COMMON FIXED POINTS FOR GENERALIZED WEAK CONTRACTION MAPPINGS IN MODULAR SPACES

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Abstract. In this paper, we prove the existence of common fixed points for a generalized weak contractive mapping in modular spaces. Moreover, we prove existence of some fixed point theorem without the $\Delta_2 - condition$.

1 Introduction

The famous Banach's contraction principle has been generalized in many ways over the years ([1, 3, 27, 28, 32, 33, 34, 35, 36]). One of the most interesting and studies is extension of Banach's contraction principle to case of weakly contraction.

A mapping $T : X \to X$ where (X, d) is a metric space, is said to be *weakly contraction* if

$$d(T(x), T(y)) \leq d(x, y) - \phi(d(x, y)), \tag{1.1}$$

where $\phi : [0, \infty) \to [0, \infty)$ is a continuous and nondecreasing function such that $\phi(t) = 0$ if and only if $t = 0$. In 2008, Dutta and Choudhury [5] introduced a new generalization of contraction in metric spaces and proved the following theorem;

Theorem 1.1. *Let (X, d) be a complete metric space, $T : X \to X$ be a self-mapping satisfying the inequality*

$$\psi(d(Tx, Ty)) \quad \leq \quad \psi(d(x, y)) - \phi(d(x, y)), \tag{1.2}$$

where $\psi, \phi : [0, \infty) \to [0, \infty)$ are both continuous and monotone nondecreasing functions with $\psi(t) = \phi(t) = 0$ if and only if $t = 0$. Then T has a unique fixed point.

We note that, if one takes $\psi(t) = t$, then (1.2) reduces to (1.1).

In 2009, Zhang and Song [37] used generalized φ-weak contractions which is defined for two mappings and gave conditions for existence of a common fixed point:

Theorem 1.2. *Let (X, d) be a complete metric space and $T, S : X \to X$ two mapping such that for all $x, y \in X$*

$$d(Tx, Sy) \quad \leq \quad M(x, y) - \varphi(M(x, y)), \tag{1.3}$$

where $\varphi : [0, \infty) \to [0, \infty)$ is lower semi-continuous function with $\varphi(t) > 0$ for $t \in (0, \infty)$ and $\varphi(0) = 0$, $M(x, y) = \max\{d(x, y), d(Tx, x), d(Sy, y), \frac{d(y,Tx)+d(x,Sy)}{2}\}$. Then there exists the unique point $u \in X$ such that $u = Tu = Su$.

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And later, D. Dorić [4] generalized Theorem 1.2 and proved the following theorem:

Theorem 1.3. *Let (X, d) be a complete metric space and $T, S : X \to X$ be two self-mappings such that for all $x, y \in X$*

$$\psi(d(Tx, Sy)) \quad \leq \quad \psi(M(x, y)) - \varphi(M(x, y)), \tag{1.4}$$

where

(a) $\psi : [0, \infty) \to [0, \infty)$ is a continuous monotone nondecreasing function with $\psi(t) = 0$ if and only if $t = 0$,

(b) $\varphi : [0, \infty) \to [0, \infty)$ is a lower semi-continuous function with $\varphi(t) = 0$ if and only if $t = 0$,

(c) M is defined in Theorem 1.2.

Then there exists the unique point $u \in X$ such that $u = Tu = Su$.

On the other hand, the notion of modular space was introduced by Nakano in 1950 [29] in connection with the theory of order spaces and redefined and generalized by Luxemburg [14, 15, 16, 17, 18, 19, 20, 21, 22, 23] and Orlicz in 1959 [25]. These spaces were developed following the successful theory of Orlicz spaces. For a current review of the theory of Musielak-Orlicz spaces and modular spaces, the reader is referred to the books of Musielak and Orlicz [25] and Kozłowski [9]. The existence of fixed point theorems in modular spaces has been studied [7, 8, 10, 11, 12, 13, 31]. In 2009, Razani and Moradi [30] studied fixed point theorems for ρ-compatible maps of integral type in modular spaces. Afterward, Beygmohammadi and Razani [2] proved the existence for mapping defined on a complete modular space satisfying contractive inequality of integral type. Recently Mongkolkaha and Kumam [26], studied existence of fixed point theorems for weak contractions mapping of integral type in modular spaces.

In objective of this paper we introduce some generalized weak contraction mapping and investigate generalized weak contractions mappings of Theorem 1.3 in the sense of modular spaces.

2 Preliminaries

First, we start with a brief recollection of basic concepts and facts in modular space;

Definition 2.1. *Let X be a vector space over $\mathbb{R}$(or $\mathbb{C}$). A functional $\rho : X \to [0, \infty]$ is called a* modular *if for arbitrary x and y, elements of X, it satisfies the following conditions :*

(1) $\rho(x) = 0$ if and only if $x = 0$;

(2) $\rho(\alpha x) = \rho(x)$ for all scalar α with $|\alpha| = 1$;

(3) $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$, whenever $\alpha, \beta \geq 0$ and $\alpha + \beta = 1$.

If we replace (3) by

(4) $\rho(\alpha x + \beta y) \leq \alpha^s \rho(x) + \beta^s \rho(y)$, for $\alpha, \beta \geq 0$, $\alpha^s + \beta^s = 1$ with an $s \in (0, 1]$, then the modular ρ is called s-convex modular, and if $s = 1$, ρ is called convex modular.

If ρ is a modular in X, then the set defined by

$$X_\rho = \{x \in X : \rho(\lambda x) \to 0 \quad as \quad \lambda \to 0\}, \tag{2.1}$$

is called a modular space. X_ρ is a vector subspace of X.

Proposition 2.2. ([24])

(1) $\rho(\alpha x)$ *is a nondecreasing function of* $\alpha \geq 0$*;*

(2) If ρ *is* $s-$*convex then* $\alpha^{-s}\rho(\alpha x)$ *is a nondecreasing function of* $\alpha \geq 0$.

Definition 2.3. *A modular* ρ *is said to satisfy the* Δ_2-*condition if* $\rho(2x_n) \to 0$ *as* $n \to \infty$*, whenever* $\rho(x_n) \to 0$ *as* $n \to \infty$.

Definition 2.4. *Let* X_ρ *be a modular space.*

(1) The sequence $(x_n)_{n\in\mathbb{N}}$ *in* X_ρ *is said to be* $\rho-$*convergent to* $x \in X_\rho$ *if* $\rho(x_n - x) \to 0$*, as* $n \to \infty$.

(2) The sequence $(x_n)_{n\in\mathbb{N}}$ *in* X_ρ *is said to be* $\rho-$*Cauchy if* $\rho(x_n - x_m) \to 0$*, as* $n, m \to \infty$.

(3) A subset C *of* X_ρ *is said to be* $\rho-$*closed if the* $\rho-$*limit of a* ρ*-convergent sequence of* C *always belongs to* C.

(4) A subset C *of* X_ρ *is said to be* $\rho-$*complete if any* $\rho-$*Cauchy sequence in* C *is* $\rho-$*convergent sequence and its limit is in* C.

(5) A subset C *of* X_ρ *is said to be* $\rho-$*bounded if* $\delta_\rho(C) = \sup\{\rho(x-y); x, y \in C\} < \infty$.

Definition 2.5. *Let* C *be a subset of* X_ρ *and* $T : C \to C$ *be an arbitrary mapping.* T *is called a* $\rho-$*contraction if for each* $x, y \in X_\rho$ *there exists* $0 \leq k < 1$ *such that*

$$\rho(T(x) - T(y)) \leq k\rho(x-y). \tag{2.2}$$

Definition 2.6. [30, Definition 2.1]*Let* X_ρ *be a modular space, where* ρ *satisfies the* Δ_2-*condition. Two self-mappings* T *and* f *of* X_ρ *are called* $\rho-$*compatible if* $\rho(Tfx_n - fTx_n) \to 0$ *as* $n \to \infty$*, whenever* $\{x_n\}_{n\in\mathbb{N}}$ *is a sequence in* X_ρ *such that* $fx_n \to z$ *and* $Tx_n \to z$ *for some point* $z \in X_\rho$.

3 A Generalized Weak Contraction in Modular Spaces

Theorem 3.1. *Let* X_ρ *be a* $\rho-$*complete modular space, where* ρ *satisfies the* Δ_2*-condition. Let* $c, l \in \mathbb{R}^+$*,* $c > l$ *and* $T, f : X_\rho \to X_\rho$ *are two* $\rho-$*compatible mappings such that* $T(X_\rho) \subseteq f(X_\rho)$ *and satisfying the inequality*

$$\psi(\rho(c(Tx - Ty))) \quad \leq \quad \psi(\rho(l(fx - fy))) - \phi(\rho(l(fx - fy))), \tag{3.1}$$

for all $x, y \in X_\rho$*, where* $\psi, \phi : [0, \infty) \to [0, \infty)$ *are both continuous and monotone nondecreasing functions with* $\psi(t) = \phi(t) = 0$ *if and only if* $t = 0$*. If one of* T *or* f *is continuous, then there exists a unique common fixed point of* T *and* f.

Proof. Let $x \in X_\rho$ and generate inductively the sequence $\{Tx_n\}_{n\in\mathbb{N}}$ as follow: $Tx_n = fx_{n+1}$. First, we prove that the sequence $\{\rho(c(Tx_n - Tx_{n-1}))\}$ converges to 0. By (3.1), we have

$$\begin{aligned} \psi(\rho(c(Tx_n - Tx_{n-1}))) &\leq \psi(\rho(l(fx_n - fx_{n-1}))) - \phi(\rho(l(fx_n - fx_{n-1}))) \\ &\leq \psi(\rho(l(fx_n - fx_{n-1}))). \end{aligned} \tag{3.2}$$

By monotone nondecreasing of ψ and Proposition 2.2 with $c > l$, we have

$$\begin{aligned} \rho(c(Tx_n - Tx_{n-1}))) &\leq \rho(l(fx_n - fx_{n-1}))) \\ &= \rho(l(Tx_{n-1} - Tx_{n-2})) \\ &< \rho(c(Tx_{n-1} - Tx_{n-2})). \end{aligned} \tag{3.3}$$

This means that the sequence $\{\rho(c(Tx_n - Tx_{n-1}))\}$ is nonincreasing and bounded below. Hence there exists $r \geq 0$ such that

$$\lim_{n\to\infty} \rho(c(Tx_n - Tx_{n-1})) = r. \tag{3.4}$$

If $r > 0$, taking $n \to \infty$ in the inequality (3.3), we get

$$\lim_{n\to\infty} \rho(l(fx_n - fx_{n-1})) = r. \tag{3.5}$$

Since

$$\psi(\rho(c(Tx_n - Tx_{n-1}))) \leq \psi(\rho(l(fx_n - fx_{n-1}))) - \phi(\rho(l(fx_n - fx_{n-1}))), \tag{3.6}$$

from (3.4), (3.5) and (3.6), it follow that $\psi(r) \leq \psi(r) - \phi(r)$ which is a contradiction, thus $r = 0$. That is

$$\lim_{n\to\infty} \rho(c(Tx_n - Tx_{n-1})) = 0. \tag{3.7}$$

Next, we prove that the sequence $\{cTx_n\}_{n\in\mathbb{N}}$ is a $\rho - Cauchy$. Suppose that $\{cTx_n\}_{n\in\mathbb{N}}$ is not $\rho - Cauchy$, then there exists $\varepsilon > 0$ and subsequence $\{x_{m_k}\}, \{x_{n_k}\}$ with $m_k > n_k \geq k$ such that

$$\rho(c(Tx_{m_k} - Tx_{n_k})) \geq \varepsilon \quad for \quad k = 1, 2, 3, ..., \tag{3.8}$$

where we can assume that

$$\rho(c(Tx_{m_k - 1} - Tx_{n_k})) < \varepsilon. \tag{3.9}$$

Let m_k be the smallest number exceeding n_k for which (3.8) holds, and set

$$\theta_k = \{m \in \mathbb{N} | \exists n_k \in \mathbb{N}; \rho(c(Tx_m - Tx_{n_k})) \geq \varepsilon, m > n_k \geq k\}. \tag{3.10}$$

Since $\theta_k \subset \mathbb{N}$ and clearly, $\theta_k \neq \emptyset$. By well ordering principle, the minimum element of θ_k is denoted by m_k and obviously (3.9) holds. Now, let $\alpha \in \mathbb{R}^+$ such that $\frac{l}{c} + \frac{1}{\alpha} = 1$, then we have

$$\begin{aligned} \psi(\rho(c(Tx_{m_k} - Tx_{n_k}))) &\leq \psi(\rho(l(fx_{m_k} - fx_{n_k}))) - \phi(\rho(l(fx_{m_k} - fx_{n_k}))) \\ &\leq \psi(\rho(l(fx_{m_k} - fx_{n_k}))) \\ &= \psi(\rho(l(Tx_{m_k - 1} - Tx_{n_k - 1}))), \end{aligned} \tag{3.11}$$

and

$$
(3.12)\quad \begin{aligned}
\rho(l(Tx_{m_k-1}-Tx_{n_k-1})) &= \rho(l(Tx_{m_k-1}-Tx_{n_k}+Tx_{n_k}-Tx_{n_k-1}))\\
&= \rho(\tfrac{l}{c}c(Tx_{m_k-1}-Tx_{n_k})+\tfrac{1}{\alpha}\alpha l(Tx_{n_k}-Tx_{n_k-1})))\\
&\le \rho(c(Tx_{m_k-1}-Tx_{n_k}))+\rho(\alpha l(Tx_{n_k}-Tx_{n_k-1}))\\
&< \varepsilon+\rho(\alpha l(Tx_{n_k}-Tx_{n_k-1})).
\end{aligned}
$$

Using the Δ_2-condition and (3.7), we get

$$\lim_{k\to\infty}\rho(\alpha l(Tx_{n_k}-Tx_{n_k-1}))=0$$

and thus

$$(3.13)\quad \lim_{k\to\infty}\psi(\rho(l(Tx_{m_k-1}-Tx_{n_k-1}))) \;<\; \psi(\varepsilon).$$

¿From (3.8), (3.11), (3.12) and (3.13), it follows that

$$
(3.14)\quad \begin{aligned}
\psi(\varepsilon) &\le \lim_{k\to\infty}\psi(\rho(l(Tx_{m_k-1}-Tx_{n_k-1})))\\
&< \psi(\varepsilon),
\end{aligned}
$$

which is a contradiction. Hence, $\{cTx_n\}_{n\in\mathbb{N}}$ is $\rho-Cauchy$ and by the Δ_2-condition $\{Tx_n\}_{n\in\mathbb{N}}$ is $\rho-Cauchy$. Since X_ρ is $\rho-complete$ there exists a point $u\in X_\rho$ such that $\rho(Tx_n-u)\to 0$ as $n\to\infty$, that is $Tx_n\to u$ and implies that $fx_n\to u$ as $n\to\infty$. If T is continuous, then $T^2x_n\to Tu$ and $Tfx_n\to Tu$ as $n\to\infty$. By ρ -compatible, $\rho(c(fTx_n-Tfx_n))\to 0$ as $n\to\infty$, thus, $fTx_n\to Tu$ as $n\to\infty$. Next, we prove that u is a unique fixed point of T. Indeed,

$$
(3.15)\quad \begin{aligned}
\psi(\rho(c(T^2x_n-Tx_n))) &= \psi(\rho(c(T(Tx_n)-Tx_n))).\\
&\le \psi(\rho(l(fTx_n-fx_n)))-\phi(\rho(l(fTx_n-fx_n))).
\end{aligned}
$$

Taking $n\to\infty$ in the inequality (3.15), we have

$$
\begin{aligned}
\psi(\rho(c(Tu-u))) &\le \psi(\rho(l(Tu-u)))-\phi(\rho(l(Tu-u)))\\
&\le \psi(\rho(l(Tu-u))).
\end{aligned}
$$

By monotone nondecreasing of ψ and Proposition 2.2 with $c>l$ we have $\rho(c(Tu-u))=0$ and $Tu=u$. Since $T(X_\rho)\subseteq f(X_\rho)$, then there exists a point u_1 such that $u=Tu=fu_1$. The inequality,

$$\psi(\rho(c(T^2x_n-Tu_1)))\le\psi(\rho(l(fTx_n-fu_1)))-\phi(\rho(l(fTx_n-fu_1)))$$

as $n\to\infty$, yields :

$$\psi(\rho(c(Tu-Tu_1)))\le\psi(\rho(l(Tu-fu_1)))-\phi(\rho(l(Tu-fu_1)))$$

and thus

$$
\begin{aligned}
\psi(\rho(c(u-Tu_1))) &\le \psi(\rho(l(u-fu_1)))-\phi(\rho(l(u-fu_1)))\\
&\le \psi(\rho(l(u-fu_1)))\\
&= \psi(\rho(l(u-u)))\\
&= 0.
\end{aligned}
$$

which implies that, $u = Tu_1 = fu_1$ and also $fu = fTu_1 = Tfu_1 = Tu = u$(see[6]). If f is continuous, then by a similar argument, one can prove $fu = Tu = u$. Finally, suppose that there exists $v \in X_\rho$ such that $Tv = v = fv$ and $v \neq u$, we have

$$\begin{aligned} \psi(\rho(c(u-v))) &= \psi(\rho(c(Tu-Tv))) \\ &\leq \psi(\rho(l(fu-fv))) - \phi(\rho(l(fu-fv))) \\ &< \psi(\rho(l(u-v))), \end{aligned} \tag{3.16}$$

again by monotone nondecreasing of ψ and Proposition 2.2 with $c > l$, we have

$$\rho(c(u-v)) < \rho(l(u-v)) < \rho(c(u-v))$$

which is a contradiction. Hence $u = v$ and the proof is complete. □

Corollary 3.2. *Let X_ρ be a $\rho-$complete modular space, where ρ satisfies the Δ_2-condition. Let $c, l \in \mathbb{R}^+$, $c > l$ and $T, f : X_\rho \to X_\rho$ are two $\rho-$compatible mappings such that $T(X_\rho) \subseteq f(X_\rho)$ and satisfying the inequality*

$$\rho(c(Tx-Ty)) \leq \rho(l(fx-fy)) - \phi(\rho(l(fx-fy))), \tag{3.17}$$

for all $x, y \in X_\rho$, where $\phi : [0,\infty) \to [0,\infty)$ is a continuous and monotone nondecreasing function with $\phi(t) = 0$ if and only if $t = 0$. If one of T or f is continuous, then there exists a unique common fixed point of T and f.

Proof. Take $\psi(t) = t$, we obtain the Corollary 3.2. □

Theorem 3.3. *Let X_ρ be a $\rho-$complete modular space and let $T, S : X_\rho \to X_\rho$ be mappings satisfying the inequality*

$$\psi(\rho((Tx-Sy))) \leq \psi(M(x,y)) - \phi(M(x,y)) \tag{3.18}$$

for all $x, y \in X_\rho$, where $M(x,y) = \max\{\rho(x-y), \rho(x-Tx), \rho(y-Sy), \frac{\rho(\frac{1}{2}(y-Tx))+\rho(\frac{1}{2}(x-Sy))}{2}\}$ and $\psi, \phi : [0,\infty) \to [0,\infty)$ are both continuous and monotone nondecreasing functions with $\psi(t) = \phi(t) = 0$ if and only if $t = 0$. Then there exists the unique point $u \in X_\rho$ such that $u = Tu = Su$.

Proof. Let $x_0 \in X_\rho$ we construct the sequence $\{x_n\}$ for $n \geq 0$ by $x_{2n+1} = Sx_{2n}$ and $x_{2n+2} = Tx_{2n+1}$. First, we prove that the sequence $\{\rho(x_{n+1} - x_n)\}$ converges to 0. If n is odd, then we have

$$\begin{aligned} \psi(\rho(x_{n+1}-x_n)) &= \psi(\rho(Tx_n - Sx_{n-1})) \\ &\leq \psi(M(x_n, x_{n-1})) - \phi(M(x_n, x_{n-1})). \\ &\leq \psi(M(x_n, x_{n-1})). \end{aligned} \tag{3.19}$$

By monotone nondecreasing of ψ, we have

$$\rho(x_{n+1}-x_n) \leq M(x_n, x_{n-1}). \tag{3.20}$$

Since

$$\begin{aligned} M(x_n, x_{n-1}) &= \max\{\rho(x_n - x_{n-1}), \rho(x_{n+1}-x_n), \tfrac{\rho(\frac{1}{2}(x_{n-1}-x_{n+1}))}{2}\} \\ &\leq \max\{\rho(x_n - x_{n-1}), \rho(x_{n+1}-x_n), \\ &\qquad \tfrac{\rho(x_{n-1}-x_n)+\rho(x_n-x_{n+1})}{2}\} \\ &\leq \max\{\rho(x_n - x_{n-1}), \rho(x_{n+1}-x_n)\}. \end{aligned} \tag{3.21}$$

If $\rho(x_{n+1} - x_n) > \rho(x_n - x_{n-1})$, then $\rho(x_{n+1} - x_n) \leq M(x_n, x_{n-1}) \leq \rho(x_{n+1} - x_n)$, so we have $M(x_n, x_{n-1}) = \rho(x_{n+1} - x_n)$, which implies that

$$\begin{array}{rcl} \psi(\rho(x_{n+1} - x_n)) & = & \psi(\rho(Tx_n - Sx_{n-1})) \\ & \leq & \psi(M(x_n, x_{n-1})) - \phi(M(x_n, x_{n-1})) \\ & = & \psi(\rho(x_{n+1} - x_n)) - \phi(\rho(x_{n+1} - x_n)) \\ & < & \psi(\rho(x_{n+1} - x_n)), \end{array} \tag{3.22}$$

which is a contradiction, thus

$$\rho(x_{n+1} - x_n) \quad \leq \quad \rho(x_n - x_{n-1}). \tag{3.23}$$

Similarly, in the case n is an even number we can obtain inequalities (3.23). So, we have the sequence $\{\rho(x_{n+1} - x_n)\}$ is nonincreasing and bounded below. Hence there exists $r \geq 0$ such that

$$\lim_{n\to\infty} \rho(x_{n+1} - x_n) \quad = \quad r. \tag{3.24}$$

Assume that $r > 0$. Since

$$\begin{array}{rcl} \psi(\rho(x_{n+1} - x_n)) & = & \psi(\rho(Tx_n - Sx_{n-1})) \\ & \leq & \psi(M(x_n, x_{n-1})) - \phi(M(x_n, x_{n-1})) \\ & = & \psi(\rho(x_n - x_{n-1})) - \phi(\rho(x_n - x_{n-1})), \end{array} \tag{3.25}$$

taking $n \to \infty$ in the inequality (3.25), we have $\psi(r) \leq \psi(r) - \phi(r)$ which is a contradiction, thus $r = 0$. So, we have

$$\lim_{n\to\infty} \rho(x_{n+1} - x_n) \quad = \quad 0. \tag{3.26}$$

Next, we prove that the sequence $\{x_n\}$ is $\rho - Cauchy$. Let $m, n \in \mathbb{N}$. Without loss of generality we can assume that m is odd and n is even. In fact, we have

$$\begin{array}{rcl} \rho(x_m - x_n) & \leq & M(x_m, x_n) \\ & = & \max\{\rho(x_m - x_n), \rho(x_m - x_{m+1}), \rho(x_n - x_{n+1}) \\ & & \frac{\rho(\frac{1}{2}(x_n - x_{m+1})) + \rho(\frac{1}{2}(x_m - x_{n+1}))}{2}\} \\ & \leq & \max\{\rho(x_m - x_n), \rho(x_m - x_{m+1}), \rho(x_n - x_{n+1}) \\ & & \frac{\rho(x_n - x_m) + \rho(x_m - x_{m+1}) + \rho(x_m - x_n) + \rho(x_n - x_{n+1})}{2}\}. \end{array} \tag{3.27}$$

Taking $m, n \to \infty$ in the inequality (3.27), we get

$$\lim_{m,n\to\infty} M(x_m, x_n) = \lim_{m,n\to\infty} \rho(x_m - x_n) := d \geq 0 \tag{3.28}$$

Since

$$\begin{array}{rcl} \psi(\rho(x_{m+1} - x_{n+1})) & = & \psi(\rho(Tx_m - Sx_n)) \\ & \leq & \psi(M(x_m, x_n)) - \phi(M(x_m, x_n)), \end{array} \tag{3.29}$$

from (3.28) and (3.29), it follow that $\psi(d) \leq \psi(d) - \phi(d)$, if $d > 0$, it is impossible. Thus $d = 0$. That is, the sequence $\{x_n\}_{n\in\mathbb{N}}$ is $\rho - Cauchy$. Since X_ρ is ρ- complete there exists a point $u \in X_\rho$ such that $\rho(x_n - u) \to 0$ as $n \to \infty$. Moreover, $\rho(x_{2n} - u) \to 0$ and $\rho(x_{2n+1} - u) \to 0$ as $n \to \infty$. Now we prove that $u = Tu = Su$. Suppose that $Tu \neq u$, then $\rho(u - Tu) > 0$ and there exists $N_1 \in \mathbb{N}$ such that for any $n > N_1$,

$$\begin{array}{c} \rho(x_{2n+1} - u) < \frac{1}{4}\rho(u - Tu), \qquad \rho(x_{2n} - u) < \frac{1}{4}\rho(u - Tu) \\ \text{and} \quad \rho(x_{2n} - x_{2n+1}) < \frac{1}{4}\rho(u - Tu). \end{array} \tag{3.30}$$

By the inequalities (3.30) and property of ρ, we have

$$
\begin{array}{rcl}
\rho(u-Tu) & \leq & M(u,x_{2n}) \\
& = & \max\{\rho(u-x_{2n}),\rho(u-Tu),\rho(x_{2n}-x_{2n+1}) \\
& & \frac{\rho(\frac{1}{2}(x_{2n}-Tu))+\rho(\frac{1}{2}(u-x_{2n+1}))}{2}\} \\
& \leq & \max\{\frac{1}{4}\rho(u-Tu),\rho(u-Tu),\frac{1}{4}\rho(u-Tu) \\
& & \frac{\rho(x_{2n}-u)+\rho(u-Tu)+\rho(x_{2n}-u)+\rho(x_{2n}-x_{2n+1})}{2}\} \\
& \leq & \max\{\frac{1}{4}\rho(u-Tu),\rho(u-Tu),\frac{7}{8}\rho(u-Tu)\} \\
& = & \rho(u-Tu),
\end{array} \tag{3.31}
$$

that is, $M(u,x_{2n})=\rho(u-Tu)$. Thus,

$$
\begin{array}{rcl}
\psi(\rho(Tu-x_{2n+1})) & = & \psi(\rho(Tu-Sx_{2n})) \\
& \leq & \psi(M(u,x_{2n}))-\phi(M(u,x_{2n})) \\
& = & \psi(\rho(Tu-u))-\phi(\rho(Tu-u))
\end{array} \tag{3.32}
$$

Taking $n\to\infty$ in the inequality (3.32), we have

$$\psi(\rho(Tu-u))\leq\psi(\rho(Tu-u))-\phi(\rho(Tu-u))<\psi(\rho(Tu-u)).$$

By monotone nondecreasing of ψ, we have

$$\rho(Tu-u)<\rho(Tu-u)$$

which is a contradiction. Hence $\rho(u-Tu)=0$ and $Tu=u$. If $Su\neq u$, then $\rho(Su-u)>0$. Using u is fixed point of T and property of ρ, we have

$$
\begin{array}{rcl}
\psi(\rho(u-Su)) & = & \psi(\rho(Tu-Su) \\
& \leq & \psi(M(u,u))-\phi(M(u,u)) \\
& = & \psi(\rho(u-Su))-\phi(\rho(u-Su)) \\
& < & \psi(\rho(u-Su)).
\end{array} \tag{3.33}
$$

Again by monotone nondecreasing of ψ, we have $\rho(u-Su)<\rho(u-Su)$ which is a contradiction. Thus $u=Tu=Su$. If there exists point $v\in X_\rho$ such that $Tv=v=Sv$ and $u\neq v$, then from

$$
\begin{array}{rcl}
\psi(\rho(u-v)) & = & \psi(\rho(Tu-Sv)) \\
& \leq & \psi(M(u,v))-\phi(M(u,v)) \\
& \leq & \psi(\rho(u-v))-\phi(\rho(u-v)),
\end{array} \tag{3.34}
$$

we conclude that $u=v$ and the proof is complete. □

Corollary 3.4. *Let X_ρ be a $\rho-$complete modular space and let $T,S:X_\rho\to X_\rho$ be mappings satisfying the inequality*

$$\rho(Tx-Sy)\quad\leq\quad M(x,y)-\phi(M(x,y)) \tag{3.35}$$

for all $x,y\in X_\rho$, where $M(x,y)=\max\{\rho(x-y),\rho(x-Tx),\rho(y-Sy),\frac{\rho(\frac{1}{2}(y-Tx))+\rho(\frac{1}{2}(x-Sy))}{2}\}$ and $\phi:[0,\infty)\to[0,\infty)$ is continuous and monotone nondecreasing function with $\phi(t)=0$ if and only if $t=0$. Then there exists the unique point $u\in X_\rho$ such that $u=Tu=Su$.

Corollary 3.5. *Let X_ρ be a $\rho-$ complete modular space and let $T: X_\rho \to X_\rho$ be a mapping satisfying the inequality*

$$\psi(\rho((Tx-Ty))) \quad \leq \quad \psi(M(x,y)) - \phi(M(x,y)) \tag{3.36}$$

for all $x, y \in X_\rho$, where $M(x,y) = \max\{\rho(x-y), \rho(x-Tx), \rho(y-Ty), \frac{\rho(\frac{1}{2}(y-Tx))+\rho(\frac{1}{2}(x-Ty))}{2}\}$ and $\psi, \phi : [0,\infty) \to [0,\infty)$ are both continuous and monotone nondecreasing functions with $\psi(t) = \phi(t) = 0$ if and only if $t = 0$. Then T has a unique fixed point.

Corollary 3.6. *Let X_ρ be a $\rho-$ complete modular space and let $T: X_\rho \to X_\rho$ be a mapping satisfying the inequality*

$$\rho(Tx-Ty) \quad \leq \quad M(x,y) - \phi(M(x,y)) \tag{3.37}$$

for all $x, y \in X_\rho$, where $M(x,y) = \max\{\rho(x-y), \rho(x-Tx), \rho(y-Ty), \frac{\rho(\frac{1}{2}(y-Tx))+\rho(\frac{1}{2}(x-Ty))}{2}\}$ and $\phi : [0,\infty) \to [0,\infty)$ is continuous and monotone nondecreasing function with $\phi(t) = 0$ if and only if $t = 0$. Then T has a unique fixed point.

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