OPERATOR Q-CLASS FUNCTIONS

Jun Ichi Fujii[1], Mohsen Kian[2] and Mohammad Sal Moslehian[3]

Received November 16, 2010

Abstract. We introduce the notion of operator Q-class function. Every non-negative operator convex function is of operator Q-class, but the converse is not true in general. Some inequalities for the operator Q-class functions are presented. In particular, we consider some conditions under which the operator Q-class functions have the operator monotonicity property.

1 Introduction

A function $f : J \to \mathbb{R}$ is said to be a Q-class function if

$$f(\lambda x + (1-\lambda)y) \le \frac{f(x)}{\lambda} + \frac{f(y)}{1-\lambda}.$$

for all $x, y \in J$ and all $\lambda \in (0, 1)$. This notion is introduced by Godunova and Levin [7].

Let D be a subset of $\mathbb{R}$ with at least two elements. A function $f : D \to \mathbb{R}$ is said to be a Schur function if

$$f(t)(t-s)(t-u) + f(s)(s-t)(s-u) + f(u)(u-t)(u-s) \ge 0,$$

for all s, t, u in D. In [7] Godunova and Levin showed that the class of Schur functions and the Q-class functions coincide. Many properties of classical Q-class functions can be found in [4, 5, 10, 11]. It is easy to see that every non-negative monotone function or convex function is of Q-class.

Let $\mathcal{H}$ be a Hilbert space and $\mathrm{B}(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. We say that an operator A in $\mathrm{B}(\mathcal{H})$ is positive and write $A \ge 0$ if $\langle Ax, x\rangle \ge 0$ for all $x \in \mathcal{H}$. Let I be the identity operator on $\mathcal{H}$ and $A, B \in \mathrm{B}(\mathcal{H})$. By $A \le B$ we mean that $B - A \ge 0$. We denote $A > 0$ if A is a invertible positive operator or equivalently, there exists a number $m > 0$ such that $A \ge mI$. Also in this case, $A > B$ means $A - B > 0$. The spectrum of an operator $A \in \mathrm{B}(\mathcal{H})$ is denoted by $\sigma(A)$.

In this paper we introduce operator Q-class functions and state some relations between the operator Q-class functions and the operator monotonicity property. In particular, we show that if $\alpha > 0$ and $f : (0, 1/\alpha) \to \mathbb{R}$ is a continuous function with $f(t) \le t^{-\beta}$ for $\beta > 1$ such that f is of operator Q-class on $(0, 1/\alpha)$, then it is operator decreasing on $(0, 1/\alpha)$. Other types of such conditions implying the operator monotonicity are discussed.

2 Main results

We start our work with the following definition.

Definition 2.1. Let f be a continuous real valued function defined on an interval J. We say that f is an operator Q-class function on J if

$$f(\lambda A + (1-\lambda)B) \le \frac{f(A)}{\lambda} + \frac{f(B)}{1-\lambda}, \tag{2.1}$$

for all self-adjoint operators A, B with spectra in J and all $\lambda \in (0, 1)$.

2010 *Mathematics Subject Classification.* Primary 47A63; Secondary 47B10, 47A30.

Key words and phrases. Operator Q-class, Operator monotonicity, Jensen inequality, Operator convexity.

Using induction one can easily extend (2.1) to n-tuples of operators and scalars, which is a Jensen type inequality, see [8]:

$$f\left(\sum_{k=1}^{n}\lambda_k A_k\right) \le \sum_{k=1}^{n}\frac{f(A_k)}{\lambda_k}$$

for all self-adjoint operators A_k with $\sigma(A_k)\subseteq J$ $(1\le k\le n)$ and positive numbers λ_k $(1\le k\le n)$ with $\sum_{k=1}^{n}\lambda_k=1$.

The next lemma gives a useful property of Q-class functions.

Lemma 2.2. *Let $f:J\to\mathbb{R}$ be a continuous function. If f is of Q-class, in particular if f is of operator Q-class, then $f(t)\ge 0$ for all $t\in J$.*

Proof. Let $x,y\in I$ and $\lambda\in(0,1)$. Then $f(\lambda x+(1-\lambda)y)\le\frac{f(x)}{\lambda}+\frac{f(y)}{1-\lambda}$. Multiplying each side by $\lambda(1-\lambda)$ we get $\lambda(1-\lambda)f(\lambda x+(1-\lambda)y)\le(1-\lambda)f(x)+\lambda f(y)$. Letting $\lambda\to 0$ we obtain $f(x)\ge 0$. □

It is trivial that every non-negative operator convex function is of operator Q-class. Also if a function $f:(0,\infty)\to(0,\infty)$ is operator monotone, then $\frac{1}{f}$ is operator convex [1, Corollary V.2.6] and so $\frac{1}{f}$ is of operator Q-class. Here we give an example for operator Q-class functions that is not operator convex (another example is also given in Example 2.7(1).

Example 2.3. Consider the function $f(x)=3-x^2$ on the interval $(0,1)$. Then it is not operator convex but operator concave. Since $2\le f(x)\le 3$ and $2\le\max\{\frac{1}{\lambda},\frac{1}{1-\lambda}\}$, we have

$$3-\big(\lambda A+(1-\lambda)B\big)^2\le 3<2\times 2\le\frac{3-A^2}{\lambda}+\frac{3-B^2}{1-\lambda},$$

for all self-adjoint operators A and B with spectra in $(0,1)$, that is, f is of operator Q-class.

Theorem 2.4. *Let f be a continuous Q-class function with $f(0)=0$. Then*

$$(G(A)-A)[G(A)f(G(A))-Af(A)]\ge 0 \tag{2.2}$$

for all selfadjoint A with $\sigma(A)\subseteq\mathrm{dom}f$ and functions G with $G(\mathrm{dom}f)\subseteq\mathrm{dom}f$.

Proof. Since f is of Q-class,

$$f(t)(t-s)(t-u)+f(s)(s-t)(s-u)+f(u)(u-s)(u-t)\ge 0 \tag{2.3}$$

for all real numbers $s,t,u\in\mathrm{dom}f$. Putting $u=0$ and $s=G(A)$ in (2.3), we obtain

$$tf(t)(t-G(A))+G(A)f(G(A))(G(A)-t)\ge 0. \tag{2.4}$$

Since A commutes with $G(A)$, we can put $t=A$ to get

$$Af(A)(A-G(A))+G(A)f(G(A))(G(A)-A)\ge 0,$$

which is (2.2). □

Corollary 2.5. *Let f be a continuous Q-class function with $f(0)=0$. Then*

$$\langle Ax,x\rangle\langle Af(A)x,x\rangle\le\langle A^2f(A)x,x\rangle, \tag{2.5}$$

for all selfadjoint operators A with $\sigma(A)\subseteq\mathrm{dom}f$ and all unit vectors x.

Proof. Putting the constant function $G(t) = \langle Ax, x\rangle$ in the above theorem, we have

$$B \equiv (\langle Ax, x\rangle - A)[\langle Ax, x\rangle f(\langle Ax, x\rangle) - Af(A)] \geq 0. \tag{2.6}$$

It follows from

$$\begin{aligned} 0 \leq \langle Bx, x\rangle &= \langle Ax, x\rangle^2 f(\langle Ax, x\rangle) - \langle Ax, x\rangle\langle Af(A)x, x\rangle \\ &\quad - \langle Ax, x\rangle^2 f(\langle Ax, x\rangle) + \langle A^2 f(A)x, x\rangle \\ &= \langle A^2 f(A)x, x\rangle - \langle Ax, x\rangle\langle Af(A)x, x\rangle, \end{aligned}$$

that (2.5) holds true. □

Now we discuss the relations between operator Q-class functions and operator monotone ones.

Theorem 2.6. *Let $\alpha > 0$, $\beta > 1$ and $f : (\alpha, \infty) \longrightarrow \mathbb{R}$ be a continuous function with $f(t) \leq t^\beta$. If $f(t^{-1})$ is of operator Q-class on $(0, 1/\alpha)$, then f is operator monotone on (α, ∞).*

Proof. Let $0 < \alpha < A$ and $0 < \varepsilon \leq B_\varepsilon = B + \varepsilon$ for any positive operator B. Take positive invertible operators $C = (A+B_\varepsilon)^{-1} < A^{-1} < 1/\alpha$ and $D = A^{-1}-(A+B_\varepsilon)^{-1} < A^{-1} < 1/\alpha$. Since $f(t^{-1})$ is of operator Q-class on $(0, 1/\alpha)$ and

$$\lambda(C + D) = \lambda C + (1 - \lambda)\frac{\lambda}{1 - \lambda}D,$$

we have

$$\begin{aligned} f((\lambda(C + D))^{-1}) &\leq \frac{f(C^{-1})}{\lambda} + \frac{1}{1-\lambda} f\left(\left(\frac{\lambda}{1-\lambda}D\right)^{-1}\right) \\ &\leq \frac{f(C^{-1})}{\lambda} + \frac{1}{1-\lambda}\left(\frac{1-\lambda}{\lambda}\right)^\beta D^{-\beta} \\ &= \frac{f(C^{-1})}{\lambda} + \frac{(1-\lambda)^{\beta-1}}{\lambda^\beta} D^{-\beta}. \end{aligned}$$

Letting $\lambda \longrightarrow 1$, we obtain

$$f(A) = f((C + D)^{-1}) \leq f(C^{-1}) = f(A + B_\varepsilon).$$

As $\varepsilon \longrightarrow 0$, we have

$$f(A) \leq f(A + B)$$

for all positive operators B. We therefore conclude that f is operator monotone on (α, ∞). □

Example 2.7. The following functions satisfy the conditions of Theorem 2.6.

1. $f(t) = t^r$, $0 \leq r \leq 1$ on $(1, +\infty)$. Then $t^r \leq t^2$ for $t \in (1, +\infty)$. Since f is operator concave, by taking inverses for the Jensen operator inequality and the arithmetic-harmonic mean inequality, we have

$$\begin{aligned} \big(\lambda A + (1 - \lambda)B\big)^{-r} &\leq (\lambda A^r + (1 - \lambda)B^r)^{-1} \\ &\leq \lambda A^{-r} + (1 - \lambda)B^{-r} \leq \frac{A^{-r}}{\lambda} + \frac{B^{-r}}{1-\lambda}, \end{aligned}$$

   which implies $f(t^{-1})$ is an operator Q-class function.

2. $f(t) = \ln t$ on $(1, +\infty)$. In this domain, we have $\log t \le t \le t^\beta$. Since $f(t^{-1}) = -\ln t$ is nonnegative operator convex on $(0,1)$ and hence of operator Q-class on $(0,1)$.

Contrastively we have similar conditions for operator decreasing:

Theorem 2.8. *Let $\alpha > 0$ and $f : (0, 1/\alpha) \to \mathbb{R}$ be a continuous function with $f(t) \le t^{-\beta}$ for some $\beta > 1$. If f is of operator Q-class on $(0, 1/\alpha)$, then it is operator decreasing on $(0, 1/\alpha)$.*

Proof. Theorem 2.6 implies that $f(t^{-1})$ is operator monotone on (α, ∞).
Let $0 < A \le B < \frac{1}{\alpha}$. Then $\alpha < B^{-1} \le A^{-1}$ whence $f(B) \le f(A)$. □

Taking $\alpha \longrightarrow 0$, we have a variation for functions on $(0, \infty)$:

Corollary 2.9. *Let $f : (0, \infty) \to \mathbb{R}$ be a continuous function with $f(t) \le t^{-\beta}$ for some $\beta > 1$. If f is of operator Q-class on $(0, \infty)$, then it is operator decreasing on $(0, \infty)$.*

Remark 2.10. Since an operator monotone (resp., operator decreasing) function on $(0, \infty)$ is operator concave (resp., operator convex), such functions satisfy

$$f(\Phi(A)) \ge \Phi(f(A)) \quad (\text{resp., } f(\Phi(A)) \le \Phi(f(A)))$$

for all unital positive linear maps Φ, which is the so-called Jensen operator inequality due to Davis–Choi, see [2, 3, 6]. In particular, a function in Corollary 2.9 is operator convex and hence satisfies

$$f\left(\sum_{i=1}^n \lambda_i A_i\right) \le \sum_{i=1}^n \lambda_i f(A_i) \tag{2.7}$$

for all $\lambda_i \ge 0 \quad (1 \le i \le n)$ with $\sum_{i=1}^n \lambda_i = 1$.

By this remark and that f is operator decreasing, we have some inequalities including the subadditivity:

Corollary 2.11. *Let f be a function as in Corollary 2.9. Then*

$$f\left(\sum_{i=1}^n A_i\right) \le f\left(\sum_{i=1}^n \lambda_i A_i\right) \le \sum_{i=1}^n \lambda_i f(A_i) \le \sum_{i=1}^n f(A_i)$$

for all $\lambda_i \ge 0 \quad (1 \le i \le n)$ with $\sum_{i=1}^n \lambda_i = 1$.

Moreover we have an inequality for weights, whose sum is greater than 1:

Corollary 2.12. *Let f be a function as in Corollary 2.9 and $p_i > 0 \quad (1 \le i \le n)$ with $\sum_{i=1}^n p_i \ge 1$. Then*

$$f\left(\sum_{i=1}^n p_i A_i\right) \le \sum_{i=1}^n p_i f(A_i). \tag{2.8}$$

Proof. By $\sum_{i=1}^n p_i A_i \ge \sum_{i=1}^n \frac{p_i}{\sum_k p_k} A_i$, decreasingness and the Jensen inequality (2.7) we have

$$f\left(\sum_{i=1}^n p_i A_i\right) \le f\left(\sum_{i=1}^n \frac{p_i}{\sum_k p_k} A_i\right) \le \sum_{i=1}^n \frac{p_i}{\sum_k p_k} f(A_i) \le \sum_{i=1}^n p_i f(A_i).$$

□

Finally we have a variation of Theorem 2.8:

Theorem 2.13. *Let $f : (0,\infty) \to (0,\infty)$ be a function with the property that $tf(t) \le f(t^{-1})$ and $\lim_{t\to 0^+} f(t) = 0$. If f is of operator Q-class, then it is operator decreasing.*

Proof. Let $0 < A \le B$ and $\epsilon > 0$. It follows from

$$\lambda(B+\epsilon) = \lambda A + (1-\lambda)\frac{\lambda}{1-\lambda}(B-A+\epsilon)$$

that

$$\begin{aligned} f(\lambda(B+\epsilon)) &\le \frac{f(A)}{\lambda} + \frac{f(\frac{\lambda}{1-\lambda}(B-A+\epsilon))}{(1-\lambda)} \\ &\le \frac{f(A)}{\lambda} + \frac{1}{1-\lambda}\frac{1-\lambda}{\lambda}(B-A+\epsilon)^{-1} f\left(\frac{1-\lambda}{\lambda}(B-A+\epsilon)^{-1}\right) \\ &= \frac{f(A)}{\lambda} + \frac{1}{\lambda}(B-A+\epsilon)^{-1} f\left(\frac{1-\lambda}{\lambda}(B-A+\epsilon)^{-1}\right). \end{aligned}$$

Letting $\lambda \to 1$ and $\epsilon \to 0$ we obtain $f(B) \le f(A)$. □

Acknowledgement. The first author was partially supported by the Ministry of Education, Science, Sports and Culture, Grant-in-Aid for Scientific Research (C), 20540166, 2008.

References

[1] R. Bhatia, *Matrix Analysis*, Springer-Verlag, New York, 1997.

[2] M.-D. Choi, *A Schwarz inequality for positive linear maps on C^*-algebras*, Illinois J. Math. **18** (1974), 565–574.

[3] C. Davis, *A Schwarz inequality for convex operator functions*, Proc. Amer. Math. Soc. **8** (1957), 42–44.

[4] S.S. Dragomir and C.E.M. Pearce, *On Jensen's inequality for a class of functions of Godunova and Levin*, Period. Math. Hungar. **33** (1996), no. 2, 93–100.

[5] S.S. Dragomir, J. Pečarić and L.E. Persson, *Some inequalities of Hadamard type*, Soochow J. Math. **21** (1995), no. 3, 335–341.

[6] J.I. Fujii and M. Fujii, *Jensen's inequalities on any interval for operators*, Proc. 3-rd Int. Conf. on Nonlinear Analysis and Convex Analysis, (2004) 29–39.

[7] E.K. Godunova and V.I. Levin, *Inequalities for functions of a broad class that contains convex, monotone and some other forms of functions*, (Russian) Numerical mathematics and mathematical physics (Russian), 138–142, 166, Moskov. Gos. Ped. Inst., Moscow, 1985.

[8] D.S. Mitrinović and J. Pečarić, *Note on a class of functions of Godunava and Levin*, C. R. Math. Rep. Acad. Sci. Canada, **12** (1990) 33–36.

[9] J. Pečarić T. Furuta, H. Mićić and Y. Seo, *Mond–Pecaric Method in Ooperator Inequalities*, Zagreb: ELEMENT, 2005.

[10] M. Radulescu, S. Radulescu and P. Alexandrescu, *On the Godunova–Levin–Schur class of functions*, Math. Inequal. Appl. **12** (2009), no. 4, 853–862.

[11] M. Radulescu, S. Radulescu and P. Alexandrescu, *On Schur inequality and Schur functions*, Annals of University of Craiova, Math. Comp. Sci. Ser. **32** (2005), 202–208.

[1] Department of Art and Sciences (Information Science), Osaka Kyoiku University, Asahigaoka, Kashiwara, Osaka 582-8582, Japan.
E-mail address: fujii@cc.osaka-kyoiku.ac.jp

[2] Department of Pure Mathematics, Ferdowsi University of Mashhad, P.O. Box 1159, Mashhad 91775, Iran.; Tusi Mathematical Research group (TMRG), Mashhad, Iran.
E-mail address: kian_tak@yahoo.com

[3] Department of Pure Mathematics, Center of Excellence in Analysis on Algebraic Structures (CEAAS), Ferdowsi University of Mashhad, P.O. Box 1159, Mashhad 91775, Iran.
E-mail address: moslehian@ferdowsi.um.ac.ir and moslehian@ams.org