A CHARACTERIZATION OF MINIMAL REAL HYPERSURFACES OF TYPE (A_2) IN A COMPLEX PROJECTIVE SPACE IN TERMS OF THEIR GEODESICS

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Abstract. We characterize minimal real hypersurfaces M^{2n-1} of type (A_2) in a complex projective space by observing some geodesics on M. Note that there do *not* exist minimal real hypersurfaces M^{2n-1} of type (A_2) in a complex hyperbolic space.

1. Introduction

We denote by $\mathbb{C}P^n(c)$ a complex n-dimensional complex projective space of constant holomorphic sectional curvature $c(>0)$. In this paper we consider real hypersurfaces M^{2n-1} of $\mathbb{C}P^n(c)$ furnished with the canonical Kähler structure J and the standard Riemannian metric g through an isometric immersion.

Among real hypersurfaces in $\mathbb{C}P^n(c)$ the following hypersurfaces are typical examples:

(A_1) A geodesic sphere of radius r $(0 < r < \pi/\sqrt{c}\,)$ in $\mathbb{C}P^n(c)$;

(A_2) A tube of radius r $(0 < r < \pi/\sqrt{c}\,)$ around a totally geodesic Kähler submanifold $\mathbb{C}P^{\ell}(c)\,(1 \leqq \ell \leqq n-2)$ in $\mathbb{C}P^n(c)$.

These real hypersurfaces are said to be of type (A_1) and of type (A_2), respectively.

The following theorem shows the importance of these hypersurfaces.

Theorem A ([5]). *For each real hypersurface M^{2n-1} of $\mathbb{C}P^n(c), n \geqq 2$, the length of the derivative of the shape operator A of M satisfies $\|\nabla A\|^2 \geqq c^2(n-1)/4$. The equality holds on M if and only if M is locally congruent to one of real hypersurfaces of type (A_1) and type (A_2).*

Real hypersurfaces of type (A_1) have two distinct constant principal curvatures in $\mathbb{C}P^n(c)$. It is well-known that $\mathbb{C}P^n(c)$ does *not* admit totally umbilic real hypersurfaces and that a real hypersurface M^{2n-1} of $\mathbb{C}P^n(c), n \geqq 3$ is of type (A_1) if and only if M has at most two distinct principal curvatures at each point of M. These imply that real hypersurfaces of type (A_1) are the simplest examples of real hypersurfaces in $\mathbb{C}P^n(c)$ and that there exist no real hypersurfaces M all of whose geodesics are mapped to circles in $\mathbb{C}P^n(c)$.

Motivated by these facts, we characterize real hypersurfaces of type (A_1) in $\mathbb{C}P^n(c)$.

Theorem B ([4]). *A connected real hypersurface M^{2n-1} of $\mathbb{C}P^n(c), n \geqq 2$ is locally congruent to a real hypersurface of type (A_1) of radius r $(0 < r < \pi/\sqrt{c}\,)$ if and only if there exist orthonormal vectors $v_1, v_2, \ldots, v_{2n-2}$ perpendicular to the characteristic vector ξ_x at each point $x \in M$ satisfying the following two conditions*:

(*i*) *All geodesics $\gamma_i = \gamma_i(s)$ on M^{2n-1} with $\gamma_i(0) = x$ and $\dot{\gamma}_i(0) = v_i$ $(1 \leqq i \leqq 2n-2)$ are mapped to circles of positive curvature in $\mathbb{C}P^n(c)$;*

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(*ii*) *All geodesics* $\gamma_{ij} = \gamma_{ij}(s)$ *on* M^{2n-1} *with* $\gamma_{ij}(0) = x$ *and* $\dot\gamma_{ij}(0) = (v_i + v_j)/\sqrt{2}$ ($1 \leqq i < j \leqq 2n-2$) *are mapped to circles of positive curvature in* $\mathbb{C}P^n(c)$.

The purpose of this paper is to characterize *minimal* real hypersurfaces of type (A_2) in $\mathbb{C}P^n(c)$ from the viewpoint of Theorem B (see Theorem).

2. Preliminaries

Let M^{2n-1} be a real hypersurface with a unit normal local vector field $\mathcal{N}$ of $\mathbb{C}P^n(c)$ furnished with the standard Riemannian metric g and the canonical Kähler structure J. The Riemannian connections $\widetilde\nabla$ of $\mathbb{C}P^n(c)$ and ∇ of M are related by the following formulas of Gauss and Weingarten:

$$\widetilde\nabla_X Y = \nabla_X Y + g(AX, Y)\mathcal{N}, \tag{2.1}$$

$$\widetilde\nabla_X \mathcal{N} = -AX \tag{2.2}$$

for arbitrary vector fields X and Y on M, where g is the Riemannian metric of M induced from the ambient space $\mathbb{C}P^n(c)$ and A is the shape operator of M in $\mathbb{C}P^n(c)$. An eigenvector of the shape operator A is called a *principal curvature vector* of M in $\mathbb{C}P^n(c)$ and an eigenvalue of A is called a *principal curvature* of M in $\mathbb{C}P^n(c)$. We set $V_\lambda = \{v \in TM \mid Av = \lambda v\}$ which is called the principal distribution associated to the principal curvature λ.

It is known that M admits an almost contact metric structure (ϕ, ξ, η, g) induced from the Kähler structure J of $\mathbb{C}P^n(c)$. The characteristic vector field ξ of M is defined as $\xi = -J\mathcal{N}$ and this structure satisfies

$$\phi^2 = -I + \eta\otimes\xi,\ \eta(\xi) = 1 \quad \text{and} \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),$$

where I denotes the identity map of the tangent bundle TM of M. It follows from (2.1), (2.2) and $\widetilde\nabla J = 0$ that

$$(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi, \tag{2.3}$$

$$\nabla_X \xi = \phi AX. \tag{2.4}$$

The following is the so-called equation of Codazzi:

$$(\nabla_X A)Y - (\nabla_Y A)X = (c/4)\big(\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi\big). \tag{2.5}$$

We usually call M a *Hopf hypersurface* if the characteristic vector ξ of M is a principal curvature vector at each point of M. The following is useful for Hopf hypersurfaces in $\mathbb{C}P^n(c)$.

Proposition A ([5]). *Suppose that* ξ *is a principal curvature vector at each point of* M^{2n-1} *in* $\mathbb{C}P^n(c)$ *and the corresponding principal curvature is* δ. *Then* δ *is locally constant on* M. *In addition,* $A\phi X = \big((\delta\lambda + (c/2))/(2\lambda - \delta)\big)\phi X$ *holds for any* $X \in V_\lambda$ *which is perpendicular to* ξ.

In Proposition A, we remark that $2\lambda - \delta \neq 0$, since $c > 0$. Furthermore, every tube of sufficiently small constant radius around each Kähler submanifold of $\mathbb{C}P^n(c)$ is a Hopf hypersurface. This fact means that the notion of Hopf hypersurfaces is natural in the theory of real hypersurfaces in $\mathbb{C}P^n(c)$.

In $\mathbb{C}P^n(c)$ ($n \geqq 2$), a Hopf hypersurface all of whose principal curvatures are constant is locally congruent to one of the following (cf. [3, 6, 7]):

(A_1) A geodesic sphere of radius r, where $0 < r < \pi/\sqrt{c}$;
(A_2) A tube of radius r around a totally geodesic $\mathbb{C}P^\ell(c)$ ($1 \leqq \ell \leqq n-2$), where $0 < r < \pi/\sqrt{c}$;
(B) A tube of radius r around a complex hyperquadric $\mathbb{C}Q^{n-1}$, where $0 < r < \pi/(2\sqrt{c}\,)$;
(C) A tube of radius r around $\mathbb{C}P^1(c) \times \mathbb{C}P^{(n-1)/2}(c)$, where $0 < r < \pi/(2\sqrt{c}\,)$ and n ($\geqq 5$) is odd;
(D) A tube of radius r around a complex Grassmann $\mathbb{C}G_{2,5}$, where $0 < r < \pi/(2\sqrt{c}\,)$ and $n = 9$;
(E) A tube of radius r around a Hermitian symmetric space $\mathrm{SO}(10)/\mathrm{U}(5)$, where $0 < r < \pi/(2\sqrt{c}\,)$ and $n = 15$.

These real hypersurfaces are said to be of types (A_1), (A_2), (B), (C), (D) and (E). Summing up, real hypersurfaces of types (A_1) and (A_2), we call them hypersurfaces of type (A). The numbers of distinct principal curvatures of these real hypersurfaces are $2, 3, 3, 5, 5, 5$, respectively.

A direct calculation yields the following lemma.

Lemma 1. *Every real hypersurface of types* $(\mathrm{A}_1), (\mathrm{A}_2), (B), (C), (D)$ *and* (E), *which is a tube of radius* r, *is minimal in the following cases*:

(A_1) $\cot(\sqrt{c}\,r/2) = 1/\sqrt{2n-1}\,$;
(A_2) $\cot(\sqrt{c}\,r/2) = \sqrt{(2\ell+1)/(2n-2\ell-1)}\,$;
(B) $\cot(\sqrt{c}\,r/2) = \sqrt{n} + \sqrt{n-1}\,$;
(C) $\cot(\sqrt{c}\,r/2) = (\sqrt{n} + \sqrt{2}\,)/\sqrt{n-2}\,$;
(D) $\cot(\sqrt{c}\,r/2) = \sqrt{5}\,$;
(E) $\cot(\sqrt{c}\,r/2) = (\sqrt{15} + \sqrt{6}\,)/3$.

At the end of this section we review the definition of circles in Riemannian geometry. A real smooth curve $\gamma = \gamma(s)$ parametrized by its arclength s in a Riemannian manifold M with Riemannian connection ∇ is called a *circle* of curvature k if it satisfies the ordinary differential equations $\nabla_{\dot\gamma}\dot\gamma = kY_s$ and $\nabla_{\dot\gamma}Y_s = -k\dot\gamma$, where k is a nonnegative constant and Y_s is the unit normal vector of γ. A circle of null curvature is nothing but a geodesic. The definition of circles is equivalent to the equation

$$\nabla_{\dot\gamma}(\nabla_{\dot\gamma}\dot\gamma) + g(\nabla_{\dot\gamma}\dot\gamma, \nabla_{\dot\gamma}\dot\gamma)\dot\gamma = 0. \tag{2.6}$$

3. Statements of results

Theorem. *A connected minimal real hypersurface* M^{2n-1} *of* $\mathbb{C}P^n(c), n \geqq 3$ *is locally congruent to a tube of radius* $r = (2/\sqrt{c}\,)\cot^{-1}\sqrt{(2\ell+1)/(2n-2\ell-1)}$ $(0 < r < \pi/\sqrt{c}\,)$ *around a totally geodesic* $\mathbb{C}P^\ell(c)$ *with* $1 \leqq \ell \leqq n-2$ *if and only if there exist a function* $d : M \to \mathbb{N}$ *and orthonormal vectors* $v_1, v_2, \ldots, v_{2n-2}$ *perpendicular to the characteristic vector* ξ_x *at each point* $x \in M$ *satisfying the following two conditions*:

(*i*) *All geodesics* $\gamma_i = \gamma_i(s)$ *on* M^{2n-1} *with* $\gamma_i(0) = x$ *and* $\dot\gamma_i(0) = v_i$ $(1 \leqq i \leqq 2n-2)$ *are mapped to circles of positive curvature in* $\mathbb{C}P^n(c)$;
(*ii*) *All geodesics* $\gamma_{ij} = \gamma_{ij}(s)$ *on* M^{2n-1} *with* $\gamma_{ij}(0) = x$ *and* $\dot\gamma_{ij}(0) = av_i + \sqrt{1-a^2}\,v_j$ $(1 \leqq i \leqq d_x < j \leqq 2n-2)$ *are mapped to geodesics in* $\mathbb{C}P^n(c)$, *where* $a = \sqrt{(2\ell+1)/(2n)}$.

In this case, d *is automatically expressed as* $d = 2\ell$.

Proof. We first investigate the "only if" part of our Theorem. It is known that a real hypersurface M of type (A_2) with radius r $(0 < r < \pi/\sqrt{c}\,)$ has three distinct constant principal curvatures $\lambda_1 = (-\sqrt{c}/2)\tan(\sqrt{c}\,r/2), \lambda_2 = (\sqrt{c}/2)\cot(\sqrt{c}\,r/2)$ and $\delta = \sqrt{c}\cot(\sqrt{c}\,r) = \lambda_1 + \lambda_2$. As our real hypersurface M of type (A_2) is minimal, the principal curvatures λ_1 and λ_2 are expressed as follows (see Lemma 1):

$$\lambda_1 = -\frac{\sqrt{c}}{2}\sqrt{\frac{2n-2\ell-1}{2\ell+1}} \quad \text{and} \quad \lambda_2 = \frac{\sqrt{c}}{2}\sqrt{\frac{2\ell+1}{2n-2\ell-1}}\,. \tag{3.1}$$

Take orthonormal vectors $v_1, v_2, \ldots, v_{2n-2}$ orthogonal to ξ at an arbitrary point x of M in such a way that $v_1, v_2, \ldots, v_{2\ell}$ and $v_{2\ell+1}, \ldots, v_{2n-2}$ are principal curvature vectors with principal curvatures λ_1 and λ_2, respectively. Then by virtue of Lemma in [4] we find that these vectors satisfy Condition (i). That is, we have the following:

(i) All geodesics $\gamma_i = \gamma_i(s)$ on M with $\gamma_i(0) = x$ and $\dot\gamma_i(0) = v_i$ $(1 \leqq i \leqq 2\ell)$ are circles of positive curvature $|\lambda_1|$ in $\mathbb{C}P^n(c)$;
(ii) All geodesics $\gamma_i = \gamma_i(s)$ on M with $\gamma_i(0) = x$ and $\dot\gamma_i(0) = v_i$ $(2\ell+1 \leqq i \leqq 2n-2)$ are circles of positive curvature λ_2 in $\mathbb{C}P^n(c)$.

We next take the geodesic $\gamma_{ij} = \gamma_{ij}(s)$ on M^{2n-1} with $\gamma_{ij}(0) = x$ and $\dot\gamma_{ij}(0) = av_i + \sqrt{1-a^2}\,v_j$ $(1 \leqq i \leqq d_x = 2\ell < j \leqq 2n-2)$, where $a = \sqrt{(2\ell+1)/(2n)}$. It is well-known that the shape operator A of our real hypersurface M satisfies (cf. [5]):

$$g((\nabla_X A)X, X) = 0 \quad \text{for each } X \in TM. \tag{3.2}$$

It follows from (2.1), (3.1) and (3.2) that

$$g(\widetilde{\nabla}_{\dot\gamma_{ij}}\dot\gamma_{ij},\mathcal{N}) = g(A\dot\gamma_{ij}(s),\dot\gamma_{ij}(s)) = g(A\dot\gamma_{ij}(0),\dot\gamma_{ij}(0)) \\ = a^2\lambda_1 + (1-a^2)\lambda_2 = 0,$$

which yields Condition (ii).

We shall investigate the "if" part of our Theorem. We consider a connected real hypersurface M^{2n-1} satisfying Conditions (i) and (ii). We explain the discussion in [1] in detail. We first concentrate our attention on Condition (i). We study on an open dense subset

$$\mathcal{U} = \left\{ x \in M^{2n-1} \,\middle|\, \begin{matrix}\text{the multiplicity of each principal curvature of } M^{2n-1} \text{ in} \\ \mathbb{C}P^n(c) \text{ is constant on some neighborhood } \mathcal{V}_x(\subset \mathcal{U}) \text{ of } x\end{matrix} \right\}$$

of M^{2n-1}. We take the geodesic $\gamma_i = \gamma_i(s)$ $(1 \leqq i \leqq 2n-2)$ on $\mathcal{U}$ with initial vector v_i given by Condition (i). Since the curve γ_i, considered as a curve in $\mathbb{C}P^n(c)$, is a circle of positive curvature (, say) k_i, Equation (2.6) shows

$$\widetilde{\nabla}_{\dot\gamma_i}\widetilde{\nabla}_{\dot\gamma_i}\dot\gamma_i = -k_i^2\dot\gamma_i. \tag{3.3}$$

On the other hand, using (2.1) and (2.2), we see that

$$\widetilde{\nabla}_{\dot\gamma_i}\widetilde{\nabla}_{\dot\gamma_i}\dot\gamma_i = -g(A\dot\gamma_i,\dot\gamma_i)A\dot\gamma_i + g((\nabla_{\dot\gamma_i}A)\dot\gamma_i,\dot\gamma_i)\mathcal{N}. \tag{3.4}$$

Comparing the tangential components of Equations (3.3) and (3.4), we have

$$g(A\dot\gamma_i,\dot\gamma_i)A\dot\gamma_i = k_i^2\dot\gamma_i.$$

This, together with $k_i \neq 0$, shows that at $s = 0$ either $Av_i = k_iv_i$ or $Av_i = -k_iv_i$ holds for $i = 1, 2, \ldots, 2n-2$. This means that our real hypersurface M^{2n-1} is a Hopf hypersurface with $A\xi = \delta\xi$ and that the linear subspace $T_x^0M^{2n-1} = \{v \in T_xM^{2n-1} | v \perp \xi_x\}$ of T_xM^{2n-1} is decomposed as:

$$T_x^0M^{2n-1} = \{v \in T_x^0M \mid Av = -k_{i_1}v\} \oplus \{v \in T_x^0M \mid Av = k_{i_1}v\} \\ \oplus \cdots \oplus \{v \in T_x^0M \mid Av = -k_{i_g}v\} \oplus \{v \in T_x^0M \mid Av = k_{i_g}v\},$$

where $0 < k_{i_1} < k_{i_2} < \ldots < k_{i_g}$ and g is the number of distinct positive k_i $(i = 1, \ldots, 2n-2)$. We decompose $T_x^0M^{2n-1}$ in such a way at each point $x \in \mathcal{U}$.

Note that each k_{i_j} is a smooth function on $\mathcal{V}_x$ for each $x \in \mathcal{U}$. We shall show the constancy of each k_{i_j}. It suffices to check the case of $Av_{i_j} = k_{i_j}v_{i_j}$. As k_{i_j} is a constant function along the curve γ_{i_j} in the ambient space $\mathbb{C}P^n(c)$, we have $v_{i_j}k_{i_j} = 0$. For any v_ℓ $(1 \leqq \ell \neq i_j \leqq 2n-2)$, since A is symmetric, we have

$$g((\nabla_{v_{i_j}}A)v_\ell, v_{i_j}) = g(v_\ell, (\nabla_{v_{i_j}}A)v_{i_j}). \tag{3.5}$$

In order to compute Equation (3.5) easily, we extend the vectors $v_\ell, v_{i_j} (\in T_x^0M)$ on some sufficiently small neighborhood $\mathcal{W}_x(\subset \mathcal{V}_x)$ in the following manner.

We define a smooth vector field V_ℓ on $\mathcal{W}_x$ satisfying that $(V_\ell)_x = v_\ell$ and V_ℓ is perpendicular to ξ. Next we shall define V_{i_j}. First we define a smooth unit vector field W_{i_j} on some "sufficiently small" neighborhood $\mathcal{W}_x(\subset \mathcal{V}_x)$ by using parallel displacement for the vector v_{i_j} along each geodesic with origin x. We note that in general W_{i_j} is not principal on $\mathcal{W}_x$, but $AW_{i_j} = k_{i_j}W_{i_j}$ on the geodesic $\gamma_{i_j} = \gamma_{i_j}(s)$ with $\gamma_{i_j}(0) = x$ and $\dot\gamma_{i_j}(0) = v_{i_j}$. We here define the vector field U_{i_j} on $\mathcal{W}_x$ as: $U_{i_j} = \left(\prod_{\alpha\neq k_{i_j}}(A - \alpha I)\right)W_{i_j}$, where α runs over the set of all distinct principal curvatures of M^{2n-1} except for the principal curvature k_{i_j}. We remark that $U_{i_j} \neq 0$ on the neighborhood $\mathcal{W}_x$, because $(U_{i_j})_x \neq 0$. Moreover, the vector field U_{i_j} satisfies $AU_{i_j} = k_{i_j}U_{i_j}(\perp \xi)$ on $\mathcal{W}_{i_j}$. We define V_{i_j} by normalizing U_{i_j} in some sense. That is, when $\prod_{\alpha\neq k_{i_j}}(k_{i_j} - \alpha)(x) > 0$ (resp. $\prod_{\alpha\neq k_{i_j}}(k_{i_j} - \alpha)(x) < 0$), we define $V_{i_j} = U_{i_j}/\|U_{i_j}\|$ (resp. $V_{i_j} = -U_{i_j}/\|U_{i_j}\|$). Then we know that $AV_{i_j} = k_{i_j}V_{i_j}$ on $\mathcal{W}_x$ and $(V_{i_j})_x = v_{i_j}$. Furthermore, our construction shows that the integral curve of V_{i_j} through the point x is a geodesic on M^n, so that in particular $\nabla_{V_{i_j}}V_{i_j} = 0$ at the point x.

Since the Codazzi equation (2.5) yields that $g((\nabla_X A)Y, Z) = g((\nabla_Y A)X, Z)$ for any $X, Y, Z(\perp \xi)$, at the point x we have

$$\begin{aligned}(\text{the left-hand side of (3.5)}) &= g((\nabla_{v_\ell} A)v_{i_j}, v_{i_j})\\ &= g((\nabla_{V_\ell} A)V_{i_j}, V_{i_j})\\ &= g(\nabla_{V_\ell}(k_{i_j}V_{i_j}) - A\nabla_{V_\ell}V_{i_j}, V_{i_j})\\ &= g((V_\ell k_{i_j})V_{i_j} + (k_{i_j}I - A)\nabla_{V_\ell}V_{i_j}, V_{i_j})\\ &= v_\ell k_{i_j}\end{aligned}$$

and

$$\begin{aligned}(\text{the right-hand side of (3.5)}) &= g(V_\ell, (\nabla_{V_{i_j}} A)V_{i_j})\\ &= g(V_\ell, \nabla_{V_{i_j}}(k_{i_j}V_{i_j}) - A\nabla_{V_{i_j}}V_{i_j})\\ &= g(v_\ell, (v_{i_j}k_{i_j})v_{i_j}) = 0.\end{aligned}$$

Thus we can see that $Xk_{i_j} = 0$ for any $X(\perp \xi) \in T_xM$. Next, we shall show that $\xi k_{i_j} = 0$. It follows from (2.4) and Proposition A that

$$\begin{aligned}(\nabla_\xi A)V_{i_j} - (\nabla_{V_{i_j}} A)\xi &= \nabla_\xi(AV_{i_j}) - A\nabla_\xi V_{i_j} - \nabla_{V_{i_j}}(\delta\xi) + A\nabla_{V_{i_j}}\xi\\ &= \nabla_\xi(k_{i_j}V_{i_j}) - A\nabla_\xi V_{i_j} - \delta\phi AV_{i_j} + A\phi AV_{i_j}\\ &= (\xi k_{i_j})V_{i_j} + (k_{i_j}I - A)\nabla_\xi V_{i_j} - k_{i_j}\Big(\delta - \frac{\delta k_{i_j} + (c/2)}{2k_{i_j} - \delta}\Big)\phi V_{i_j}.\end{aligned}$$

On the other hand, the Codazzi equation (2.5) implies

$$g((\nabla_\xi A)V_{i_j} - (\nabla_{V_{i_j}} A)\xi, V_{i_j}) = 0.$$

Hence, $\xi k_{i_j} = 0$. Therefore we can see that the differential dk_{i_j} of k_{i_j} vanishes at the point x, which shows that every k_{i_j} (> 0) is constant on $\mathcal{W}_x$, since we can take the point x as an arbitrarily fixed point of $\mathcal{W}_x$. So the principal curvature function k_{i_j} is locally constant on the open dense subset $\mathcal{U}$ of M^{2n-1}. This, together with the continuity of k_{i_j} and the connectivity of M^{2n-1}, implies that k_{i_j} is constant on the hypersurface M^{2n-1}. Hence all principal curvatures of M^{2n-1} are constant if M^{2n-1} satisfies Condition (i).

Next, we consider Condition (ii). Since the above argument tells us that every v_i $(1 \leqq i \leqq 2n-2)$ is principal, we can set $Av_i = \mu_i v_i$. On the other hand, Condition (ii) shows that $g(A\dot\gamma_{ij}(0), \dot\gamma_{ij}(0)) = 0$, so that

$$a^2\mu_i + (1-a^2)\mu_j = 0 \quad \text{for } 1 \leqq \forall i \leqq d_x < \forall j \leqq 2n-2. \tag{3.6}$$

This, combined with $0 < a^2 = (2\ell+1)/(2n) < 1$, implies that M is a Hopf hypersurface with three distinct constant principal curvatures δ, μ_i and μ_j satisfying Equation (3.6). Hence M is of either type (A$_2$) or type (B). Needless to say, all minimal real hypersurfaces of type (A$_2$) satisfy Equation (3.6) (see the "only if" part of the proof of our Theorem).

Finally we shall check the case of type (B). We know that a real hypersurface M of type (B) with radius r $(0 < r < \pi/(2\sqrt{c}\,))$ has three distinct constant principal curvatures $\lambda_1 = (\sqrt{c}/2)\cot\big((\sqrt{c}\,r)/2 - \pi/4\big), \lambda_2 = (\sqrt{c}/2)\cot\big((\sqrt{c}\,r)/2 + \pi/4\big)$ and $\delta = \sqrt{c}\cot(\sqrt{c}\,r)$. As our real hypersurface M of type (B) is minimal, the principal curvatures λ_1 and λ_2 are expressed as (see Lemma 1):

$$\lambda_1 = -\frac{\sqrt{c}}{2}\frac{1+\sqrt{n}}{\sqrt{n-1}} \quad \text{and} \quad \lambda_2 = \frac{\sqrt{c}}{2}\frac{\sqrt{n}-1}{\sqrt{n-1}}. \tag{3.7}$$

The rest of the proof is to show that the principal curvatures λ_1 and λ_2 in (3.7) satisfy neither $a^2\lambda_1 + (1-a^2)\lambda_2 = 0$ nor $a^2\lambda_2 + (1-a^2)\lambda_1 = 0$. Suppose that $a^2\lambda_1 + (1-a^2)\lambda_2 = 0$. Then we have $-(2\ell+1)(1+\sqrt{n}\,) + (2n-2\ell-1)(\sqrt{n}-1) = 0$, so that $\sqrt{n} = n - 2\ell - 1$. Hence we can set $\sqrt{n} = p$ for some $p \in \mathbb{N}$, which implies that $p = p^2 - 2\ell - 1$. Thus we obtain the equality $p(p-1) = 2\ell + 1$, which is a contradiction. We next suppose that $a^2\lambda_2 + (1-a^2)\lambda_1 = 0$. By easy computation we get $(2\ell+1)(\sqrt{n}-1) - (2n-2\ell-1)(1+\sqrt{n}\,) = 0$, so that $-\sqrt{n} = n - 2\ell - 1$.

Then by the same discussion as in the case of $\sqrt{n} = n - 2\ell - 1$ we also obtain a contradiction in this case. □

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