REFINEMENTS OF THE HARDY AND MORGAN UNCERTAINTY PRINCIPLES

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ABSTRACT. Various generalizations of Hardy’s theorem and Morgan’s theorem, which assert that a function on \( \mathbb{R} \) and its Fourier transform cannot both be very small, are known. We give two theorems which improve various generalizations known so far.

1 Introduction

For an integrable function \( f \) on \( \mathbb{R} \), we define the Fourier transform \( \hat{f} \) by

\[
\hat{f}(y) = \int_{-\infty}^{+\infty} f(x) e^{-ixy} dx, \quad y \in \mathbb{R}.
\]

Classical Hardy’s theorem [4] reads as follows: if \( a, b > 0, \ ab = 1/4 \), and if \( f \) is a measurable function on \( \mathbb{R} \) such that \( f(x)e^{ax^2} \in L^\infty(\mathbb{R}) \) and \( \hat{f}(y)e^{by^2} \in L^\infty(\mathbb{R}) \), then \( f \) is a constant multiple of \( e^{-ax^2} \).

An immediate corollary of this theorem is the following: if \( a, b > 0, \ ab > 1/4 \), and if \( f \) is a measurable function on \( \mathbb{R} \) satisfying (1), then \( f = 0 \) almost everywhere. The examples \( f(x) = e^{ax^2}P(x) \) with \( P(x) \) polynomials show that there are infinitely many \( f \)'s that satisfy (1) for \( ab < 1/4 \).

Morgan [6] proved the following variant of Hardy’s theorem: if \( 1 < \beta < 2 < \alpha < \infty, 1/\alpha + 1/\beta = 1, \ a, b > 0, \) and

\[
(aa)^{1/\alpha}(b\beta)^{1/\beta} > (\sin(\pi(\beta - 1)/2))^{1/\beta},
\]

and if \( f \) is a measurable function on \( \mathbb{R} \) satisfying

\[
f(x)e^{a|x|^\alpha} \in L^\infty(\mathbb{R}) \quad \text{and} \quad \hat{f}(y)e^{b|y|^\beta} \in L^\infty(\mathbb{R}),
\]

then \( f = 0 \) almost everywhere. He also obtained that the condition (2) is optimal; if \( (aa)^{1/\alpha}(b\beta)^{1/\beta} = (\sin(\pi(\beta - 1)/2))^{1/\beta} \), then for any \( m \in \mathbb{R} \) and \( m' = (2m - \alpha + 2)/(2\alpha - 2) \), there exists a measurable function \( f \) on \( \mathbb{R} \) such that \( (1 + |x|)^{-m}f(x)e^{a|x|^\alpha} \in L^\infty(\mathbb{R}) \) and \( (1 + |y|)^{-m'}\hat{f}(y)e^{b|y|^\beta} \in L^\infty(\mathbb{R}) \). Therefore, there are infinitely many \( f \)'s that satisfy (3).

Various generalizations of Hardy’s theorem and Morgan’s theorem are known. Cowling and Price [2] proved that, if in Hardy’s theorem the assumption (1) is replaced by

\[
f(x)e^{ax^2} \in L^p(\mathbb{R}) \quad \text{and} \quad \hat{f}(y)e^{dy^2} \in L^q(\mathbb{R})
\]

with \( 1 \leq p, q \leq \infty \) and with at least one of \( p \) and \( q \) finite, then \( f = 0 \). The third author proved that (see [5], Theorem 1), if \( a, b > 0, \ ab = 1/4 \), and if \( f \) is a measurable function on \( \mathbb{R} \) such that

\[
f(x)e^{ax^2} \in L^1(\mathbb{R}) + L^\infty(\mathbb{R}) \quad \text{and} \quad \int_{-\infty}^{+\infty} \log^+ |\hat{f}(y)e^{by^2}| dy < \infty
\]
for some $C > 0$, then $f$ is a constant multiple of $e^{-ax^2}$. Here $L^1(\mathbb{R}) + L^\infty(\mathbb{R})$ is the set of functions of the form $f = f_1 + f_2$, $f_1 \in L^1(\mathbb{R})$, $f_2 \in L^\infty(\mathbb{R})$, and $\log^+ x = \log x$ if $x > 1$ and $\log^+ x = 0$ if $x \leq 1$. Ben Farah and Mokni [1] proved that, if we replace $L^\infty$ in the assumptions of Morgan’s theorem by $L^p$ and $L^q$, $1 \leq p, q \leq \infty$, then $f = 0$ and the condition (2) is optimal.

The purpose of the present paper is to give further generalizations of the above theorems. Our results are the following two theorems.

**Theorem 1** Let $1 < \alpha, \beta < \infty$, $1/\alpha + 1/\beta = 1$, $a, b > 0$, and
\[(a\alpha)^{1/\alpha}(b\beta)^{1/\beta} > c(\alpha, \beta)\]
with
\[c(\alpha, \beta) = \begin{cases} (\sin(\pi(\beta - 1)/2))^{1/\beta} & \text{if } \beta < 2, \\ (\sin(\pi(\alpha - 1)/2))^{1/\alpha} & \text{if } \beta > 2. \end{cases}\]

Suppose $f$ is a measurable function on $\mathbb{R}$ such that
\[e^{a|x|^\alpha}f(x) \in L^1(\mathbb{R}) + L^\infty(\mathbb{R})\]
and
\[\int_{-\infty}^{+\infty} \log^+ \frac{|\hat{f}(y)|e^{b|y|^\beta}}{C} \frac{dy}{1 + |y|} < \infty\]
for some $C > 0$. Then $f = 0$ almost everywhere.

**Theorem 2** If $a, b > 0$, $ab = 1/4$, and if $f$ is a measurable function on $\mathbb{R}$ that satisfies (6) and (7) with $\alpha = \beta = 2$, then $f(x)$ is a constant multiple of $e^{-ax^2}$.

**Remark 3** (a) If the conditions (4) and (6) are satisfied and if we take $a' < a$ sufficiently near to $a$, then (4) is still satisfied with $a'$ in place of $a$ and the condition (6) implies
\[f(x)e^{a'|x|^\alpha} = f(x)e^{a|x|^\alpha}e^{(a'-a)|x|^\alpha} \in L^1(\mathbb{R}).\]

Hence the essential claim of Theorem 1 remains unchanged if the assumption (6) is replaced by the seemingly stronger assumption $f(x)e^{a|x|^\alpha} \in L^1(\mathbb{R})$.
(b) It is easy to see that (3) or its $L^p$-$L^q$-version implies (6) and (7). Therefore, $L^p$-$L^q$ Morgan’s theorem follows from Theorem 1.
(c) Theorem 2 is an improvement of the third author’s Theorem 1 in [5], where the condition (7) was assumed with $dy$ instead of $dy/(1 + |y|)$.
(d) Similarly as Morgan’s result, the condition (4) is optimal.

In §3 we shall prove Theorems 1 and 2. Part of the argument will be only a slight modification of that of [5]. Since the paper [5] was published in a proceedings of a local seminar in Japan and is not easy to refer, we shall repeat some argument of [5] for convenience of the reader.
2 Key lemmas For $-\infty < \alpha < \beta < \infty$, we write
\[ D(\alpha, \beta) = \{ z \mid \alpha < \arg z < \beta \}, \]
which is the domain in the Riemann surface of $\log z$. We shall give three lemmas. The first lemma is an improvement of Lemma 1 of [5], where the integral (8) below is taken with respect to $ds$ instead of $ds/s$.

**Lemma 4** Let $-\infty < \alpha < \beta < \infty$ and $f$ be a bounded holomorphic function on $D(\alpha, \beta)$. Then for each $\theta$ with $\alpha < \theta < \beta$,
\begin{align*}
(8) \quad \sup_{0 < r < \infty} \log |f(re^{i\theta})| & \leq c_+(\alpha, \beta, \theta) \int_0^\infty \log^+ |f(se^{i\alpha})| \frac{ds}{s} + c_-(\alpha, \beta, \theta) \int_0^\infty \log^+ |f(se^{i\beta})| \frac{ds}{s},
\end{align*}
where
\[ c_+(\alpha, \beta, \theta) = \frac{1 \pm \cos \frac{\pi(\theta - \alpha)}{\beta - \alpha}}{2(\beta - \alpha) \sin \frac{\pi(\theta - \alpha)}{\beta - \alpha}} \]
and $f(se^{i\alpha})$ and $f(se^{i\beta})$ denote the nontangential boundary values of $f(z)$.

**Proof.** Let $\delta = (\beta - \alpha)/\pi$. For $z = re^{i\theta} \in D(\alpha, \beta)$, we make a change of variables as $z = e^{i\alpha}w^\delta$. Then $w \in D(0, \pi)$ and $g(w) = f(z) = f(e^{i\alpha}w^\delta)$ is a bounded holomorphic function on the upper half plane. Let $P_w(t) = \Im w/\pi w - t^2$ be the Poisson kernel for the upper half plane. Then Jensen’s inequality (cf. [3], Chap. II, §4, p.65) gives
\[ \log |f(z)| = \log |g(w)| \leq \int_{-\infty}^\infty P_w(t) \log |g(t)| dt \]
\[ \leq \int_{-\infty}^\infty P_w(t) \log^+ |g(t)| dt \]
\[ = \int_{-\infty}^\infty P_w(t) \log^+ |f(e^{i\alpha}t^\delta)| dt \]
\[ = \int_0^\infty P_w(t) \log^+ |f(e^{i\alpha}t^\delta)| dt + \int_0^\infty P_w(-t) \log^+ |f(e^{i\beta}t^\delta)| dt \]
\[ = \frac{1}{\delta} \int_0^\infty P_w(t^{1/\delta})t^{1/\delta} \log^+ |f(e^{i\alpha}t)| \frac{dt}{t} \]
\[ + \frac{1}{\delta} \int_0^\infty P_w(-t^{1/\delta})t^{1/\delta} \log^+ |f(e^{i\beta}t)| \frac{dt}{t}. \]
If we write $w = (re^{i(\theta - \alpha)})^{1/\delta} = u + iv$, then
\[ \max_{0 < s < \infty} \{ sP_w(\pm s) \} = \frac{us}{\pi((u + s)^2 + v^2)} \big|_{s = \sqrt{u^2 + v^2}} \]
\[ = \frac{v}{2\pi(\sqrt{u^2 + v^2} + u)} = \frac{\sqrt{u^2 + v^2} \pm u}{2\pi v} = \delta c_+(\alpha, \beta, \theta). \]
Hence the desired inequality follows. 

**Lemma 5** Let $0 < \beta - \alpha < \pi/\rho$ and $f$ be a holomorphic function on $D(\alpha, \beta)$. Suppose that there exist constants $A, B > 0$ such that
\[ |f(z)| \leq Ae^{B|z|^\rho} \]
for all $z \in D(\alpha, \beta)$. Then (8) holds for each $\theta$ with $\alpha < \theta < \beta$. 

\[ \text{REFINEMENTS OF THE HARDY AND MORGAN UNCERTAINTY PRINCIPLES 3} \]
Proof. By a rotation of the variable of the variable, we may suppose that \( \alpha = -\beta \) and \( 0 < \beta < \pi/(2\rho) \).

Take a \( \gamma \) such that \( \gamma > \rho \) and \( \gamma \beta < \pi/2 \). For \( \epsilon > 0 \), set \( f_\epsilon(z) = f(z)e^{-\epsilon z^2} \). Then \( f_\epsilon \) is holomorphic on \( D = D(-\beta, \beta) \). Moreover, if \( z \in D \) and \( \phi = \arg z \), then

\[
|f_\epsilon(z)| = |f(z)|e^{-\epsilon |z|^2} \cos \gamma \phi \leq Ae^{B|z|^\gamma \cos \gamma \phi}.
\]

Since \( \gamma > \rho \) and \( \cos \gamma \beta > 0 \), it follows that \( f_\epsilon \) is bounded on \( D \). Hence (8) holds with \( f \) replaced by \( f_\epsilon \). We note that \( |f_\epsilon(z)| \leq |f(z)| \) on \( D \) and \( f_\epsilon(z) \to f(z) \) as \( \epsilon \to 0 \). Hence, letting \( \epsilon \to 0 \), we have the desired inequality.

The last lemma is well known as the Phragmén-Lindelöf theorem, which can be proved by an application of Lemma 5 to \( f(z)/M \).

Lemma 6 Let \( \alpha, \beta, \rho \) and \( f \) satisfy the same assumptions as in Lemma 5. Assume in addition that there exists a constant \( M \) such that \( |f(z)| \leq M \) on the boundary of \( D(\alpha, \beta) \). Then \( |f(z)| \leq M \) for all \( z \in D(\alpha, \beta) \).

3 Proof of Theorem 1 We shall use the notation

\[
l(\theta) = \{re^{i\theta} \mid r > 0\}, \quad \theta \in \mathbb{R}.
\]

Let \( a, b, \alpha, \beta \), and \( f \) satisfy the assumptions of Theorem 1. As noted in Remark 3 (a), by replacing \( a \) with a smaller constant if necessary, we may assume that \( f(t)e^{a|t|^\alpha} \in L^1(\mathbb{R}) \). Thus \( f(t), t \in \mathbb{R}, \) is of the form \( f(t) = f_1(t)e^{-a|t|^\alpha} \) with \( f_1 \in L^1(\mathbb{R}) \).

We define \( \hat{f}(z) \) for \( z \in \mathbb{C} \) by

\[
\hat{f}(z) = \int_{-\infty}^{+\infty} f(t)e^{-izt}dt.
\]

For \( z = x + iy \in \mathbb{C} \),

\[
|\hat{f}(z)| \leq \int_{-\infty}^{\infty} |f_1(t)|e^{-a|t|^\alpha} e^{yt}dt.
\]

Using Young’s inequality \( u^\alpha/\alpha + v^\beta/\beta \geq uv \) for \( u, v > 0 \) with \( u = (\alpha a)^{1/\alpha}|t| \) and \( v = |y|/(\alpha a)^{1/\alpha} \), we have \( a|t|^\alpha + |y|^\beta/(\beta(\alpha a)^{\beta/\alpha}) \geq |y||t| \) and thus

\[
\int_{-\infty}^{\infty} |f_1(t)|e^{-a|t|^\alpha} e^{yt}dt \leq e^{\|e^{\beta/(\beta(\alpha a)^{\beta/\alpha})}\|f_1\|_1}.
\]

Combining the above inequalities, we see that there exists a constant \( c \) such that

\[
|\hat{f}(x + iy)| \leq ce^{A|y|^\alpha}, \quad A = 1/(\beta(\alpha a)^{\beta/\alpha}).
\]

It is also easy to see that \( \hat{f}(z) \) is an entire holomorphic function.

We shall consider the two cases \( \beta < 2 \) and \( \beta > 2 \) separately.

Case I: \( 1 < \beta < 2 \). In this case the condition (4) with (5) implies

\[
A(-\cos \pi \beta/2) < b.
\]

Since \( -\cos \pi \beta/2 > 0 \), we can take a sufficiently small \( \epsilon > 0 \) such that \( 0 < \epsilon < \pi/2\beta \) and

\[
A < (-\cos \pi \beta/2)^{-1}b\left(\frac{\tan(\pi \beta/2 + \beta \epsilon)}{\tan \pi \beta/2} \sin^2 \pi \beta/2 + \cos^2 \pi \beta/2\right)
= -b \frac{\tan(\pi \beta/2 + \beta \epsilon)}{\tan \pi \beta/2} \sin \pi \beta/2 - b \cos \pi \beta/2
= v \sin \pi \beta/2 - b \cos \pi \beta/2,
\]
where we set

\[(12) \quad v = -b \tan(\pi \beta / 2 + \beta \epsilon).\]

We set

\[\theta_e = \pi / 2 - \pi / 2 \beta + \epsilon.\]

Notice that \(0 < \theta_e < \pi / 2\).

We shall prove that \(\hat{f}\) is bounded on \(l(\theta_e)\). To prove this, consider the function

\[g(z) = \hat{f}(z)e^{(b+iv)z^\beta}, \quad z \in D(0, \pi / 2).\]

By (10), there exists \(B > 0\) such that

\[(13) \quad |g(z)| \leq ce^{B|z|^\beta}\]

for \(z \in D(0, \pi / 2)\). Since \(g(x)\), \(x \in \mathbb{R}\), is bounded on a neighborhood of \(x = 0\), the condition (7) implies that there exists a constant \(C' > 0\) such that

\[(14) \quad \int_0^\infty \log + \frac{|g(x)|}{C'} \frac{dx}{x} < \infty.\]

For \(z = re^{i\pi / 2}, r > 0\), from (10) and (11) we have

\[(15) \quad |g(z)| \leq ce^{r^\beta(A+b \cos \pi \beta / 2 - v \sin \pi \beta / 2)} \leq c.\]

Since \(\pi / 2 < \pi / \beta\), we can apply Lemma 5 to \(g\) on \(D(0, \pi / 2)\) to see that \(g(z)\) is bounded on each half line \(l(\theta)\) with \(0 < \theta < \pi / 2\). For \(z = re^{i\theta}, r > 0\), (12) gives

\[|\hat{f}(z)| = |g(z)||e^{-(b+iv)z^\beta} = |g(z)|e^{-r^\beta\{b \cos \beta \theta_e - v \sin \beta \theta_e\}} = |g(z)|.\]

Thus, since \(g\) is bounded on \(l(\theta_e)\), \(\hat{f}\) is bounded on \(l(\theta_e)\).

Applying the same argument to \(\hat{f}(-z)\), \(\hat{f}(-z)\), we see that \(\hat{f}\) is also bounded on \(l(-\theta_e), l(\theta_e + \pi)\), and \(l(-\theta_e + \pi)\). By (10), \(\hat{f}\) is also bounded on \(l(0)\) and \(l(\pi)\). Notice that the 6 half lines \(l(\pm \theta_e), l(\pm \theta_e + \pi), l(0), l(\pi)\) divide the complex plane into 6 sectors each of which has angle less than \(\pi / \beta\). Thus using Lemma 6, we conclude that \(\hat{f}\) is bounded on the whole plane. Thus by Liouville’s theorem \(\hat{f}\) is a constant. Obviously the constant must be 0 and hence \(\hat{f} = 0\) and \(f = 0\). This completes the proof for the case \(\beta < 2\).

Case II: \(2 < \beta < \infty\). Define \(v\) by

\[(16) \quad v = A(\sin \pi / 2 \beta)^\beta.\]

Consider

\[g(z) = \hat{f}(z)e^{(b+iv)z^\beta}, \quad z \in D(0, \pi / 2 \beta).\]

By (10) and (7), there exist constants \(B\) and \(C'\) for which \(g\) satisfies (13) for \(z \in D(0, \pi / 2 \beta)\) and (14). For \(z = re^{i\pi / 2}, r > 0\), it follows from (10) and (16) that

\[|g(z)| \leq ce^{r^\beta\{A(\sin \pi / 2 \beta)^\beta - v\}} = c.\]
Hence, by Lemma 5, \( g \) is bounded on \( l(\theta) \) for each \( \theta \in (0, \pi/2\beta) \). Thus we proved

\[
\sup_{r > 0} \{|\hat{f}(re^{i\theta})|e^{r^2(b \cos \beta \theta - v \sin \beta \theta)}\} < \infty
\]

for each \( \theta \in (0, \pi/2\beta) \).

Applying the same argument with \( \tilde{\hat{f}}(\tilde{\zeta}) \) in place of \( \hat{f}(\zeta) \), we also have

\[
\sup_{r > 0} \{|\tilde{\hat{f}}(re^{-i\theta})|e^{r^2(b \cos \beta \theta - v \sin \beta \theta)}\} < \infty
\]

for each \( \theta \in (0, \pi/2\beta) \).

Take a \( \theta_0 \) satisfying \( 0 < \theta_0 < \pi/2\beta \) and set

\[
b' = b - v \tan \beta \theta_0.
\]

Consider the function \( h(z) = \hat{f}(z)e^{b'z^\beta} \) on \( D_0 = D(-\theta_0, \theta_0) \). For \( z = re^{i\theta_0}, r > 0 \), we have

\[
|h(z)| = |\hat{f}(re^{i\theta_0})|e^{r^2b' \cos \beta \theta_0} = |\hat{f}(re^{i\theta_0})|e^{r^2(b \cos \beta \theta_0 - v \sin \beta \theta_0)}.
\]

Thus, by (17) and (18), the function \( h(z) \) is bounded on \( l(\pm \theta_0) \). By (10), \( h(z) \) satisfies the global estimate \( |h(z)| \leq ce^{B|z|^s} \) on \( D_0 \). Since \( 2\theta_0 < \pi/\beta \), we can use Lemma 6 to see that \( h(z) \) is bounded on \( D_0 \). Thus, in particular, \( \hat{f}(y)e^{b'y^\beta} \) is bounded for \( y > 0 \).

Applying the same argument to \( \hat{f}(-z) \), we see that \( \hat{f}(-y)e^{b'y^\beta} \) is also bounded for \( y > 0 \). Thus we conclude that \( \hat{f}(y)e^{b'|y|^{\alpha}} \) is bounded for \( y \in \mathbb{R} \).

Now the conditions (6) and (7) are satisfied with \( f, \alpha, \beta, a, b \) replaced by \( \hat{f}, \beta, \alpha, b', a \). Notice that \( b' \rightarrow b \) as \( \theta_0 \rightarrow 0 \). Hence if we take \( \theta_0 \) sufficiently small the condition (4) is satisfied with \( \alpha, \beta, a, b \) replaced by \( \beta, \alpha, b', a \). Therefore, applying the result of Case 1, we conclude that \( f = 0 \). This completes the proof of Theorem 1.

### 4 Proof of Theorem 2

By dilation of variables, we may assume that \( a = b = 1/2 \). We define \( \hat{f}(z) \) by (9). From (6) with \( a = 1/2 \) and \( \alpha = 2 \), it follows that, for \( z = x + iy \in \mathbb{C} \),

\[
|\hat{f}(z)| \leq \int_{-\infty}^{\infty} |f(t)|e^{t^2}dt = e^{y^2/2} \int_{-\infty}^{\infty} |f(t)|e^{t^2/2}e^{-(t-y)^2/2}dt \leq ce^{y^2/2},
\]

where \( c \) is a constant independent of \( z \). It is also easy to see that \( \hat{f} \) is an entire holomorphic function. We consider \( g(z) = \hat{f}(z)e^{z^2/2} \), which is also an entire function. We shall prove that \( g(z) \) is bounded.

For \( \epsilon \in (0, \pi/2) \), we set

\[
v_{\epsilon} = (\tan \epsilon)/4 = (\sin \epsilon)^2/2 \sin 2\epsilon, \quad \theta_{\epsilon} = \pi/2 - \epsilon
\]

and

\[
g_{\epsilon}(z) = \hat{f}(z)e^{(1/2 + iv_{\epsilon})z^2}.
\]

By (19), there exists a constant \( B_{\epsilon} \) such that

\[
|g_{\epsilon}(z)| \leq ce^{B_{\epsilon}|z|^2}, \quad z \in \mathbb{C}.
\]
For $x \in \mathbb{R}$, $|g_\epsilon(x)| = |\hat{f}(x)e^{x^2/2}|$ satisfies (14) for some sufficiently large $C'$ which is independent of $\epsilon$. For $z = re^{i\theta}$, $r > 0$, (19) implies

$$
|g_\epsilon(z)| \leq ce^{(r^2/2)((\sin \theta_\epsilon)^2+\cos 2\theta_\epsilon-2v_\epsilon \sin 2\theta_\epsilon)}
= ce^{(r^2/2)((\cos \theta_\epsilon)^2-2v_\epsilon \sin 2\theta_\epsilon)}
= ce^{(r^2/2)((\sin \epsilon)^2-2v_\epsilon \sin 2\epsilon)} = c.
$$

If $0 < \theta < \theta_\epsilon$, then using Lemma 5 we have

$$
\sup_{r>0}|g_\epsilon(re^{i\theta})| \leq c(\theta, \epsilon),
$$

where the constant $c(\theta, \epsilon)$ remains bounded if $\theta \in (0, \pi/2)$ is fixed and $\epsilon \to 0$. Since, as $\epsilon \to 0$, $v_\epsilon \to 0$ and $g_\epsilon(z) \to g(z)$, we conclude that $g(z)$ is bounded on each half line $l(\theta)$ with $0 < \theta < \pi/2$.

Applying the same argument to $\bar{g}(-\bar{z})$, $g(-z)$, $\bar{g}(\bar{z})$, we see that $g$ is also bounded on the half lines $l(\theta)$ for $\pi/2 < \theta < \pi$, $\pi < \theta < 3\pi/2$, and $3\pi/2 < \theta < 2\pi$. Thus we can find, say, 5 half lines that divide the complex plane into 5 sectors each of which has angle less than $\pi/2$ and $g(z)$ is bounded on each half line. Thus, using Lemma 6, we can conclude that $g$ is bounded on the whole complex plane. Since $g$ is entire, it must be constant and thus $f(x)$ is a constant multiple of $e^{-x^2/2}$.

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