebra.

The axioms $(ii) - (iv)$ are obvious. Since by the assumption the axiom (i) is satisfied for all $x, y, z \in G$, we must verify it only in the case when at least one of x, y, z is equal to θ. But this is a routine calculation. Also it is not difficult to verify that $(G'; \star, 0)$ is solid if $(G; \cdot, 0)$ is solid. □

It can be noticed that the above construction saves the number of branches. Indeed, $\theta \in B(0)$ since $0 < \theta < y$ for every $y \in B(0)$. So, $(G; \cdot, 0)$ and $(G'; \star, 0)$ have the same initial elements and the same branches determined by non-zero initial elements. The branch $B(0)$ has in $(G'; \star, 0)$ one element more than in $(G; \cdot, 0)$.

Theorem 5.12. *Any BCK-algebra can be embedded into a solid weak BCC-algebra as its $B(0)$ branch.*

Proof. Let $(G; \cdot, 0)$ be a BCK-algebra and let $\theta \notin G$ be a fixed element. Then, as it is not difficult to see, $(G'; \star, 0)$ with the operation

$$x \star y = \begin{cases} xy & \text{for } x, y \in G, \\ \theta & \text{for } x \in G,\ y = \theta, \\ \theta & \text{for } x = \theta,\ y \in G, \\ 0 & \text{for } x = y = \theta \end{cases}$$

is a solid weak BCC-algebra containing $(G; \cdot, 0)$ as its subalgebra. This weak BCC-algebra contains two branches: $B(0) = G$ and $B(\theta) = \{\theta\}$. □

Proposition 5.13. *Any BCK-algebra can be embedded into a solid weak BCC-algebra without trivial branches.*

Proof. Let $(G;\cdot,0)$ be a BCK-algebra and $(H;*,0)$ a solid weak BCC-algebra without trivial branches such that $G\cap H=\{0\}$. On $G\cup H$ we define a common operation $\star$ by putting

$$x\star y=\begin{cases} xy & \text{if } x,y\in G,\\ x*y & \text{if } x,y\in H,\\ 0*y & \text{if } x\in G,\ y\in H-\{0\},\\ x & \text{if } x\in H,\ y\in G.\end{cases}$$

Then long but simple calculations show that $(G\cup H;\star,0)$ is a solid weak BCC-algebra. The natural order of $(G\cup H;\star,0)$ coincides on G with the natural order of $(G;\cdot,0)$, and on H with the natural order of $(H;*,0)$. Each element of G is smaller than each non-zero element of the branch $B(0)$ of a weak BCC-algebra $(H;*,0)$. Elements of G and elements of other branches of H are incomparable. □

Corollary 5.14. *Any BCC-algebra can be embedded into a weak BCC-algebra without trivial branches.*

Proof. We can use the same construction. Obtained weak BCC-algebra will be solid only in the case when the starting BCC-algebra will be a BCK-algebra. □

The idea of the above construction is based on gluing graphs presented in the following example.

Example 5.15. Consider a BCK-algebra $(G;\cdot,0)$:

$\cdot$	0	1	2	3
0	0	0	0	0
1	1	0	1	0
2	2	2	0	0
3	3	3	3	0

Table 5.1.

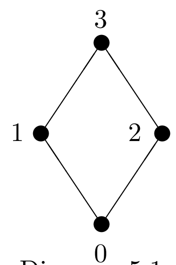

Diagram 5.1.

and a solid weak BCC-algebra $(H;*,0)$:

$*$	0	a	b	c	d
0	0	0	b	b	b
a	a	0	b	b	b
b	b	b	0	0	0
c	c	b	a	0	a
d	d	b	a	a	0

Table 5.2.

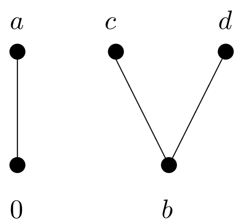

Diagram 5.2.

The above construction gives the following solid weak BCC-algebra:

$\star$	0	1	2	3	a	b	c	d
0	0	0	0	0	0	b	b	b
1	1	0	1	0	0	b	b	b
2	2	2	0	0	0	b	b	b
3	3	3	3	0	0	b	b	b
a	a	a	a	a	0	b	b	b
b	b	b	b	b	b	0	0	0
c	c	c	c	c	b	a	0	a
d	d	d	d	d	b	a	a	0

Table 5.3.

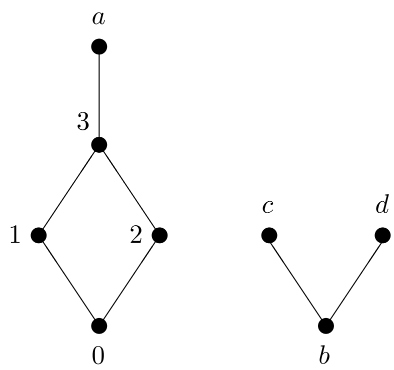

Diagram 5.3. □

Theorem 5.16. *Any weak BCC-algebra can be embedded into a BCC-algebra.*

Proof. Let $(G;\cdot,0)$ be a weak BCC-algebra and let $G' = G \cup \{\theta\}$, where $\theta \notin G$. Then, as it is not difficult to see, $(G';\star,\theta)$ with the operation

$$x \star y = \begin{cases} xy & \text{if} \quad xy \neq 0, \\ \theta & \text{if} \quad xy = 0, \\ \theta & \text{if} \quad x = \theta,\ y \in G', \\ x & \text{if} \quad x \in G',\ y = \theta \end{cases}$$

is a BCC-algebra. □

Example 5.17. Using the last construction we can extend the weak BCC-algebra defined by Table 2.1.2 (Example 5.15) into the following BCC-algebra:

$*$	0	a	b	c	d	θ
0	θ	θ	b	b	b	0
a	a	θ	b	b	b	a
b	b	b	θ	θ	θ	b
c	c	b	a	θ	a	c
d	d	b	a	a	θ	d
θ	θ	θ	θ	θ	θ	θ

Table 5.4.

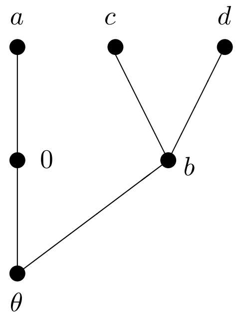

Diagram 5.4. □

Corollary 5.18. *Any BCI-algebra can be embedded into a BCK-algebra.*

Proof. According to the definition, any BCI-algebra is a solid weak BCC-algebra. So, starting from a BCI-algebra $(G;\cdot,0)$ and using the construction proposed in the proof of Theorem 5.16 we obtain a BCC-algebra $(G';\star,0)$ which is a BCK-algebra. Indeed, $(G';\star,0)$ is a BCC-algebra and, by the assumption, the condition $(x \star y) \star z = (x \star z) \star y$ is satisfied by all $x, y, z \in G$. It is not difficult to verify that it is also satisfied in the case when at least one element of x, y, z is equal to θ. Thus, it is satisfied for all $x, y, z \in G'$. Therefore, $(G';\star,0)$ is a BCK-algebra. □

Corollary 5.19. *Any group-like weak BCC-algebra can be embedded into a BCK-algebra containing only atoms.*

Proof. Let G be a group-like weak BCC-algebra. Using the construction from the proof of Theorem 5.16 we obtain a BCC-algebra G' in which elements of G are comparable only with θ since in this construction we have $\theta \star x = \theta$ for all $x \in G'$. Also $x \star \theta = x$. Thus the condition $(x \star y) \star z = (x \star z) \star y$ is satisfied if at least one of x, y, z is equal to θ. Let $x, y, z \in G$. Then by definition of $\star$, $\theta \leqslant y$ and so by (3), $x \star y \leqslant x \star \theta$. Since x is comparable only with θ and x, then we have $x \star y = \theta$ or $x \star y = x$. In the first case $x = y$ and $(x \star x) \star z = \theta \star z = \theta = (x \star z) \star x$. In the second $(x \star y) \star z = x \star z = x = x \star z = (x \star z) \star y$. This proves that this BCC-algebra is a BCK-algebra. □

6 Quasi-commutative weak BCC-algebras As it is widely known (cf. for example [19]), commutative BCC-algebras, i.e., BCC-algebras satisfying the identity $x \cdot xy = y \cdot yx$, form a variety, but the class of all BCC-algebras is not a variety (cf. [18]). Also the class of all weak BCC-algebras is not a variety. Similarly, the class of all BCI-algebras. However, the so-called quasi-commutative BCI-algebras form a variety (cf. [13]). In this section we prove that analogous result is valid for quasi-commutative weak BCC-algebras.

In a weak BCC-algebra G for non-negative integers m, n we define a polynomial $Q_{m,n}(x, y)$ by putting:

$$Q_{m,n}(x, y) = (x \cdot xy)(xy)^m \cdot (yx)^n.$$

Definition 6.1. A weak BCC-algebra G is called *quasi-commutative of type* $(m, n; i, j)$ if there exist two pairs of non-negative integers i, j and m, n such that

$$Q_{m,n}(x, y) = Q_{i,j}(y, x),$$

or equivalently

$$(x \cdot xy)(xy)^m \cdot (yx)^n = (y \cdot yx)(yx)^i \cdot (xy)^j,$$

holds for all $x, y \in G$. If the above identity holds only for all x, y belonging to the same branch, then we say that this weak BCC-algebra is *branchwise quasi-commutative* (shortly: *b-quasi-commutative*).

Exchanging x and y in $Q_{m,n}(x, y) = Q_{i,j}(y, x)$, we see that *a weak BCC-algebra is quasi-commutative of type* $(i, j; m, n)$ *if and only if it is quasi-commutative of type* $(m, n; i, j)$.

Example 6.2.
(1) A group-like weak BCC-algebra is b-quasi-commutative of any type since each its branch has only one element.
(2) A medial weak BCC-algebra is quasi-commutative of type $(0, 1; 0, 0)$ because it satisfies the identity $x \cdot xy = y$.
(3) A weak BCC-algebra is branchwise commutative (commutative) if and only if it b-quasi-commutative (quasi-commutative) of type $(0, 0; 0, 0)$. □

Proposition 6.3. *A b-quasi-commutative solid weak BCC-algebra G of type* $(0, k; 0, 0)$ *is branchwise commutative.*

Proof. Let G be a weak BCC-algebra satisfying the assumption. Then

$$Q_{0,k}(x, y) = (x \cdot xy)(yx)^k = y \cdot yx = Q_{0,0}(y, x)$$

for x, y belonging to the same branch. For $k = 0$ it is obviously branchwise commutative.

Let $k > 0$. Then $yx \in B(0)$. Hence $0 \leqslant yx$, and consequently $(x \cdot xy)(yx) \leqslant x \cdot xy$, by (3). Thus

$$y \cdot yx = (x \cdot xy)(yx)^k \leqslant (x \cdot xy)(yx)^{k-1} \leqslant \ldots \leqslant (x \cdot xy)(yx) \leqslant x \cdot xy,$$

i.e., $y \cdot yx \leqslant x \cdot xy$.

Interchanging x and y we get $x \cdot xy = y \cdot yx$. □

Proposition 6.4. *In solid weak BCC-algebras the following inequalities*

(1) $Q_{n-1,n}(x,y) \geqslant Q_{n,n}(x,y) \geqslant Q_{n,n+1}(x,y) \geqslant Q_{n+1,n+1}(x,y)$,

(2) $Q_{n-1,n}(x,y) \geqslant Q_{n,n}(y,x) \geqslant Q_{n,n+1}(x,y) \geqslant Q_{n+1,n+1}(y,x)$

are valid for all natural n and x, y belonging to the same branch.

Proof. (1) Observe that $x \cdot xy \in B(a)$ and $(x \cdot xy)(xy)^k \in B(a)$ for every k and $x, y \in B(a)$. The first is a consequence of Proposition 3.12, the second follows from the fact that $0 \leqslant xy$ implies $a \cdot xy \leqslant a$, i.e., $a \cdot xy = a$ because $a \in I(G)$. Therefore $a = a \cdot (xy)^k \leqslant (x \cdot xy)(xy)^k$. Thus using (i') and (2) we obtain

$$\begin{aligned} Q_{n,n}(x,y) \cdot Q_{n-1,n}(x,y) &= ((x \cdot xy)(xy)^n \cdot (yx)^n) \cdot ((x \cdot xy)(xy)^{n-1} \cdot (yx)^n) \\ &\leqslant ((x \cdot xy)(xy)^n \cdot (yx)^{n-1}) \cdot ((x \cdot xy)(xy)^{n-1} \cdot (yx)^{n-1}) \\ &\leqslant ((x \cdot xy)(xy)^n \cdot (yx)^{n-2}) \cdot ((x \cdot xy)(xy)^{n-1} \cdot (yx)^{n-2}) \\ &\leqslant \ldots \leqslant (x \cdot xy)(xy) \cdot (x \cdot xy) \\ &= (x \cdot xy)(x \cdot xy) \cdot xy = 0 \cdot xy = 0. \end{aligned}$$

Thus

$$Q_{n,n}(x,y) \cdot Q_{n-1,n}(x,y) = 0,$$

which proves

$$Q_{n,n}(x,y) \leqslant Q_{n-1,n}(x,y).$$

Similarly,

$$\begin{aligned} Q_{n,n+1}(x,y) \cdot Q_{n,n}(x,y) &= ((x \cdot xy)(xy)^n \cdot (yx)^{n+1}) \cdot ((x \cdot xy)(xy)^n \cdot (yx)^n) \\ &\leqslant ((x \cdot xy)(xy)^n \cdot (yx)^n) \cdot ((x \cdot xy)(xy)^n \cdot (yx)^{n-1}) \\ &\leqslant \ldots \leqslant ((x \cdot xy)(xy)^n \cdot yx) \cdot (x \cdot xy)(xy)^n \\ &= ((x \cdot xy)(xy)^n \cdot (x \cdot xy)(xy)^n) \cdot yx = 0 \cdot yx = 0. \end{aligned}$$

Hence

$$Q_{n,n+1}(x,y) \leqslant Q_{n,n}(x,y).$$

The last inequality of (1) is a consequence of the first.

(2) If $x, y \in B(a)$, then $xy, yx \in B(0)$ and $x \cdot xy, y \cdot yx \in B(a)$ by Corollary 3.10 and Proposition 3.12. From this, analogously as in the proof of (1), we can deduce that $(x \cdot xy)(xy)^{n-1}$ and $(x \cdot xy)(xy)^{n-1} \cdot (yx)^n$ are in $B(a)$ for every natural n. Therefore

$$\begin{aligned}
Q_{n,n}(y,x)\cdot Q_{n-1,n}(x,y) &= ((y\cdot yx)(yx)^n\cdot(xy)^n)\cdot((x\cdot xy)(xy)^{n-1}\cdot(yx)^n)\\
&= ((y\cdot yx)(yx)^n\cdot((x\cdot xy)(xy)^{n-1}\cdot(yx)^n))\cdot(xy)^n\\
&\leqslant ((y\cdot yx)\cdot(x\cdot xy)(xy)^{n-1})\cdot(xy)^n\\
&= (y\cdot yx)(xy)^n\cdot(x\cdot xy)(xy)^{n-1}\\
&\leqslant (y\cdot yx)(xy)\cdot(x\cdot xy)\leqslant (y\cdot yx)x=0.
\end{aligned}$$

Hence

$$Q_{n,n}(y,x)\leqslant Q_{n-1,n}(x,y).$$

Analogously,

$$\begin{aligned}
Q_{n,n+1}(x,y)\cdot Q_{n,n}(y,x) &= ((x\cdot xy)(xy)^n\cdot(yx)^{n+1})\cdot((y\cdot yx)(yx)^n\cdot(xy)^n)\\
&= ((x\cdot xy)(xy)^n\cdot((y\cdot yx)(yx)^n\cdot(xy)^n))\cdot(yx)^{n+1}\\
&\leqslant ((x\cdot xy)\cdot(y\cdot yx)(yx)^n)\cdot(yx)^{n+1}\\
&= (x\cdot xy)(yx)^{n+1}\cdot(y\cdot yx)(yx)^n\\
&\leqslant (x\cdot xy)(yx)\cdot(y\cdot yx)\leqslant (x\cdot xy)y=0.
\end{aligned}$$

This proves that

$$Q_{n,n+1}(x,y)\leqslant Q_{n,n}(y,x).$$

The last inequality of (2) is a consequence of the first. □

Theorem 6.5. *Every solid weak BCC-algebra which is decomposed into a finite number of finite branches is b-quasi-commutative of some type of the form $(m,m;m,m+1)$.*

Proof. Each branch $B(a)$ of G is finite, hence for each pair of elements $x,y\in B(a)$ the sequence (2) from Proposition 6.4 is finite. This means that for all $x,y\in B(a)$ there exists natural $n'=n(x,y)$ such that $Q_{n,n}(x,y)=Q_{n,n+1}(y,x)$ for all $n\geqslant n'$. Since $I(G)$ is finite for every

$$m\geqslant \max\{n(x,y): x,y\in B(a),\ a\in I(G)\}$$

and x,y belonging to the same branch we have $Q_{m,m}(x,y)=Q_{m,m+1}(y,x)$, which shows that G is quasi-commutative of type $(m,m;m,m+1)$. □

Corollary 6.6. *Any finite solid weak BCC-algebra is b-quasi-commutative of some type $(m,m;m,m+1)$.* □

Theorem 6.7. *If a proper weak BCC-algebra is quasi-commutative of type $(i,j;m,n)$, then $i-j+m-n+1\neq\pm1$.*

Proof. Since, by the assumption, a weak BCC-algebra G is proper, it has at least two branches, i.e., there exists $a\in I(G)$ such that $a\neq 0$. For this a we have $Q_{i,j}(0,a)\cdot Q_{m,n}(a,0)=0$ because G is quasi-commutative of type $(i,j;m,n)$.

By Corollary 2.5 $I(G)$ is a subalgebra of G. By Theorems 2.4 and 4.3 it is a group-like subalgebra. Hence (Theorem 4.2) there exists a group $(I(G);*,0)$ such that $xy=x*y^{-1}$ for $x,y\in I(G)$. Thus,

$$\begin{aligned}
0=Q_{i,j}(0,a)\cdot Q_{m,n}(a,0) &= ((0\cdot 0a)(0a)^i\cdot(a0)^j)\cdot((a\cdot a0)(a0)^m\cdot(0a)^n)\\
&= ((a\cdot(0a)^i)\cdot a^j)\cdot((0\cdot a^m)\cdot(0a)^n)\\
&= (a^{1+i}*a^{-j})*(a^{-m}*a^n)^{-1}\\
&= a^{1+i-j+m-n}.
\end{aligned}$$

For $i-j+m-n+1=\pm 1$, from the above we obtain $a^{\pm 1}=0$, which implies $a=0$. But this contradicts to our assumption on a. Therefore, it must be $i-j+m-n+1\neq\pm 1$. □

Theorem 6.8. *For $i-j+m-n+1\neq\pm 1$ there exists a group-like quasi-commutative weak BCC-algebra of type $(i,j;m,n)$.*

Proof. Let $k=|i-j+m-n+1|$. By Theorem 6.7 we have $k\neq 1$. Consider a group-like weak BCC-algebra $(G;\cdot,0)$ induced by an abelian group $(G;*,0)$. Then, as it is not difficult to verify,

$$Q_{i,j}(x,y)\cdot Q_{m,n}(y,x)=(x^{-1}*y)^{i-j+m-n+1}=(x^{-1}*y)^{\pm k}.$$

This means that for $k=0$ each group-like weak BCC-algebra induced by an abelian group is quasi-commutative of type $(i,j;m,n)$. For $k>1$ such weak BCC-algebra should be induced by a cyclic group of order k. □

Theorem 6.9. *An algebra $(G;\cdot,0)$ of type $(2,0)$ is a quasi-commutative weak BCC-algebra of type $(i,j;m,n)$ if and only if it satisfies the following three identities:*

(a) $(xy\cdot zy)\cdot xz=0$,

(b) $x0=x$,

(c) $Q_{i,j}(x,y)=Q_{m,n}(y,x)$.

Proof. The necessity is obvious. To show the sufficiency, we only need to verify two axioms from the definition of weak BCC-algebras: (ii) and (iv), because (i) coincides with (a), (iii) with (b).

Using (a) and (b) we obtain $xx=xx\cdot 0=(x0\cdot x0)\cdot 00=0$, which proves ($ii$). If $xy=yx=0$, then $Q_{i,j}(x,y)=(x\cdot xy)(xy)^{i}\cdot(yx)^{j}=x$ and $Q_{m,n}(y,x)=(y\cdot yx)(yx)^{m}\cdot(xy)^{n}=y$. This, by ($c$), implies $x=y$ and completes the proof. □

Corollary 6.10. *The class of quasi-commutative weak BCC-algebras of a fixed type is a variety.* □

The class of quasi-commutative weak BCC-algebras of a fixed type can also be defined by two identities.

Theorem 6.11. *An algebra $(G;\cdot,0)$ of type $(2,0)$ is a quasi-commutative weak BCC-algebra of type $(i,j;m,n)$ if and only if it satisfies the following identities:*

(α) $u\cdot((xy\cdot zy)\cdot xz)=u$,

(β) $Q_{i,j}(x,y)=Q_{m,n}(y,x)\cdot 0$.

Proof. The necessity is obvious. To prove sufficiency we will show that any algebra $(G;\cdot,0)$ satisfying the conditions (α), (β), also satisfies the conditions (a), (b), (c) from the previous theorem.

Let $\theta=(00\cdot 00)\cdot 00$. Then, by ($\alpha$), we have

$$\theta\theta=\theta\cdot((00\cdot 00)\cdot 00)=\theta.$$

Using (α) once again, for every $u\in G$ we obtain

$$u\cdot((\theta\theta\cdot\theta\theta)\cdot\theta\theta)=u,$$

which, in view of $\theta\theta = \theta$, gives $u\theta = u$. Now, putting $y = z = \theta$ in (α) and applying just proved identity $u\theta = u$ we get $u \cdot xx = u$ for all $x, u \in G$. This means that

$$u \cdot (xx)^k = u \tag{5}$$

for any natural k. In particular $0 \cdot (00)^k = 0$. Hence

$$Q_{i,j}(0,0) = (0 \cdot 00)(00)^i \cdot (00)^j = 0 \cdot (00)^j = 0.$$

Similarly, $Q_{m,n}(0,0) = 0$. This, by (β), implies $00 = 0$. Consequently, $u0 = u \cdot 00 = u$ for every $u \in G$. So, the condition (b) from Theorem 6.9 is satisfied. Combining (b) and (β) we obtain the condition (c).

Observe that (5) for $u = xx$ implies $(xx)^{k+1} = xx$ for any natural k. ¿From (5) we also obtain $0 \cdot (xx)^k = 0$ for any natural k. Hence

$$\begin{aligned} Q_{i,j}(xx,0) &= (xx \cdot (xx \cdot 0))(xx \cdot 0)^i \cdot (0 \cdot xx)^j \\ &= (xx \cdot xx)(xx)^i \cdot 0^j = (xx)^{i+2} = xx \end{aligned}$$

and

$$\begin{aligned} Q_{m,n}(0,xx) &= (0 \cdot (0 \cdot xx))(0 \cdot xx)^m \cdot (xx \cdot 0)^n \\ &= (00 \cdot 0^m) \cdot (xx)^n = 0 \cdot (xx)^n = 0, \end{aligned}$$

which together with just proved (c) gives $xx = 0$ for every $x \in G$. Now, putting $u = (xy \cdot zy) \cdot xz$ in (α) we have

$$u = u \cdot (xy \cdot zy) \cdot xz = uu = 0.$$

This means that $(xy \cdot zy) \cdot xz = 0$, so any algebra $(G; \cdot, 0)$ satisfying (α), (β) satisfies also (c), and consequently it is a quasi-commutative weak BCC-algebra of type $(i, j; m, n)$. □

Theorem 6.12. *If a solid weak BCC-algebra G is quasi-commutative of type $(i, j; m, n)$, then its branch $B(0)$ is a quasi-commutative BCK-algebra of one of the following three types: $(i, i; i, i)$, $(j, j; j, j)$ and $(n, j; j, n)$.*

The proof of this theorem is based on the following lemma.

Lemma 6.13. *In a quasi-commutative solid weak BCC-algebra of type $(i, j; m, n)$ we have*

(1) $xy^{i+1} = xy^{n+1}$,

(2) $xy^{j+1} = xy^{m+1}$

for $x, y \in B(0)$.

Proof. According to [10] $B(0)$ is the greatest BCC-algebra contained in G. Since G is solid, for all $x, y, z \in B(0)$ we have $xy \cdot z = xz \cdot y$. Thus, $B(0)$ is a BCK-algebra.

Observe first that

$$x(x \cdot xy)^k = xy^k$$

for $x, y \in B(0)$ and any natural k.

Indeed, for $k = 1$ it is valid by Lemma 5.8. If it is valid for some k, then for $k+1$ we have

$$\begin{aligned} x(x \cdot xy)^{k+1} &= x(x \cdot xy)^k \cdot (x \cdot xy) \\ &= xy^k \cdot (x \cdot xy) && \text{by the assumption on } k \\ &= (xy^{k-1} \cdot (x \cdot xy)) \cdot y \\ &= (xy^{k-2} \cdot (x \cdot xy)) \cdot y^2 \\ &= \ldots = (x \cdot (x \cdot xy)) \cdot y^k = xy \cdot y^k && \text{by Lemma 5.8} \\ &= xy^{k+1}. \end{aligned}$$

Then it is valid for every natural k.

Hence

$$\begin{aligned} Q_{i,j}(x, xy) &= (x \cdot (x \cdot xy))(x \cdot xy)^i \cdot (xy \cdot x)^j \\ &= x(x \cdot xy)^{i+1} \cdot 0^j && \text{because } \ xy \cdot x = 0 \\ &= x(x \cdot xy)^{i+1} = xy^{i+1}. \end{aligned}$$

Likewise,

$$\begin{aligned} Q_{m,n}(xy, x) &= (xy \cdot (xy \cdot x))(xy \cdot x)^m \cdot (x \cdot xy)^n \\ &= xy \cdot (x \cdot xy)^n = x(x \cdot xy)^n \cdot y = xy^{n+1}. \end{aligned}$$

Further, since G is quasi-commutative of type $(i, j; m, n)$, we have

$$Q_{i,j}(x, xy) = Q_{m,n}(xy, x).$$

Thus, $xy^{i+1} = xy^{n+1}$. This proves the first identity.

The second follows from the fact that any quasi-commutative weak BCC-algebra of type $(i, j; m, n)$ is also quasi-commutative of type $(m, n; i, j)$. □

Proof of Theorem 6.12. Let a solid weak BCC-algebra G be quasi-commutative of type $(i, j; m, n)$. Then, in particular,

$$(x \cdot xy)(xy)^i \cdot (yx)^j = (y \cdot yx)(yx)^m \cdot (xy)^n$$

for $x, y \in B(0)$. Since $yx \in B(0)$, the second identity of Lemma 6.13 shows that

$$(y \cdot yx)(yx)^m = y \cdot (yx)^{m+1} = y \cdot (yx)^{j+1} = (y \cdot yx)(yx)^j.$$

Thus

$$(x \cdot xy)(xy)^i \cdot (yx)^j = (y \cdot yx)(yx)^j \cdot (xy)^n$$

for all $x, y \in B(0)$. Hence $Q_{i,j}(x, y) = Q_{j,n}(y, x)$ for $x, y \in B(0)$. So, $B(0)$ is quasi-commutative of type $(i, j; j, n)$. Obviously, it also is quasi-commutative of type $(j, n; i, j)$. Repeating the above procedure we can show that $B(0)$ is quasi-commutative of type $(j, n; n, j)$. This implies that it is quasi-commutative of type $(n, j; j, n)$. For $j = n$ it is quasi-commutative of type $(j, j; j, j)$. Thus in a solid weak BCC-algebra quasi-commutative of type $(i, j; m, j)$ the branch $B(0)$ is quasi-commutative of type $(j, j; j, j)$.

Finally let us consider the case $i = j$, i.e., the quasi-commutativity of type $(i, i; m, n)$. From the first part of this proof it follows that in this case $B(0)$ is quasi-commutative of type $(i, i; i, n)$. Thus for $x, y \in B(0)$ and $i = j$ we have

$$(x \cdot xy)(xy)^i \cdot (yx)^i = (y \cdot yx)(yx)^i \cdot (xy)^n.$$

Since

$$(y \cdot yx)(yx)^i \cdot (xy)^n \leqslant (y \cdot yx)(yx)^i \cdot (xy)^i$$

for $i \leqslant n$ and $x, y \in B(0)$, the above implies

$$(x \cdot xy)(xy)^i \cdot (yx)^i \leqslant (y \cdot yx)(yx)^i \cdot (xy)^i.$$

Exchanging x and y we obtain

$$(y \cdot yx)(yx)^i \cdot (xy)^i \leqslant (x \cdot xy)(xy)^i \cdot (xy)^i,$$

which together with the previous inequality gives

$$(x \cdot xy)(xy)^i \cdot (yx)^i = (y \cdot yx)(yx)^i \cdot (xy)^i.$$

Therefore in this case $B(0)$ is quasi-commutative of type $(i, i; i, i)$. □

Corollary 6.14. *Suppose that G is a quasi-commutative BCK-algebra of type $(i, j; m, n)$. Then its type of quasi-commutativity can be reduced to one of the following types: $(i, i; i, i)$, $(j, j; j, j)$ and $(n, j; j, n)$.* □

References

[1] W.M. Bunder, *BCK and related algebras and their corresponding logics*, J. Non-classical Logic, 7 (1983), 15 − 24.

[2] M.A. Chaudhry, *Branchwise commutative BCI-algebras*, Math. Japonica, 37 (1992), 163 − 170.

[3] M.A. Chaudhry, *On two classes of BCI-algebras*, Math. Japonica, 53 (2001), 269 − 278.

[4] M.A. Chaudhry, *On branchwise implicative BCI-algebras*, International J. Math. Math. Sci., 29 (2002), 417 − 425.

[5] W.A. Dudek, *On group-like BCI-algebras*, Demonstratio Math., 21 (1988), 369 − 376.

[6] W.A. Dudek, *On BCC-algebras*, Logique et Analyse, 129-130 (1990), 103 − 111.

[7] W.A. Dudek, *On proper BCC-algebras*, Bull. Inst. Math. Acad. Sinica, 20 (1992), 137 − 150.

[8] W.A. Dudek, *Remarks on the axioms system for BCI-algebras*, Prace Naukowe WSP w Czestochowie, ser. Matematyka, 2 (1996), 46 − 61.

[9] W.A. Dudek, *Solid weak BCC-algebras*, Intern. J. Computer Math. 88 (2011), 2915-2925.

[10] W.A. Dudek, B. Karamdin, S.A. Bhatti, *Branches and ideals of weak BCC-algebras*, Algebra Coll., 18 (Special) (2011), 899-914.

[11] W.A. Dudek, X.H. Zhang, *On ideals and congruences in BCC-algebras*, Czechoslovak Math. J., 48(123) (2000), 21 − 29.

[12] W.A.Dudek, X.H.Zhang, Y.Q.Wang, *Ideals and atoms of BZ-algebras*, Math. Slovaca, 59 (2009), 387 − 404.

[13] Y.S. Huang, *BCI-algebra*, Science Press, Beijing 2006.

[14] K. Iséki, S. Tanaka, *Ideal theory of BCK-algebras*, Math. Japonica, 21 (1976), 351 − 366.

[15] Y. Imai, K. Iséki, *On axiom system of propositional calculi*, Proc. Japan Acad., 42 (1966), 19 − 22.

[16] K. Iséki, *An algebra related with a propositional calculus*, Proc. Japan Acad., 42 (1966), 26 − 29.

[17] K. Iséki, S. Tanaka, *An introduction to the theory of BCK-algebras*, Math. Japonica 23 (1978), 1 − 26.

[18] Y. Komori, *The class of BCC-algebras is not variety*, Math. Japonica, 29 (1984), 391 − 394.

[19] J. Meng, Y.B. Jun, *BCK-algebras*, Kyung Moon SA, Seoul, 1994.

[20] L. Tiande, X. Changchang, *p-radicals in BCI-algebras*, Math. Japonica, 30 (1985), 511 − 517.

[21] A. Wroński, *BCK-algebras do not form a variety*, Math. Japonica, 28 (1983), 211 − 213.

[22] X.H. Zhang, *BIK^+-logic and non-commutative fuzzy logics*, Fuzzy Systems Math. 21 (2007), 31 − 36.

[23] X.H. Zhang, W.A. Dudek, *Fuzzy BIK^+-logic and non-commutative fuzzy logics*, Fuzzy Systems Math. 23 (2009), 9 − 20.

[24] X.H. Zhang, R. Ye, *BZ-algebras and groups*, J. Math. Phys. Sci., 29 (1995), 223 − 233.

communicated by *Mariko Yasugi*

* Department of Mathematics, University of the Punjab, Quaid-e-Azam Campus, Lahore-54590, Pakistan
E-mail address: ayeshafatima5@hotmail.com

** Hinueberstrasse 13 A, 30175 Hannover, Germany
E-mail address: janus.thomys@htp-tel.de