n-FOLD BCI-POSITIVE IMPLICATIVENESS

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Received January 13, 2012

Abstract.

In this paper we introduce the notion of n-fold BCI-positive implicative ideals in BCI-algebras and study some properties. We also establish the extension properties of n-fold BCI-positive implicative ideals. We use the notion of n-fold BCI-positive implicative ideals to completely describe n-fold BCI-algebras. This work generalizes the corresponding results in BCK/BCI-algebras.

1 Introduction BCI/BCK-algebras were introduced by K. Iséki and Y. Imai in the mid 60's and since then, have been intensely studied with applications in both applied and pure Mathematics. Some of the most recent studies of these concepts can be found in [11] and [1].

Liu and Zhan [9], Wei and Jun [12] independently introduced BCI-positive implicative ideals and used these to completely describe positive implicative BCI-algebras (namely weakly positive implicative BCI-algebras). It arises the natural question of whether or not it is possible to generalize this notion and obtain a proper class of BCI algebra which also contains the corresponding class of BCK-algebras.

In this paper we answer this question positively and initiate the study of $n-$fold BCI-positive implicativeness. We introduce n-fold positive implicative BCI-algebra, a class of algebras generalizing both positive implicative BCI-algebras and n-fold positive implicative BCK-algebras. We also introduce generalized associative BCI-algebra and study the relationship between these and the generalized positive implicative BCI-algebras as treated in [6]. We also establish that generalized associative BCI-algebra form a proper subclass of nil BCI-algebras. We also introduce the notion of n-fold BCI-positive implicative ideals, as a generalization of the notion of BCI-positive implicative ideals. Furthermore, some

[0] *2010 AMS Classifications*: 06F35, 13A15, 8A72, 03B52, 03E72.

Key Words and phrases : BCI-algebras, n-fold BCI-positive implicative ideal, closed ideal, nil BCI-algebra, G-associative BCI-algebra.

characterizations of n-fold BCI-positive implicative ideals are obtained. Using these characterizations, the extension property of n-fold BCI-positive implicative ideals is established. Finally the n-fold positive implicative BCI-algebras are completely described by n-fold BCI-positive implicative ideals. It should be pointed out that our techniques in the present work are similar to those used by C. Lele and al. in studying the concept of foldness on other types of algebras [7].

2 Preliminaries

Definition 2.1. *[8] An algebra $X =< X; \star, 0 >$ of type $< 2, 0 >$, is said to be a BCI-algebra if it satisfies the following conditions for all $x, y, z \in X$:*

- *BCI1-* $((x \star y) \star (x \star z)) \star (z \star y) = 0$
- *BCI2-* $(x \star (x \star y)) \star y = 0$
- *BCI3-* $x \star x = 0$
- *BCI4-* $x \star y = 0$ *and* $y \star x = 0$ *imply* $x = y$

If a BCI-algebra X satisfies the condition

∘ BCI5- $0 \star x = 0$ for all $x \in X$; then X is called a BCK-algebra. Hence, BCK-algebra form a subclass of BCI-algebra.

Proposition 2.2. *[[8], [4]]On every BCI-algebra X, there is a natural order called the BCI-ordering defined by $x \leq y$ if and only if $x \star y = 0$. Under this order, the following properties hold for all $x, y, z \in X$.*

(a) $x \star 0 = x$.

(b) $x \leq y$, *then* $x \star z \leq y \star z$ *and* $z \star y \leq z \star x$.

(c) $(x \star y) \star z = (x \star z) \star y$.

(d) $x \star (x \star (x \star y)) = x \star y$.

(e) $(x \star z) \star (y \star z) \leq x \star y$.

(f) $0 \star (x \star y) = (0 \star x) \star (0 \star y)$.

(g) $0 \star (x \star y) = 0 \star (0 \star (y \star x))$.

(h) $0 \star x = 0 \star (y \star (y \star x))$.

(i) $(0 \star (x \star y)) \star (y \star x) = 0$.

(j) *If* $x \leq y$, *then* $0 \star x = 0 \star y$

(k) *If* $x \leq 0 \star y$, *then* $x = 0 \star y$

(l) *If* $u = x \star (y \star (y \star x))$, *then* $0 \star u = 0$

Let n be a positive integer. Throughout this paper we appoint that $X = (X, \star, 0)$ denotes a BCI-algebra; $x \star y^n = (...((x \star y) \star y) \star ...) \star y$, in which y occurs n times; $x \star y^0 = x$ and $x \star \prod_{i=1}^{n} y_i$ denotes $(...((x \star y_1) \star y_2) \star ...) \star y_n$ where $x, y, y_i \in X$;

Definition 2.3. *[8] A nonempty subset I of a BCI-algebra is called an ideal of X if it satisfies*

(I_1) $0 \in I$, and

(I_2) $x \star y \in I$ and $y \in I$ imply $x \in I$

Remark 2.4. *Let $\{I_k\}_{k \in K}$ be a family of ideals of a BCI-algebra X. Then $\bigcap_{k \in K} I_k$ is an ideal of X*

Definition 2.5. *Let S be a subset of a BCI-algebra X. The ideal $< S >$ of X generated by S is the intersection of all the ideals of X containing S.*

It is easy to see that

$< S > := \{0\} \cup \{a \in X,\ \exists n \in \mathbb{N};\ \exists x_1, x_2, x_3, ..., x_n \in S / a \star \prod_{i=1}^{n} x_i = 0\}$

Definition 2.6. *[5] Let X, Y be two BCI-algebras.*

(i) A map $f : X \to Y$ is called a BCI-homomorphism if: $f(x \star y) = f(x) \star f(y)$ for all $x, y \in X$.

(ii) A bijective BCI-homomorphism is called a BCI-isomorphism. Two BCI-algebras X, Y are said to be isomorphic and we write $X \cong Y$ if there exists a BCI-isomorphism between X and Y.

Definition 2.7. *[5] A BCI-algebra X is said to be associative if for all $x, y, z \in X$, $(x \star y) \star z = x \star (y \star z)$.*

Definition 2.8. *[5] A BCI-algebra X is said to be p-semisimple if $0 \star (0 \star x) = x$ for all $x \in X$.*

Proposition 2.9. *[5] The following properties also hold in any p-semisimple BCI-algebra X.*

For all $x, y, z \in X$, we have:

(1) $x \star y = 0$ *implies* $x = y$

(2) $(x \star z) \star (z \star y) = x \star y$

(3) $x \star (x \star y) = y$

(4) $x \star y = x \star z$ *implies* $y = z$

(5) $x \star y = z \star y$ *implies* $x = z$

3 $n-$Fold Positive Implicative BCI-algebra In this section, we introduce the notion of $n-$fold positive implicative BCI-algebra and study some important properties. We start by recalling the definition of the corresponding notion for BCK-algebras.

Definition 3.1. *[3] Let X be a BCK-algebra and n be a positive integer. Then X is called an $n-$fold positive implicative BCK-algebra if $x \star y^{n+1} = x \star y^n$; for all $x, y \in X$.*

Definition 3.2. *[4] Let X be a BCI-algebra. Then X is called a positive implicative BCI-algebra if $(x \star (x \star y)) \star (y \star x) = x \star (x \star (y \star (y \star x)))$ for all $x, y \in X$.*

The following result and Definition 3.1 motivate our definition of n-fold positive implicative BCI-algebras.

Proposition 3.3. *[4, Lemma 5] A BCI-algebra X is positive implicative if and only if it satisfies the condition: $x \star y = (x \star y^2) \star (0 \star y)$ for all $x, y \in X$.*

Definition 3.4. *Let X be a BCI-algebra and n be a positive integer. Then X is called an $n-$fold positive implicative BCI-algebra if $\left(x \star y^{n+1}\right) \star (0 \star y) = x \star y^n$; for all $x, y \in X$.*

The following example shows that the newly introduced class $n-$fold positive implicative BCI-algebras $n > 1$ is not empty.

Example 3.5. *Let $X = \{0, 1, 2, 3, 4, 5\}$ with the operation $\star$ defined by*

$\star$	0	1	2	3	4	5
0	0	0	0	3	3	3
1	1	0	1	3	3	3
2	2	2	0	3	3	3
3	3	3	3	0	0	0
4	4	3	4	1	0	0
5	5	3	5	1	1	0

Then $X =< X; \star, 0 >$ is a 2-fold positive implicative BCI-algebra.

Example 3.6. (1) *Let $X = \mathbb{R} \setminus \{0\}$ with $\star$ defined by $x \star y = xy^{-1}$. Then $\mathcal{X} =< X; \star, 1 >$ is an $n-$fold positive implicative BCI-algebra, for all $n \in \mathbb{N} \setminus \{0\}$.*

(1) *is a special case of the following class of algebras.*

(2) *Every $p-$semisimple BCI-algebra is $n-$fold positive implicative for all positive integer $n > 0$. This is because for every $n > 0$, $(x \star y^{n+1}) \star (0 \star y) \leq x \star y^n$ and every element of $p-$semisimple BCI-algebra is minimal (Proposition 2.9(1)).*

Remark 3.7. *1. It is clear from Proposition 3.3 that 1-fold positive implicative BCI-algebras coincide with positive implicative BCI-algebras.*

2. n-fold positive implicative BCI-algebras generalize n-fold positive implicative BCK-algebras. This is because if X is a BCK-algebra, for every $x \in X$ $0 \star x = 0$, so the Definitions 3.1 and 3.4 agree.

Proposition 3.8. *Let X and Y be two n-fold positive implicative BCI-algebras. Then their product $X \times Y = \{(x, y) \,/\, x \in X,\ y \in Y\}$ define by $(x_1, y_1) \star (x_2, y_2) = (x_1 \star x_2, y_1 \star y_2)$ is a n-fold positive implicative BCI-algebra.*

Proof: Suppose X and Y are two n-fold positive implicative BCI-algebras.

Let $(x_1, y_1)\,; (x_2, y_2) \in X \times Y$. Then

$$
\begin{aligned}
&\Big((x_1, y_1) \star (x_2, y_2)^{n+1}\Big) \star ((0,0) \star (x_2, y_2)),\\
= \quad & \left(x_1 \star x_2^{n+1}, y_1 \star y_2^{n+1}\right) \star (0 \star x_2, 0 \star y_2),\\
= \quad & \left(\left(x_1 \star x_2^{n+1}\right) \star (0 \star x_2), \left(y_1 \star y_2^{n+1}\right) \star (0 \star y_2)\right),\\
= \quad & \left(x_1 \star x_2^{n}, y_1 \star y_2^{n}\right),\\
= \quad & (x_1, y_1) \star (x_2, y_2)^{n}.
\end{aligned}
$$

Thus $\Big((x_1, y_1) \star (x_2, y_2)^{n+1}\Big) \star ((0,0) \star (x_2, y_2)) = (x_1, y_1) \star (x_2, y_2)^{n}$. Hence $X \times Y$ is an $n-$fold positive implicative BCI-algebra □

Lemma 3.9. *Let X be a BCI-algebra and n be a positive integer.*

If X is n-fold positive implicative BCI-algebra, then X is $(n+1)$-fold positive implicative BCI-algebra.

Proof: Let X be a BCI-algebra $n-$fold positive implicative; let $x, y \in X$. Then

$\left(x \star y^{n+2}\right) \star (0 \star y) = \left[\left(x \star y^{n+1}\right) \star (0 \star y)\right] \star y = (x \star y^{n}) \star y = x \star y^{n+1}$. Hence $\left(x \star y^{n+2}\right) \star (0 \star y) = x \star y^{n+1}$. □

A simple induction argument from Lemma 3.9 yields the following.

Proposition 3.10. *Let X be a BCI-algebra and n be a positive integer.*

If X is n-fold positive implicative BCI-algebra, then X is $(n+k)-$fold positive implicative BCI-algebra, for all $k \in \mathbb{N} \backslash \{0\}$.

The converse of Proposition 3.10 is false as the following example shows.

Example 3.11. *Let n be a positive integer.*

For each $n > 1$, consider the set $X_n = \{0, 1, ..., n, \pi\}$ and define the operation $\star$ on X_n by:

$$k \star l = max(0, k - l) \ \ if \ \ 0 \leq k, l \leq n$$
$$k \star \pi = \pi \star k = \pi \ \ if \ \ k = 0, 1, 2, ..., n$$
$$\pi \star \pi = 0.$$

Then $(X_n; \star, 0)$ is an $n-$fold positive implicative BCI-algebra, but not a $(n-1)-$fold positive implicative.

Now we establish some important properties of $n-$fold positive implicative BCI-algebras.

Theorem 3.12. *Let X be a BCI-algebra and n be a positive integer.*
Then X is an $n-$fold positive implicative BCI-algebra if and only if
$(x \star y) \star z^n = (x \star z^{n+1}) \star (y \star z)$, for all $x, y, z \in X$.

Proof: Suppose that X is an $n-$fold positive implicative BCI-algebra. Let $x, y, z \in X$ and $u = x \star y$. Then $(x \star z^{n+1}) \star (0 \star z) = x \star z^n$, so

$$\begin{aligned} (x \star y) \star z^n &= u \star z^n \\ &= (u \star z^{n+1}) \star (0 \star z) \end{aligned}$$

Now it suffices to show that $(u \star z^{n+1}) \star (0 \star z) = (x \star z^{n+1}) \star (y \star z)$

$$\begin{aligned} &\left[(u \star z^{n+1}) \star (0 \star z)\right] \star [((x \star z) \star z^n) \star (y \star z)], \\ &= \left[\left((x \star z^{n+1}) \star (0 \star z)\right) \star \left((x \star z^{n+1}) \star (y \star z)\right)\right] \star y, \\ &= [(y \star z) \star (0 \star z)] \star y, \\ &= ((y \star z) \star y) \star (0 \star z), \\ &= 0. \end{aligned}$$

Hence $(u \star z^{n+1}) \star (0 \star z) \leq ((x \star z) \star z^n) \star (y \star z)$

Next by Proposition 2.2(d), $x \star y = x \star (x \star u) = u$ implies

$$\begin{aligned} &\left[(x \star z^{n+1}) \star (y \star z)\right] \star \left[\left(u \star z^{n+1}\right) \star (0 \star z)\right], \\ &= \left[(x \star z^{n+1}) \star (y \star z)\right] \star \left[\left(\left(x \star z^{n+1}\right) \star (0 \star z)\right) \star (x \star u)\right], \\ &= \left[(x \star z^{n+1}) \star (y \star z)\right] \star [(x \star z^n) \star (x \star u)], \\ &= [((x \star z^n) \star (y \star z)) \star ((x \star z^n) \star (x \star u))] \star z, \\ &\leq ((x \star u) \star (y \star z)) \star z = ((x \star z) \star (y \star z)) \star u, \\ &\leq (x \star y) \star u = 0, \end{aligned}$$

That is, $(x \star z^{n+1}) \star (y \star z) \leq \left(u \star \star z^{n+1}\right) \star (0 \star z)$.

Hence $(x \star y) \star z^n = \left(u \star \star z^{n+1}\right) \star (0 \star z) = (x \star z^{n+1}) \star (y \star z)$.

Conversely Suppose $(x \star y) \star z^n = (x \star z^{n+1}) \star (y \star z)$, and put $y = 0$.
Then $x \star z^n = (x \star z^{n+1}) \star (0 \star z)$; hence X is an $n-$fold positive implicative BCI-algebra. □

Theorem 3.13. *Let X be a BCI-algebra and n be a positive integer.*
If X is an $n-$fold positive implicative BCI-algebra, then $((x \star y^n) \star (0 \star (0 \star y)) = x \star y^{n+1}$, for all $x, y \in X$.

Proof: Let X be an $n-$fold positive implicative BCI-algebra and let $x, y, \in X$.

Then $(x \star y^{n+1}) \star (0 \star y) = x \star y^n$.

$$
\begin{aligned}
&((x \star y^n) \star (0 \star (0 \star y))) \star (x \star y^{n+1}),\\
&= (((x \star y^{n+1}) \star (0 \star y)) \star (0 \star (0 \star y))) \star (x \star y^{n+1}),\\
&= (((x \star y^{n+1}) \star (x \star y^{n+1})) \star (0 \star y)) \star (0 \star (0 \star y)),\\
&= (0 \star (0 \star y)) \star (0 \star (0 \star y)) = 0.
\end{aligned}
$$

Thus $((x \star y^n) \star (0 \star (0 \star y)) \leq x \star y^{n+1}$.

On the other hand $0 \star (0 \star y) \leq y$ implies $(x \star y^n) \star y \leq (x \star y^n) \star (0 \star (0 \star y))$.

Hence $((x \star y^n) \star (0 \star (0 \star y)) = x \star y^{n+1}$. □

Corollary 3.14. *Let X be a BCI-algebra and n be a positive integer.*
If X is an $n-$fold positive implicative BCI-algebra, then $(x \star y^n) \star (x \star y^{n+1}) = 0 \star (0 \star y)$, for all $x, y, z \in X$.

Proof: Let X be an $n-$fold positive implicative BCI-algebra, and let $x, y \in X$. Then, by Theorem 3.13, $((x \star y^n) \star (0 \star (0 \star y)) = x \star y^{n+1}$. In particular, $((x \star y^n) \star (0 \star (0 \star y)) \leq x \star y^{n+1}$ which implies that $((x \star y^n) \star (x \star y^{n+1})) \leq 0 \star (0 \star y)$. It follows from Proposition 2.2(k) that $((x \star y^n) \star (x \star y^{n+1})) = 0 \star (0 \star y)$ as needed.

□

The next example shows that converses of above results are false.

Example 3.15. *Let $X = \{0, 1, a, b, c\}$ with Cayley table as follows*

$\star$	0	1	a	b	c
0	0	0	a	a	a
1	1	0	a	a	a
a	a	a	0	0	0
b	b	a	1	0	a
c	c	a	1	1	0

Then $X =< X; \star, 0 >$ is a BCI-algebra.

(i) For all $x, y, z \in X$, $(x \star y) \star (0 \star (0 \star y)) = x \star y^2$ but $\mathcal{X}$ is not 1-fold positive implicative BCI-algebra because $\left(c \star b^2\right) \star (0 \star b) = 0 \neq 1 = c \star b$.

(ii) For all $x, y, z \in X$, $(x \star y) \star \left(x \star y^2\right)\right) = 0 \star (0 \star y)$; but $\mathcal{X}$ is not 1-fold positive implicative BCI-algebra.

4 NIL BCI-algebras In this section, we establish some properties of generalized positive implicative BCI-algebra as introduced in [6]. We prove that a generalized positive implicative BCI-algebra is nil BCI-algebra. In addition, we introduce the notion of generalized idempotent (G-idempotent) elements in BCI-algebras. We use the later to introduce generalized associative (G-associative) BCI-algebras as a generalization of associative BCI-algebras and a proper subclass of nil BCI-algebras. We prove that for any BCI-algebra X, the set $I(X)$ of its G-idempotents is a sub-algebra of X, but not an ideal of X in general.

We recall that an element x in a BCI-algebra X is called a nilpotent element if $0 \star x^n = 0$ for some positive integer n.

$N(X) = \{x \in X,\ \exists n \in \mathbb{N} \backslash \{0\}; 0 \star x^n = 0\}$. $N(X)$ is called the nil-radical of X.

If $N(X) = X$, we say that X is a nil BCI-algebra

Definition 4.1 ([6] theorem 2.6(a)). *A BCI-algebra X is a generalized positive implicative BCI-algebra if for all $x, y \in X$; $x \star y^{n+1} = x \star y$ for some positive integer n*

Example 4.2. *The set $X = \{0, 1\} \sqcup \{\sqrt{2}\}$ with the operation $\star$ defined by:*

$x \star y = max(0; x - y)$, if $x, y \in \{0, 1\}$; $\sqrt{2} \star y = \sqrt{2}$ if $y \neq \sqrt{2}$ and $x \star \sqrt{2} = 0$ is a generalized positive implicative BCI-algebra.

Proposition 4.3. *[6] Let X be any BCI-algebra and k be any positive integer. Then we have*

$(x \star z^k) \star (y \star z^k) \leq x \star y$ for any $x, y \in X$.

Proposition 4.4. *[6] Let X be a generalized positive implicative BCI-algebra and x, y, z be any elements in X. Then there must exist some positive integer n such that*

$(x \star y) \star z^n = (x \star z^n) \star (y \star z^n)$.

Proof: Let $x, y, z \in X$, then since X is generalized positive implicative, there exists a positive integer n such that $x \star z^{n+1} = x \star z$. In addition,

$$\begin{aligned}((x \star y) \star z^n) \star ((x \star z^n) \star (y \star z^n)) &= ((x \star z^n) \star y) \star ((x \star z^n) \star (y \star z^n)) \\ &\leq (y \star z^n) \star y \\ &= 0 \star z^n \\ &= (z \star z^n) \star z \\ &= z \star z^{n+1} \\ &= z \star z \\ &= 0\end{aligned}$$

and

$$\begin{aligned}&((x \star z^n) \star (y \star z^n)) \star ((x \star y) \star z^n), \\ =\ & ((x \star z) \star z^{n-1}) \star (y \star z^n)) \star ((x \star y) \star z^n)), \\ =\ & (((x \star z^{n+1}) \star z^{n-1}) \star (y \star z^n)) \star ((x \star y) \star z^n), \\ =\ & (((x \star z^n) \star z^n) \star (y \star z^n)) \star ((x \star y) \star z^n), \\ \leq\ & ((x \star z^n) \star y) \star ((x \star y) \star z^n), \\ \leq\ & ((x \star y) \star z^n) \star ((x \star y) \star z^n) = 0.\end{aligned}$$

Hence $(x \star y) \star z^n = (x \star z^n) \star (y \star z^n)$ and the proof is completed. □

Theorem 4.5. *A generalized positive implicative BCI-algebra is a nil BCI-algebra.*

Proof: Let $x \in X$, we must prove that x is nilpotent. Since X is a generalized positive implicative BCI-algebra, by the Proposition 4.4 there exists some positive integer n such that $(0 \star 0) \star x^n = (0 \star x^n) \star (0 \star x^n)$. So $0 \star x^n = 0$; hence x is nilpotent as required. □

Now we introduce the notion of generalized associative BCI-algebras. The following result motivates our definition of generalized associative BCI-algebras.

Proposition 4.6. *[5, Thm. 1.3.14] A BCI-algebra X is associative if and only if $0 \star x = x$ for all $x \in X$.*

Definition 4.7. *Let X be a BCI-algebra.*

(i) An element $x \in X$ is k-idempotent ($k \geq 1$) if $0 \star x^k = x$.

(ii) An element $x \in X$ is called G-idempotent if x is k-idempotent for some $k \geq 1$.

(iii) X is said to be G-associative if every element of X is G-idempotent.

If X is a BCI-algebra, the set of G-idempotent elements in X will be denoted by $I(X)$. So X is G-associative if $I(X) = X$.

Example 4.8. *(i) Every associative BCI-algebra is G-associative as all its elements are* 1*-idempotent by Proposition 4.6.*

(ii) Every finite p-semisimple BCI-algebra is G-associative.

Remark 4.9. *For every BCI-algebra X, $I(X) \subseteq N(X)$ because if $x \in X$ is k-idempotent, then $0 \star x^{k+1} = 0$. In particular every G-associative BCI-algebra is nilpotent.*
In addition if X is G-associative, then the BCK-part $V(0)$ of X is trivial. Therefore, it follows from [5, Thm. 1.3.2] that X is p-semisimple and by [5, Thm. 1.3.11], its adjoint Abelian group is a torsion group. It is also easy to see that if the adjoint Abelian group of a p-semisimple BCI-algebra X is a torsion group, then X is G-associative. Therefore, we have established that $G-$associative BCI-algebras and torsion Abelian groups are equivalent abstract systems.

Proposition 4.10. *Let X be a BCI-algebra, then $I(X)$ is a sub-algebra of X.*

Proof: Clearly 0 is 1-idempotent, so $0 \in I(X)$. Before, we prove the closure of $I(X)$ under $\star$, observe that if $a \in X$ is k-idempotent, then $0 \star a^{k+1} = 0$, therefore it follows by an easy induction argument that $0 \star a^{lk+l-1} = a$ for all $l \geq 1$.
Now, let $x, y \in I(X)$, then there exists $m, n \geq 1$ such that $0 \star x^m = x$ and $0 \star y^n = y$. Then by the observation above and [5, Ex. 1.1.6(2)], we get:

$$0 \star (x \star y)^{mn+m+n} = (0 \star x^{(n+1)m+n}) \star (0 \star y^{(m+1)n+m}) = x \star y$$

Hence $x \star y \in I(X)$. Therefore, $I(X)$ is a sub-algebra of X as required. □

Note however that in general, $I(X)$ is not an ideal of X. For instance, for the BCI-algebra X of Example 3.5, we have $I(X) = \{0, 3\}$ and $1 \star 3 = 3$.

Proposition 4.11. *Every G-associative BCI-algebra is generalized positive implicative.*

Proof: Suppose that X is a G-idempotent BCI-algebra. Let $x, y \in X$, then there exists positive integers m, n such that $0 \star x^m = x$ and $0 \star y^n = y$. It follows that $0 \star y^{n+2} = 0 \star y$. Hence,

$$\begin{aligned} x \star y^{n+2} &= (0 \star x^m) \star y^{n+2} \\ &= (0 \star y^{n+2}) \star x^m \\ &= (0 \star y) \star x^m \\ &= (0 \star x^m) \star y \\ &= x \star y \end{aligned}$$

hence $x \star y^{n+2} = x \star y$, and X is generalized positive implicative. □

Note however that the converse of Proposition 4.11 is false. For instance, the BCI-algebra of Example 4.2 is not G-idempotent.

5 n-fold BCI-positive implicative ideals In this section, we introduce the notion of $n-$fold BCI-positive implicative ideal, as a generalization of the notion of BCI-positive implicative ideals and $n-$fold positive implicative BCK ideals. Furthermore, a characterization of n-fold BCI-positive implicative ideal is obtained. Using this characterization, the extension property of $n-$fold BCI-positive implicative ideals is obtained. Finally the $n-$fold positive implicative BCI-algebra are completely described by $n-$fold BCI-positive implicative ideals.

Definition 5.1. *[3] A nonempty subset I of a BCK-algebra is called an $n-$fold BCK-positive implicative ideal of X if it satisfies*

(I_1) $0 \in I$ *and*

(I_3) $\left(x \star y^{n+1}\right) \star z \in I$ *and* $z \in I$ *imply* $x \star y^n \in I$; *for all* $x, y,\ z \in X$.

Proposition 5.2. *[3, Theorem 1.5] Let I be an ideal of a BCK-algebra X. Then I is $n-$fold positive implicative if and only if for any $x, y \in X$ $x \star y^{n+1} \in I$ implies $x \star y^n \in I$*

Definition 5.3. *[8] A nonempty subset I of a BCI-algebra is called a BCI-positive implicative ideal of X if it satisfies (I_1) and*

(I_4) $\left(x \star y^2\right) \star (z \star y) \in I$ *and* $z \in I$ *imply* $x \star y \in I$; *for all* $x, y,\ z \in X$.

Proposition 5.4. *[9] Let I be an ideal of BCI-algebra X. Then the following conditions are equivalent:*

i) I *is BCI-positive implicative.*

ii) $(x \star z^2) \star (y \star z) \in I$ *implies* $(x \star z) \star y \in I$.

iii) $(x \star z^2) \star (0 \star z) \in I$ *implies* $x \star z \in I$.

Definition 5.5. *A nonempty subset I of a BCI-algebra is called an $n-$fold BCI-positive implicative ideal of X if it satisfies (I_1) and*

(I_5) $\left(x \star y^{n+1}\right) \star (z \star y) \in I$ *and* $z \in I$ *imply* $x \star y^n \in I$; *for all* $x, y,\ z \in X$.

Example 5.6. *Let $X = \{0, 1, 2, 3, 4, 5\}$ with Cayley table as follows*

$\star$	0	1	2	3	4	5
0	0	0	0	3	3	3
1	1	0	1	3	3	3
2	2	2	0	3	3	3
3	3	3	3	0	0	0
4	4	3	4	1	0	0
5	5	3	5	1	1	0

Then $\mathcal{X} = <X; \star, 0>$ *is a BCI-algebra. We have* $I = \{0,1,2\}$ *is a* 2*-fold BCI-positive implicative ideal of* $\mathcal{X}$.

Remark 5.7. *It is clear from Proposition 5.4(iii) that* $1-$*fold BCI-positive implicative ideal of* X *is a BCI-positive implicative ideal of* X,

An $n-$*fold BCI-positive implicative ideal is a generalization of* n*-fold BCK-positive implicative ideal.*

Proposition 5.8. *Every* n*-fold BCI-positive implicative ideal of a BCI-algebra* X *must be an ideal of* X.

Proof: Let I be an $n-$fold BCI-positive implicative ideal in X; it is clear that $0 \in I$.

Let $x, y \in X$ such that $x \star y \in I$ and $y \in I$. We will prove that $x \in I$

$x \star y = \left(x \star 0^{n+1}\right) \star (y \star 0) \in I$ and $y \in I$, therefore since I is $n-$fold BCI-positive implicative $x = x \star 0^n \in I$ as required. □

However the following example shows that not every ideal is $n-$fold BCI-positive implicative.

Example 5.9. (1) *Let* $X = \{0, 1, a, b, c\}$ *with Cayley table as follows*

$\star$	0	1	a	b	c
0	0	0	a	a	a
1	1	0	a	a	a
a	a	a	0	0	0
b	b	a	1	0	a
c	c	a	1	1	0

Then $X = < X; \star, 0 >$ *is a BCI-algebra.* $I = \{0\}$ *is an ideal of* X, *but not a* 1*-fold BCI-positive implicative ideal of* X *because* $\left(c \star b^2\right) \star (0 \star c) = 0 \in I$ *and* $0 \in I$, *but* $c \star b = 1 \notin I$.

(2) *Let* $(X; \star, 0)$ *be a BCK-algebra that is* $n-$*fold positive implicative but not* $(n-1)-fold$ *positive implicative, where* n *is a positive integer.*

Consider $X' = X \cup \{a\}$ *where* $a \notin X$ *and define the operation* $\star'$ *on* X' *by:*

$$x \star' y = x \star y \;\; \textit{if} \;\; x, y \in X$$

$$x \star' a = a \star' x = a \;\; \textit{if} \;\; x \in X$$

$$a \star' a = 0.$$

Then $(X'; \star', 0)$ is a BCI-algebra. The zero ideal of X' is not $(n-1)-$fold positive implicative.

Remark 5.10. *It is easy to see that, in any n-fold BCI-positive implicative algebra, the concepts of the ideal and n-fold BCI-positive implicative ideal are the same.*

Theorem 5.11. *Let I be an ideal of a BCI-algebra X. Then the following conditions are equivalent:*

(i) $\left(x \star y^{n+1}\right) \star (z \star y) \in I$ implies $(x \star y^n) \star z \in I$, for all $x, y,\ z \in X$.

(ii) I is n-fold BCI-positive implicative ideal,

(iii) $\left(x \star y^{n+1}\right) \star (0 \star y) \in I$ implies $x \star y^n \in I$, for all $x, y \in X$.

Proof: $(i) \Rightarrow (ii)$. Let $x, y, z \in X$ such that $\left(x \star y^{n+1}\right) \star (z \star y) \in I$ and $z \in I$. We will prove that $x \star y^n \in I$.

Since $\left(x \star y^{n+1}\right) \star (z \star y) \in I$, we have $(x \star y^n) \star z \in I$ thus $x \star y^n \in I$; as $z \in I$.

$(ii) \Rightarrow (iii)$. It easy to obtain by setting $z = 0$

$(iii) \Rightarrow (i)$. Let $x, y, z \in X$ such that $v = \left(x \star y^{n+1}\right) \star (z \star y) \in I$. Putting $u = x \star z$, it suffice to show that $\left(u \star y^{n+1}\right) \star (0 \star y) \in I$.

$$\begin{array}{rcl} \left[\left(u \star y^{n+1}\right) \star (0 \star y)\right] \star v & = & \left[\left(\left(x \star y^{n+1}\right) \star (0 \star y)\right) \star \left(\left(x \star y^{n+1}\right) \star (z \star y)\right)\right] \star z \\ & \leq & [(z \star y) \star (0 \star y)] \star z \\ & \leq & (z \star 0) \star z = 0 \in I \end{array}.$$

Thus $\left[\left(u \star y^{n+1}\right) \star (0 \star y)\right] \star v \in I$ and as $v \in I$, it follows $\left(u \star y^{n+1}\right) \star (0 \star y) \in I$. Hence $u \star y^n = (x \star z) \star y^n = (x \star y^n) \star z \in I$ □

Theorem 5.12. *Let X be a BCI-algebra and I be a subset of X.*

If I is an $n-$fold BCI-positive implicative ideal of X, then $(x \star y^n) \star (0 \star (0 \star y)) \in I$ implies $x \star y^{n+1} \in I$.

Proof: Suppose that I is an $n-$fold BCI-positive implicative ideal of X. In order to prove that if $(x \star y^n) \star (0 \star (0 \star y)) \in I$, then $x \star y^{n+1} \in I$. putting $u = x \star y$, then it suffices to show that $(u \star y^n) \star (0 \star (0 \star y)) \in I$.

Let $x, y \in X$ such that $(x \star y^n) \star (0 \star (0 \star y)) \in I$,

$$\begin{aligned}
&((u\star y^{n+1})\star(0\star y)))\star((x\star y^{n})\star(0\star(0\star y))),\\
=\;&((((x\star y^{n})\star(0\star y))\star y)\star y)\star((x\star y^{n})\star(0\star(0\star y))),\\
=\;&((((x\star y^{n+1})\star(0\star y))\star((x\star y^{n})\star(0\star(0\star y))))\star y)\star y,\\
\leq\;&(((0\star(0\star y))\star(0\star y))\star y)\star y,\\
=\;&(((0\star(0\star y))\star y)\star(0\star y))\star y,\\
=\;&(0\star(0\star y))\star y=0\in I.
\end{aligned}$$

We have $((x\star y)\star y^{n+1})\star(0\star y)=(u\star y^{n+1})\star(0\star y)\in I$, since I is an $n-$fold BCI-positive implicative, we obtain $x\star y^{n+1}=(x\star y)\star y^{n}=u\star y^{n}\in I$. The proof is complete. □

However the converse of theorem 5.12 is not true. For instance it suffices to consider the zero ideal of example 5.9(1).

Lemma 5.13. *Let I be an ideal of a BCI-algebra X.*

If I is an $n-$fold BCI-positive implicative ideal, then I is $(n+1)$-fold BCI-positive implicative ideal.

Proof: let I be an $n-$fold BCI-positive implicative ideal of X, let $x,y\in X$ such that $\left(x\star y^{n+2}\right)\star(0\star y)\in I$. Then since I is $n-$fold BCI-positive implicative ideal of X; $\left((x\star y)\star y^{n+1}\right)\star(0\star y)=\left(x\star y^{n+2}\right)\star(0\star y)\in I$ implies $x\star y^{n+1}=(x\star y)\star y^{n}\in I$. Hence I is $(n+1)-$fold BCI-positive implicative ideal. □

Using this lemma and a simple induction argument, we obtain the following proposition.

Proposition 5.14. *Let I be an ideal of a BCI-algebra X and $k\in\mathbb{N}\backslash\{0\}$ If I is an $n-$fold BCI-positive implicative ideal, then I is $(n+k)$-fold BCI-positive implicative ideal.*

However there exists $(n+k)-$fold BCI-positive implicative ideals which are not $n-$fold BCI-positive implicative ideals

In fact the zero ideal of BCI-algebra of example3.11 is $n-$fold BCI-positive implicative, but not a $(n-1)-$fold BCI-positive implicative.

Theorem 5.15. ***(Extension Property for $n-$ fold BCI-positive implicative ideals)***

Let I, J be two ideals of a BCI-algebra X with $I\subset J$. If I is $n-$fold BCI-positive implicative then J is also $n-$fold BCI-positive implicative.

Proof: Let I,J two ideals such that $I\subset J$. Suppose that $u=\left(x\star y^{n+1}\right)\star(0\star y)\in J$. We have

$$\begin{aligned} ((x \star u) \star y^{n+1}) \star (0 \star y))) &= ((x \star y^{n+1}) \star u) \star (0 \star y)) \\ &= ((x \star y^{n+1}) \star (0 \star y)) \star u \\ &= 0 \in I \end{aligned}$$

Since I is $n-$fold BCI-positive implicative, $(x \star u) \star y^n = (x \star y^n) \star u \in I \subset J$, therefore $(x \star y^n) \star u \in J$. Note that J is an ideal of X and $u \in J$, we obtain $x \star y^n \in J$.

Hence by theorem 5.11(iii), J is an $n-$fold BCI-positive implicative ideal of X. □

Corollary 5.16. *Let I and J two ideals of BCI-algebra X. Then an ideal generated by $I \sqcup J$ denoted by $< I \sqcup J >$ is an $n-$fold BCI-positive implicative ideal if I or J is $n-$fold BCI-positive implicative ideal.*

Remark 5.17. *Let X be a BCI-algebra, I be a closed ideal (i.e.; $0 \star x \in I$ for all $x \in I$) of X. Define the relation Θ by: for all $x, y \in X$, $x\Theta y$ if and only if $x \star y \in I$ and $y \star x \in I$. Then Θ is a congruence of X and $\frac{X}{\Theta} =< \frac{X}{\Theta}; \star, [0]_\Theta >= \frac{X}{I}$; with $[x]_\Theta = I_x$; $[0]_\Theta = I$ and $[x]_\Theta \star [y]_\Theta = [x \star y]_\Theta$. So for any $a \in X$, $I_a = \{x \in X / \ x \star a \in I, \ a \star x \in I\}$. Clearly the class I_0 of 0 which is equal to I is the zero element of $\dfrac{X}{I}$.*

Theorem 5.18. *Let I be a closed ideal of a BCI-algebra X. Then I is an $n-$fold BCI-positive implicative if and only if the quotient algebra $\dfrac{X}{I}$ is n-fold positive implicative*

Proof: Suppose that I is n-fold BCI-positive implicative ideal. Let $x, y \in X$, and set $u = (x \star y^{n+1}) \star (0 \star y)$. Then $((x \star u) \star y^{n+1}) \star (0 \star y) = ((x \star y^{n+1}) \star (0 \star y)) \star u = 0 \in I$. Therefore since I is $n-$fold BCI-positive implicative , then $(x \star u) \star y^n \in I$, thus $(x \star y^n) \star u \in I$.

On the other hand $u = (x \star y^{n+1}) \star (0 \star y) \leq x \star y^n$. Hence $u \star (x \star y^n) = 0 \in I$, whence $(x \star y^n) \star u \in I$ and $u \star (x \star y^n) \in I$. Thus $I_u = I_{x \star y^n}$, that is $\left(I_x \star I_y^{n+1}\right) \star (I_0 \star I_y) = I_x \star I_y^n$. Therefore $\dfrac{X}{I}$ is an $n-$fold BCI-positive implicative as needed.

Conversely if $\dfrac{X}{I}$ is n-fold BCI-positive implicative, by the remark 5.10, the zero ideal $\{I\}$ of $\dfrac{X}{I}$ is $n-$fold positive implicative. Now, let $x, y \in X$ such that $(x \star y^{n+1}) \star (0 \star y) \in I$, then $(I_x \star I_y^{n+1}) \star (I \star I_y) = I \in \{I\}$. Since $\{I\}$ is n-fold positive implicative, then $I_x \star I_y^n = I$, thus $x \star y^n \in I$. Hence, I is n-fold positive implicative. □

Remark 5.19. *If X, Y are two isomorphic BCI-algebras, then X is n-fold positive implicative if and only if Y is.*

Now we can characterize the $n-$fold positive implicative BCI-algebras using ideals.

Corollary 5.20. *Let X be a BCI-algebra. Then X is n-fold BCI-positive implicative algebra if and only if the zero ideal of X is an $n-$fold BCI-positive implicative.*

Proof: By proposition 5.10 the necessity holds.

Conversely, if the zero ideal $\{0\}$ of X is n-fold positive implicative, then by Theorem 5.18, $\dfrac{X}{\{0\}}$ is n-fold positive implicative. But $\dfrac{X}{\{0\}} \cong X$, therefore by Remark 5.19, X is n-fold positive implicative. $\square$

Combining Corollary 5.20 and the Extension Property (Theorem 5.15), we obtain the following characterization of n-fold positive implicative BCI-algebras.

Corollary 5.21. *A BCI-algebra is n-fold BCI-positive implicative if and only if all its closed ideals are n-fold BCI-positive implicative.*

Conclusion and further suggestions To investigate the structure of an algebraic system, it is clear that ideals with special properties play an important role. The present paper introduced and studied $n-$fold BCI-positive implicative ideal. The extension property of $n-$fold BCI-positive implicative ideals was established and the $n-$fold positive implicative BCI-algebras were described completely. A characterization of $n-$fold positive implicative algebra was obtained. The main purpose of future work is to investigate the foldness of other types of ideals in BCI-algebras and the relation diagram between them similar to the one in [3]

References

[1] W. J. Blok and C. J. Van Alten, The finite embeddability property for residuated lattices, pocrims and BCK-algebras, Algebra Universalis, 43(3), (2002), $253-271$

[2] L. Chen, $n-$fold positive implicative BCK-algebras. Selected Papers on Fourth National BCK and BCI-algebraic Academic Conference, Fuzhou (1989)

[3] Y. Huang. and Z. Chen, On ideals in BCK-algebras , Math. japonica 50(1999) $211-226$

[4] Y. Huang, Characterizations of implicative BCI-algebras, Soochow Journal of Mathematics. Volume 25 No 4(1999), $375-386$.

[5] Y. Huang, BCI-Algebra, *Science Press*, **Beijing**,(2003).

[6] W. Huang, D. Sun, The semigroup characterizations of nil BCI-algebras, Southeast Asian Bulletin of Mathematics (2002)26 : $651-657$

[7] Y. B. Jun, S. Z. Song and C. Lele, Foldness of Quasi-associative Ideals in BCI-algebras, Sci. Math. Jpn, 6 (2002)$227-231$.

[8] Y. L. Liu, Y. Xu, J. Meng; BCI implicative ideals of BCI algebras, Information Sciences 22(2007) $4987-4996$

[9] Y. L. Liu and X. H. Zhang, Characterization of weakly positive implicative BCI-algebras, J. Hanzhong Teachers College (Natural) (3) (1994) $4-8$

[10] J. Meng, X. L. Xin, Positive implicative BCI-algebras, Pure Appl. Math.9(1)(1993)$19-22$.

[11] A. Torrens, Semisimplicity and the discriminator in bounded BCK-algebras, Algebra Universalis, $63(1), (2010), 1-16$

[12] S. M. Wei and Y. B. Jun, Weakly positive implicative BCI-algebras, Comm. Korean Math. Soc. 10 (1995) $815-821$

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