ON OPERATION APPROACHES OF $\alpha$-OPEN SETS AND SEMI-OPEN SETS

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Abstract. The aim of the present paper is to introduce and study the concept of operation-$\alpha$-open sets, operation-semi-open sets and some operation-closures by using operations from $SO(X, \tau)$ into $P(X)$. We obtain some operation-closure formulas. As corollaries, well known formulas on $\alpha$-closures and semi-closures are obtained.

1. Introduction

Throughout this paper, $(X, \tau)$ and $(Y, \sigma)$ represent nonempty topological spaces on which no separation axioms are assumed, unless otherwise mentioned. A subset $A$ of $X$ is semi-open [10] in $(X, \tau)$ if there exists an open set $O$ such that $O \subset A \subset Cl(O)$. It is well known that $A$ is semi-open in $(X, \tau)$ if and only if $A \subset Cl(Int(A))$ holds. The complement of a semi-open set is called a semi-closed set. Let $SO(X, \tau)$ be the collection of all semi-open sets in $(X, \tau)$. For a subset $B$ of $X$, let $sCl(B) := \bigcap \{F \mid B \subset F, X \setminus F \in SO(X, \tau)\}$ and $sInt(B) := \{x \in X \mid \text{there exists a subset } U \in SO(X, \tau) \text{ such that } x \in U \text{ and } U \subset B\}$. It is proved that $sCl(B) = \{x \in X \mid U \cap B \neq \emptyset \text{ for any semi-open set } U \text{ containing } x\}$. A subset $A$ of $X$ is said to be preopen [13] in $(X, \tau)$ if $A \subset Int(Cl(A))$ holds. We denote by $PO(X, \tau)$ the set of all preopen sets in $(X, \tau)$. Kasahara [7] defined and investigated the concept of operations on $\tau$, i.e., the function $\alpha : \tau \to P(X)$ such that $U \subset U^\alpha$ for each $U \in \tau$, where $U^\alpha$ denotes the value $\alpha(U)$ of $\alpha$ at $U$ and $P(X)$ the power set of $X$. He generalized the notion of compactness with help of operations. After the work of Kasahara, Janković [5] defined the concept of operation-closures and investigated some properties of functions with operation-closed graphs. In 1991, one of the present authors, Ogata, defined and investigated the concept of operation-open sets [15], say $\gamma$-open sets; he used the symbol $\gamma : \tau \to P(X)$ as an operation on $\tau$. He avoided a confusion between the concept of $\alpha$-sets [14] (sometimes $\alpha$-open sets) and one of “$\alpha$”-open set (where the latter symbol “$\alpha$” is an operation in the sense of Kasahara [7]). Let $\gamma : \tau \to P(X)$ be an operation on $\tau$. A nonempty subset $A$ of $X$ is said to be $\gamma$-open (in the sense of Ogata) [15] if for each point $x \in A$, there exists an open set $U$ containing $x$ such that $U^\gamma \subset A$. Recently, Krishnan et al. [9] (resp. Tran Van, Dang Xuan et al. [17]) investigated operations on the family $SO(X, \tau)$ (resp. $PO(X, \tau)$) for a topological space $(X, \tau)$.

In the present paper, we shall introduce the concept of an alternative operation-open sets, i.e., $\gamma_\alpha$-open sets (cf. Definition 3.2(i)) using the concept of operations on $SO(X, \tau)$ due to [9, Definition 2.4]. Let $\gamma_\alpha : SO(X, \tau) \to P(X)$ be a function from $SO(X, \tau)$ into $P(X)$ satisfying the following property: $V \subset V^{\gamma_\alpha}$ for any $V \in SO(X, \tau)$, where $V^{\gamma_\alpha}$ denotes the value $\gamma_\alpha(V)$ of $\gamma_\alpha$ at $V$. By [9, Definition 2.4], the function $\gamma_\alpha$ is an operation on $SO(X, \tau)$. In Section 2, we introduce and investigate the concepts of operations (cf. Definition 2.1) on
a family, say $\xi$, of subsets of a topological space $(X, \tau)$ such that $\tau \subset \xi$; this operation is denoted by $\gamma_{\xi} : \xi \to P(X)$. Furthermore, we introduce the concept of an operation-open sets and related concepts, say $\gamma_{s}$-open sets (cf. Definition 2.1 below), and investigate some fundamental properties of them (cf. Theorem 2.10). In Section 3, especially we consider the case where $\xi := SO(X, \tau)$ and introduce the concept of $\gamma_{s}$-open sets (cf. Definition 3.2); and let $\gamma_{s}\tau^{\alpha}$ be the set of all $\gamma_{s}$-open sets in $(X, \tau)$. We also investigate a formula on $\gamma_{s}\tau^{\alpha}$-closures of subsets in $(X, \tau)$ (cf. Theorem 3.7(iii)). As corollary we get the well known formula [1] on $\alpha$-closure of a subset (cf. Corollary 3.8). The concept of $\alpha$-sets (sometimes, $\alpha$-open sets) introduced by Njåstad [14]; a subset $A$ of $X$ is said to be $\alpha$-open in $(X, \tau)$ if $A \subset \text{Int}(\text{Cl}(\text{Int}(A)))$ holds in $(X, \tau)$. In Section 4, we investigate some properties on operation semi-open sets and operation semi-closures $\gamma_{s}SO(X, \tau)-\text{Cl}(B)$ of a subset $B$ of $(X, \tau)$.

2. OPERATIONS ON A FAMILY AND SOME OPERATION-OPEN SETS

Wirst we recall a concept of operations on a family $\xi$ of subsets of a space $(X, \tau)$ such that $\tau \subset \xi$. Kasahara [7] and Ogata [16] introduced and investigated a general theory on operations on $\tau$, i.e., $\xi := \tau$. Throughout this section, for a family $\xi$ we assume that $\tau \subset \xi$ holds.

**Definition 2.1.** (i) (cf. Kasahara [7], Ogata [16]) An operation on $\xi$, say $\gamma_{\xi}$, is a mapping from $\xi$ to the power set $P(X)$ of $X$ such that $U \subset U^{\xi}$ for each $U \in \xi$, where $U^{\xi}$ denotes the value $\gamma_{\xi}(U)$ of $\gamma_{\xi}$ at $U$ and $\tau \subset \xi$. The operation on $\xi$ is denoted by $\gamma_{\xi} : \xi \to P(X)$.

(ii) The restriction of $\gamma_{\xi} : \xi \to P(X)$ to $\tau$, say $\gamma_{\xi}|_{\tau} : \tau \to P(X)$, is well defined by $(\gamma_{\xi}|_{\tau})(V) := V^{\xi}$ for every $V \in \tau$.

**Remark 2.2.** (i) The following function “$\text{Cl}$” : $P(X) \to P(X)$ is well defined as follows, i.e., “$\text{Cl}(V) := \text{Cl}(V)$” for every subset $V \in P(X)$, where $\text{Cl}(E)$ is the closure of a subset $E$ in a topological space $(X, \tau)$. Since $V \subset \text{Cl}(V)$ holds for every $V \in P(X)$, the function “$\text{Cl}$” : $P(X) \to P(X)$ is an example of operations on $P(X)$ (in the sense of Definition 2.1(i)); and also for any subfamily $\xi$ of $P(X)$, the restriction to $\xi$, say “$\text{Cl}|_{\xi}$” : $\xi \to P(X)$, can be an operation on $\xi$ if $\tau \subset \xi$. This operation is called the closure operation on $\xi$ (cf. Remark 2.1(ii) below).

(ii) The following function “$\text{Int}$” : $P(X) \to P(X)$ is well defined as follows, i.e., “$\text{Int}(U) := \text{Int}(U)$” for every subset $U \in P(X)$, where $\text{Int}(E)$ is the interior of a subset $E$ in a topological space $(X, \tau)$. If $U \subset \text{Int}(U)$ holds (i.e., $U = \text{Int}(U)$) in $(X, \tau)$ for every $U \in P(X)$, then the function “$\text{Int}$” : $P(X) \to P(X)$ can be called as operation on $P(X)$. However, there may be many topological spaces $(X, \tau)$ such that $U \notin \text{Int}(U)$ for some subset $U \in P(X)$ (cf. (iii) below); and so for such space $(X, \tau)$, the function “$\text{Int}$” : $P(X) \to P(X)$ is not an operation on $P(X)$. We see easily that the restriction of “$\text{Int}$” to $\tau$, say “$\text{Int}|_{\tau}$ : $\tau \to P(X)$, is an operation on $\tau$, because $U = \text{Int}(U)$ holds for every subset $U \in \tau$ (cf. Definition 2.1(i)).

(iii) For the digital line $(\mathbb{Z}, \kappa)$ (eg. [4]), the function “$\text{Int}$” : $SO(\mathbb{Z}, \kappa) \to P(\mathbb{Z})$ is not an operation on $SO(\mathbb{Z}, \kappa)$. Indeed, for a subset $\{2m, 2m + 1\} \in SO(\mathbb{Z}, \kappa)$, “$\text{Int}((\{2m, 2m + 1\}) \ast \text{Int}((\{2m, 2m + 1\}) = \{2m + 1\}$ and so $\{2m, 2m + 1\} \notin \text{“Int”}(\{2m, 2m + 1\})$. The restriction of “$\text{Int}$” to $\kappa$, i.e., “$\text{Int}|_{\kappa}$ : $\kappa \to P(\mathbb{Z})$ is an operation on $\kappa$ (cf. (ii) above) (cf. [12]).

**Remark 2.3.** (i) For the case where $\xi := \tau$ in Definition 2.1(i), $\gamma_{\xi}$ is identical to the operation on $\tau$ in the sense of [7] (e.g. [15]).

(ii) Krishnan et al. [9] introduced an operation on $SO(X, \tau)$, say $\gamma_{s} : SO(X, \tau) \to P(X)$. This operation $\gamma_{s}$ is an operation on $\xi := SO(X, \tau)$ in the sense of Definition 2.1(i) above, because of $\tau \subset SO(X, \tau)$. Namely, we have the equality of two operations: $\gamma_{SO(X, \tau)} = \gamma_{s} : SO(X, \tau) \to P(X)$. 


(iii) Tran An and Dang Xuan et al. [17] introduced an operation on $PO(X, \tau)$, say $\gamma_p : PO(X, \tau) \to P(X)$. This operation $\gamma_p : PO(X, \tau) \to P(X)$ is an operation on $\xi := PO(X, \tau)$ in the sense of Definition 2.1, because of $\tau \subseteq PO(X, \tau)$. Namely, we have the equality of two operations: $\gamma_{PO(X, \tau)} = \gamma_p : PO(X, \tau) \to P(X)$.

Secondly, we shall recall analogous fundamental concepts on operation $\gamma_\xi : \xi \to P(X)$ as follows (we note that $\tau \subseteq \xi$).

**Definition 2.4.** (i) Let $\tau_\xi$ denotes the following collection of subsets of $(X, \tau)$:

- $\tau_\xi := \{\emptyset\} \cup \{A| there exists a set $U \in \tau$ such that $x \in U$ and $U^\gamma_\xi \subseteq A$ for each point $x \in A\}$.

(ii) A subset $A$ is said to be $\gamma_\xi$-open in $(X, \tau)$, if $A \in \tau_\xi$.

(ii) A subset $A$ of $(X, \tau)$ is $\gamma_\xi$-open in $(X, \tau)$ if and only if, $A = \emptyset$ or, for each point $x \in A$ there exists an open set $U$ containing $x$ such that $U^\gamma_\xi \subseteq A$.

Defining 2.4. (i) A subset $A$ is said to be $\gamma_\xi$-closed in $(X, \tau)$, if $X \setminus A \in \tau_\xi$, (i.e., $X \setminus A$ is $\gamma_\xi$-open in $(X, \tau)$ in the sense of (ii) above).

Using the concept of $\gamma_\xi$-open sets above, we define the operation-regularity (resp. operation-openness) of operation on $\xi$ as follows:

**Definition 2.5.** Let $\gamma_\xi : \xi \to P(X)$ be an operation on $\xi$ with $\tau \subseteq \xi$.

(i) $\gamma_\xi : \xi \to P(X)$ is said to be regular on $\xi$, if for each open neighbourhoods $U$ and $V$ of each $x \in X$, there exists an open neighbourhood $W$ of $x$ such that $W^\gamma_\xi \subseteq U^\gamma_\xi \cap V^\gamma_\xi$.

(ii) $\gamma_\xi : \xi \to P(X)$ is called to be open on $\xi$, if for each point $x \in X$ and any open neighbourhood $U$ of $x$, there exists a $\gamma_\xi$-open set $V$ containing $x$ such that $V \subseteq U^\gamma_\xi$.

**Definition 2.6.** For a subset $B$ of $(X, \tau)$, we define the following three subsets of $(X, \tau)$:

- $\text{Cl}_{\gamma_\xi}(B) := \{x \in X| U^\gamma_\xi \cap B \neq \emptyset$ for any open neighbourhood $U$ of $x\}$;
- $\text{Int}_{\gamma_\xi}(B) := \{x \in X| there exists an open neighbourhood $U$ of $x$ such that $U^\gamma_\xi \subseteq B\}$;
- $\tau_\xi$-Cl$(B) := \bigcap\{F| B \subseteq F, X \setminus F \in \tau_\xi\}$.

**Remark 2.7.** It is evident that for the subset $\emptyset$ (resp. $X$), $\text{Cl}_{\gamma_\xi}(\emptyset) = \tau_\xi$-Cl$(\emptyset) = \emptyset$ (resp. $\text{Cl}_{\gamma_\xi}(X) = \tau_\xi$-Cl$(X) = X$) hold, because $\tau \subseteq \xi$ and $\emptyset \in \tau_\xi$ and $X \in \tau_\xi$; furthermore, if $A \subseteq B$ in $(X, \tau)$, then $\text{Cl}_{\gamma_\xi}(A) \subseteq \text{Cl}_{\gamma_\xi}(B), \text{Int}_{\gamma_\xi}(A) \subseteq \text{Int}_{\gamma_\xi}(B)$ and $\tau_\xi$-Cl$(A) \subseteq \tau_\xi$-Cl$(B)$ holds.

**Definition 2.8.** Let $\gamma_\xi : \xi \to P(X)$ be an operation, where $\tau \subseteq \xi$, and $B$ subset of a topological space $(X, \tau)$.

(i) $(X, \tau)$ is said to be a $\gamma_\xi$-regular space, if for each point $x \in X$ and every open neighbourhood $U$ of $x$ there exists an open neighbourhood $W$ of $x$ such that $W^\gamma_\xi \subseteq U$.

(ii) $B$ is called $\gamma_\xi$-closed in the sense of Janković if \text{Cl}_{\gamma_\xi}(B) \subseteq B$, i.e., $B = \text{Cl}_{\gamma_\xi}(B)$ holds (cf. [5]).

Finally, the following Theorem 2.9 (resp. Theorem 2.10) is proved by definitions (resp. Theorem 2.9 and the corresponding theorems in [15]); hence the proofs are omitted.

**Theorem 2.9.** Let $\gamma_\xi : \xi \to P(X)$ be an operation on $\xi$, where $\tau \subseteq \xi$.

(i) If a subset $A$ is $\gamma_\xi$-open in $(X, \tau)$ if and only if $A$ is $\gamma_\xi|\tau$-open in $(X, \tau)$ in the sense of [15].

(ii) For a subset $B$ of $(X, \tau)$, $\text{Cl}_{\gamma_\xi|\tau}(B) = \text{Cl}_{\gamma_\xi}(B)$ and $\text{Int}_{\gamma_\xi|\tau}(B) = \text{Int}_{\gamma_\xi}(B)$ hold, where $\text{Cl}_{\gamma_\xi|\tau}(B), \text{Int}_{\gamma_\xi|\tau}(B)$ are defined by [15].

(iii) The operation $\gamma_\xi : \xi \to P(X)$ is regular on $\xi$ if and only if $\gamma_\xi|\tau : \tau \to P(X)$ is regular in the sense of [15].

(iv) The operation $\gamma_\xi : \xi \to P(X)$ is open on $\xi$ if and only if $\gamma_\xi|\tau : \tau \to P(X)$ is open in the sense of [15].
(v) A space \((X, \tau)\) is \(\gamma_\xi\)-regular if and only if \((X, \tau)\) is \(\gamma_\xi|\tau\)-regular in the sense of [15].

\[\Box\]

**Theorem 2.10.** Let \(\gamma_\xi: \xi \to P(X)\) be an operation on \(\xi\), where \(\tau \subset \xi\). Then the following properties hold in \((X, \tau)\).

(i) (cf. [15, p.176]) Every \(\gamma_\xi\)-open set is open in \((X, \tau)\), i.e., \(\tau_{\gamma_\xi} \subset \tau\).

(ii) (cf. [15, Proposition 2.3]) The union of every family of \(\gamma_\xi\)-open sets is \(\gamma_\xi\)-open.

(iii) (cf. [15, Proposition 2.9(i)(ii)]) If \(\gamma_\xi: \xi \to P(X)\) is regular on \(\xi\), then the intersection of every finite family of \(\gamma_\xi\)-open sets is \(\gamma_\xi\)-open and hence \(\tau_{\gamma_\xi}\) is a topology of \(X\).

(iv) (cf. [15, (3.3)]) For a point \(x \in X\) and a subset \(A\) of \((X, \tau)\), \(x \in \tau_{\gamma_\xi}-\text{Cl}(A)\) if and only if \(V \cap A \neq \emptyset\) for any set \(V \in \tau_{\gamma_\xi}\) such that \(x \in V\).

(v) (cf. [15, (3.4)]) For a subset \(A\) of \((X, \tau)\), \(A \subset \text{Cl}(A) \subset \text{Cl}_{\gamma_\xi}(A) \subset \tau_{\gamma_\xi}-\text{Cl}(A)\) hold.

(vi) (cf. [15, Theorem 3.6(i)]) The subset \(\text{Cl}_{\gamma_\xi}(A)\) is closed in \((X, \tau)\) and \(\text{Int}_{\gamma_\xi}(A)\) is open in \((X, \tau)\), where \(A\) is a subset of \((X, \tau)\).

(vii) \(X \setminus \text{Cl}_{\gamma_\xi}(A) = \text{Int}_{\gamma_\xi}(X \setminus A)\) holds.

(viii) (cf. [15, Theorem 3.7]) The following properties are equivalent for a subset \(A\) of \((X, \tau)\):

\begin{enumerate}
\item \(A\) is \(\gamma_\xi\)-open;
\item \(X \setminus A\) is \(\gamma_\xi\)-closed in the sense of Janković (i.e., \(\text{Cl}_{\gamma_\xi}(X \setminus A) = X \setminus A\) holds);
\item \(\tau_{\gamma_\xi}-\text{Cl}(X \setminus A) = X \setminus A\) holds;
\item \(\text{Int}_{\gamma_\xi}(A) = A\) holds.
\end{enumerate}

(ix) (cf. [15, Theorem 3.6(ii)],(v) (viii) above) If \(\gamma_\xi: \xi \to P(X)\) is open on \(\xi\), then the following properties on a subset \(A\) of \((X, \tau)\) hold.

\begin{enumerate}
\item \(\text{Cl}_{\gamma_\xi}(A) = \tau_{\gamma_\xi}-\text{Cl}(A)\) and \(\text{Cl}_{\gamma_\xi}(\text{Cl}_{\gamma_\xi}(A)) = \text{Cl}_{\gamma_\xi}(A)\) hold.
\item \(\text{Cl}_{\gamma_\xi}(A)\) is \(\gamma_\xi\)-closed in the sense of Janković (cf. Definition 2.8(ii)).
\item \(\text{Int}_{\gamma_\xi}(A)\) is \(\gamma_\xi\)-open.
\end{enumerate}

(x) (cf. [15, Lemma 3.10]) If \(\gamma_\xi: \xi \to P(X)\) is regular on \(\xi\), then \(\text{Cl}_{\gamma_\xi}(A \cup B) = \text{Cl}_{\gamma_\xi}(A) \cup \text{Cl}_{\gamma_\xi}(B)\) holds for any subsets \(A\) and \(B\) of \(X\).

(xi) (cf. [15, Corollary 3.11], (v) (iv) above) If \(\gamma_\xi: \xi \to P(X)\) is regular and open on \(\xi\), then the operation \(\text{Cl}_{\gamma_\xi}(-)\) satisfies the Kuratowski closure axiom. Namely, for any subsets \(A\) and \(B\) of \((X, \tau)\), \(A \subset \text{Cl}_{\gamma_\xi}(A), \text{Cl}_{\gamma_\xi}(A \cup B) = \text{Cl}_{\gamma_\xi}(A) \cup \text{Cl}_{\gamma_\xi}(B), \text{Cl}_{\gamma_\xi}(\text{Cl}_{\gamma_\xi}(A)) = \text{Cl}_{\gamma_\xi}(A)\) and \(\text{Cl}_{\gamma_\xi}(\emptyset) = \emptyset\) and \(\text{Cl}_{\gamma_\xi}(X) = X\) hold.

(xii) (a)(cf. [15, Theorem 3.6(ii)],(v) above) If \((X, \tau)\) is a \(\gamma_\xi\)-regular space, then \(\text{Cl}_{\gamma_\xi}(A) = \text{Cl}(A)\) holds for a subset \(A\) of \((X, \tau)\).

(b) (cf. [15, Proposition 2.4], Theorem 2.9(v)(i) above) A space \((X, \tau)\) is a \(\gamma_\xi\)-regular space if and only if \(\tau = \tau_{\gamma_\xi}\) holds. \[\Box\]

**Remark 2.11.**

(i) The identity operation on \(\xi\), say “id” : \(\xi \to P(X)\), is well defined by \(U^{\text{id}} = U\), where \(U \in \xi\). Then, a subset \(A\) of \((X, \tau)\) is “id”-open in \((X, \tau)\) if and only if \(A\) is open in \((X, \tau)\), i.e., \(\tau_{\gamma_{\text{id}}} = \tau\) holds. Then, it is shown that \(\text{Cl}_{\gamma_{\text{id}}}(A) = \text{Cl}(A), \text{Int}_{\gamma_{\text{id}}}(A) = \text{Int}(A)\) and every topological space \((X, \tau)\) is “id”-regular (cf. Theorem 2.10(xii)(b)).

(ii) The closure operation on \(\xi\), \(\text{Cl} : \xi \to P(X)\), is well defined by \(U^{\text{Cl}} = \text{Cl}(U)\), where \(U \in \xi\). Then, a subset \(A\) of \((X, \tau)\) is “Cl”-open in \((X, \tau)\) if and only if \(A\) is \(\theta\)-open in \((X, \tau)\). That is, \(\tau_{\gamma_\xi} = \tau_{\gamma_\xi}|_{\tau} = \tau_\theta\) hold, where \(\tau_\theta\) is the collection of all \(\theta\)-open sets of \((X, \tau)\) [18, cf. Theorem 2.9(i)]. Then, it is shown that \(\text{Cl}_{\gamma\text{Cl}}(A) = \text{Cl}_\theta(A)\) and \(\text{Int}_{\gamma\text{Cl}}(A) = \text{Int}_\theta(A)\) hold (cf. Theorem 2.9(ii), [18] and \((X, \tau)\) is “Cl”-regular if and only if \((X, \tau)\) is regular (i.e., \(\tau = \tau_\theta\)). We note that, in general, \(\tau_{\gamma_\xi} \text{Cl}(A) = \bigcap\{V | A \subset V, X \setminus V \in \tau_{\gamma_\xi}\} = \bigcap\{V | A \subset V, X \setminus V \in \tau_\theta\} = \tau_\theta \text{Cl}(A)\) hold for any subset \(A\) of \((X, \tau)\).

Furthermore, we note that [15, Remark 3.19, Example 2.7] shows there exists a topological space \((X, \tau)\) such that \(\text{Cl}_{\gamma_\xi}(A) \neq \tau_{\gamma_\xi} \text{Cl}(A)\) (i.e., \(\text{Cl}_\theta(A) \neq \tau_\theta \text{Cl}(A)\)).
for some set $A$ of $(X, \tau)$. Indeed, let $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$; then $\tau^\circ = \tau^\circ = \{\emptyset, X\}$, $\tau^\circ - \text{Cl}(\{a\}) = X$ and $\text{Cl}(\{a\}) = \{a, c\}$. However, it is known that, in general, for a subset $A$ of $(X, \tau)$, $\text{Cl}(A) \subset A$ (i.e., $\text{Cl}(A) = A$) holds if and only if $\tau^\circ - \text{Cl}(A) = A$ (i.e., $\tau^\circ - \text{Cl}(A) = A$) holds (cf. Theorem 2.10(viii)). The example [15, Example 2.7, Remark 3.19] shows that this operation “$\text{Cl}$” is not open.

(iii) For other examples of operations on $\xi = \text{PO}(X, \tau)$, we refer them in [17, Example 3.2(iv)-(vi)].

A function, say “$\text{Int} \circ \text{Cl}$” : $\xi \to P(X)$, is well defined by $U^\circ \text{Int} \circ \text{Cl} = \text{Int}(\text{Cl}(U))$, where $U \in \xi$.

**Remark 2.12.** Assume that $\tau \subset \xi$.

(i) The function “$\text{Int} \circ \text{Cl}$” : $\xi \to P(X)$ is an operation on $\xi$ if and only if $\xi \subset \text{PO}(X, \tau)$ holds. (The operation “$\text{Int} \circ \text{Cl}$” is called the “interior-closure” operation on $\xi$.)

   **Proof.** (Necessity) Put $\gamma := \text{Int} \circ \text{Cl}$. Let $V \in \xi$. Then, $V \subset V^\gamma$ holds, because $\gamma : \xi \to P(X)$ is an operation on $\xi$; and so $V \subset \text{Int}(\text{Cl}(V))$, i.e., $V \subset \text{PO}(X, \tau)$. Thus, we have $\xi \subset \text{PO}(X, \tau)$.

   (Sufficiency) Assume that $\xi \subset \text{PO}(X, \tau)$ holds. Let $U \in \xi$. Since $U \subset \text{PO}(X, \tau)$, $U \subset \text{Int}(\text{Cl}(U))$ holds, i.e., $U \subset U^\circ \text{Int} \circ \text{Cl}$. Hence, “$\text{Int} \circ \text{Cl}$” : $\xi \to P(X)$ is an operation on $\xi$.

(ii) Let “$\text{Int} \circ \text{Cl}$” : $\xi \to P(X)$ be the “interior-closure” operation on $\xi$, where $\tau \subset \xi \subset \text{PO}(X, \tau)$ (cf. (i) above). A subset $A$ of $(X, \tau)$ is “$\text{Int} \circ \text{Cl}$”-open in $(X, \tau)$ if and only if $A$ is $\delta$-open in $(X, \tau)$. That is, $\tau^\circ \text{Int} \circ \text{Cl} = \tau^\circ \text{Int} \circ \text{Cl} |_{\tau} = \tau_0$, where $\tau_0$ is the collection of all $\delta$-open sets of $(X, \tau)$ [18]. Then, it is shown that $\text{Int} \circ \text{Cl} = \text{Cl} \circ \text{Int}$ hold and $(X, \tau)$ is “$\text{Int} \circ \text{Cl}$”-regular if and only if $\tau = \tau_0$ holds (cf. Theorem 2.10(xii)(b)) ((X, $\tau$) is called semi-regular if $\tau = \tau_0$ holds, eg. [3, Lemma 2.1]). This operation “$\text{Int} \circ \text{Cl}$” : $\xi \to P(X)$ is an example of the open operation on $\xi$ (cf. Definition 2.5(ii)), under the assumption that $\tau \subset \xi \subset \text{PO}(X, \tau)$. Indeed, let $x \in X$ and $U$ be any open neighbourhood of $x$. Then, since $x \in \text{Int}(\text{Cl}(U))$ and $\text{Int}(\text{Cl}(U)) \subset U^\circ \text{Int} \circ \text{Cl}$ holds. Thus, “$\text{Int} \circ \text{Cl}$” : $\xi \to P(X)$ is open on $\xi$. Therefore, by Theorem 2.10(ix), $\text{Cl} \circ \text{Int} \circ \text{Cl} = \text{Int}(\text{Cl}(U))$ hold, the subset Int$(\text{Cl}(U))$ is an “$\text{Int} \circ \text{Cl}$”-open set containing $x$; furthermore, Int$(\text{Cl}(U)) \subset U^\circ \text{Int} \circ \text{Cl}$ holds. Thus, “$\text{Int} \circ \text{Cl}$” : $\xi \to P(X)$ is open on $\xi$. Therefore, by Theorem 2.10(ix), $\text{Cl} \circ \text{Int} \circ \text{Cl} = \text{Int}(\text{Cl}(U))$ hold, the subset Int$(\text{Cl}(U))$ is an “$\text{Int} \circ \text{Cl}$”-open set containing $x$; furthermore, Int$(\text{Cl}(U)) \subset U^\circ \text{Int} \circ \text{Cl}$ holds. Thus, “$\text{Int} \circ \text{Cl}$” : $\xi \to P(X)$ is open on $\xi$. Therefore, by

3. **Operation-$\alpha$-open sets**

In the present section, we introduce and study the concept of operation-$\alpha$-open sets in a topological space $(X, \tau)$. We study a formula on operation-$\alpha$-closures of subsets of $(X, \tau)$ (cf. Theorem 3.7(iii)) As a corollary of Theorem 3.7(iii), we get the well-known formula of $\tau^\alpha - \text{Cl}(E)$ of a subset $E$ of $(X, \tau)$. Firstly, we recall the concept of the semi-operation-open sets and two kinds of operation-semi-closures due to Krishnan et al. [9].

**Definition 3.1.** [9] Let $\gamma_\alpha : \text{SO}(X, \tau) \to P(X)$ be an operation on $\text{SO}(X, \tau)$ and $A$ a subset of $X$.

(i) A subset $A$ is said to be semi-$\gamma_\alpha$-open [9, Definition 2.5] in $(X, \tau)$, if for each $x \in A$ there exists a subset $U \in \text{SO}(X, \tau)$ such that $x \in U$ and $U^\gamma_\alpha \subset A$.

- The collection of all semi-$\gamma_\alpha$-open sets is denoted as $\text{SO}(X, \tau)_{\gamma_\alpha}$.

(ii) The following two operation-semi-closures of $A$ are defined as follows:

- $\text{SO}(X, \tau)_{\gamma_\alpha} - \text{Cl}(A) := \bigcap \{F \subset F, X \backslash F \in \text{SO}(X, \tau)_{\gamma_\alpha}\}$ (9, Definition 2.24);

- $\text{sCl}_\alpha(A) := \{x \in X | U^\gamma_\alpha \cap A \neq \emptyset$ for every semi-open set $U$ (i.e., $U \in \text{SO}(X, \tau)$ containing $x\}$ (9, Definition 2.23).

(iii) For a subset $A$ of $(X, \tau)$, an operation semi-interior is defined:
Let \( \gamma_s: SO(X, \tau) \rightarrow P(X) \) be an operation on \( SO(X, \tau) \) and \( A \) a subset of \( (X, \tau) \).

(i) A subset \( A \) of \( (X, \tau) \) is called a \( \gamma_s \)-\( \alpha \)-open set in \( (X, \tau) \), if there exists a \( \gamma_s \)-open set \( U \) (i.e., \( U \in \tau_\gamma \)) such that \( U \subset A \subset Cl_\gamma(U) \). The complement of a \( \gamma_s \)-\( \alpha \)-open set is called \( \gamma_s \)-\( \alpha \)-closed in \( (X, \tau) \).

(ii) \( \bullet \) The collection of all \( \gamma_s \)-\( \alpha \)-open sets in \( (X, \tau) \) is denoted by \( \gamma_s \tau^\alpha \).

**Remark 3.3.** (i) For the identity operation \( \text{"id"} : SO(X, \tau) \rightarrow P(X) \), it is known that \( [9] \ SO(X, \tau)_{\text{id}} = SO(X, \tau) \) and \( SO(X, \tau)_{\text{id}} \cdot \text{-Cl}(A) = sCl_{\text{id}}(A) = sCl(A) \) hold for any subset \( A \) of \( (X, \tau) \). Furthermore, for a subset \( A \) of \( (X, \tau) \), \( A \) is \( \text{"id"} \)-\( \alpha \)-open in \( (X, \tau) \) if and only if \( A \) is \( \alpha \)-open in \( (X, \tau) \). Namely, \( \gamma_{\text{id}} \tau^\alpha = \tau^\alpha \) holds for any topology \( \tau \) of \( X \). Indeed, for the identity operation \( \gamma_s = \text{"id"} : SO(X, \tau) \rightarrow P(X) \), a subset \( A \) is \( \text{"id"} \)-\( \alpha \)-open in \( (X, \tau) \) if and only if \( (*) \) there exists an open set \( U \) such that \( U \subset A \subset Cl(U) \) (cf. Remark 2.11(i)); it is well known that the above property \( (*) \) holds if and only if \( A \) is \( \alpha \)-open (cf. [11], [6] and the contents of MR914467(885:54909) in Math.RAMS).

(ii) By Definition 3.1(ii) and (iii), it is shown that \( sInt_\gamma(A) = X \setminus CL_\gamma(X \setminus A) \) holds in \( (X, \tau) \); furthermore, for the identity operation \( \text{"id"} : SO(X, \tau) \rightarrow P(X), sInt_{\text{id}}(A) = sInt(A) \) holds, where \( A \) is any subset of \( (X, \tau) \).

**Theorem 3.4.** Let \( A \) be a subset of \( (X, \tau) \) and \( \gamma_s: SO(X, \tau) \rightarrow P(X) \) an operation on \( SO(X, \tau) \).

(i) If \( A \in \gamma_s \tau^\alpha \), then \( A \subset Cl_\gamma(sInt_\gamma(A)) \) holds.

(ii) Suppose that \( \gamma_s: SO(X, \tau) \rightarrow P(X) \) is open on \( SO(X, \tau) \) or \( \gamma_s|\tau : \tau \rightarrow P(X) \) is open. If \( A \subset Cl_\gamma(sInt_\gamma(A)) \) holds, then \( A \in \gamma_s \tau^\alpha \).

(iii) Suppose that \( \gamma_s: SO(X, \tau) \rightarrow P(X) \) is open on \( SO(X, \tau) \) (cf. Definition 2.5(ii)). For a subset \( A \) of \( (X, \tau) \), \( A \) is \( \gamma_s \)-\( \alpha \)-open in \( (X, \tau) \) if and only if \( A \subset Cl_\gamma(sInt_\gamma(A)) \) holds. Furthermore, for a subset \( B \) of \( (X, \tau) \), \( B \) is \( \gamma_s \)-\( \alpha \)-closed in \( (X, \tau) \) if and only if \( sInt_{\gamma_\xi}(Cl_\gamma(B)) \subset B \) holds.

**Proof.** (i) It follows from assumption that there exists a set \( V \in \tau_\gamma \) such that \( V \subset A \subset Cl_\gamma(V) \). Using Theorem 2.10(viii), we have that \( V = Int_\gamma(V) \subset Int_\gamma(A) \) holds and so \( A \subset Cl_\gamma(V) \subset sCl_\gamma(sInt_\gamma(A)) \) holds.

(ii) We have \( Int_\gamma(A) \subset A \subset sCl_\gamma(sInt_\gamma(A)) \). Since \( Int_\gamma(A) \) is \( \gamma_s \)-open under assumption that \( \gamma_s \) is open (cf. Theorem 2.10(ix)(c)), it is shown that \( A \) is \( \gamma_s \)-\( \alpha \)-open in \( (X, \tau) \).

(iii) For the proof of the former, it is obvious from (i) and (ii). For the proof of the latter, it is obtained by using the former, Remark 3.3(ii) and Theorem 2.10(vii) for \( \xi = SO(X, \tau) \). □

**Proposition 3.5.** Let \( \gamma_s: SO(X, \tau) \rightarrow P(X) \) be an operation on \( SO(X, \tau) \). If \( A_i \in \gamma_s \tau^\alpha \) for each \( i \in \Omega \), then \( \bigcup \{ A_i | \ i \in \Omega \} \in \gamma_s \tau^\alpha \), where \( \Omega \) is an index set.

**Proof.** It follows from Definition 3.2(i) that, for each \( i \in \Omega \), there exists a \( \gamma_s \)-open set \( U_i \) such that \( U_i \subset A_i \subset Cl_\gamma(U_i) \). Then, \( \bigcup \{ U_i | i \in \Omega \} \subset \bigcup \{ A_i | i \in \Omega \} \subset \bigcup \{ Cl_\gamma(U_i) | i \in \Omega \} \subset Cl_\gamma \bigcup \{ U_i | i \in \Omega \} \) hold and so \( \bigcup \{ A_i | i \in \Omega \} \in \gamma_s \tau^\alpha \), because \( \bigcup \{ U_i | i \in \Omega \} \in \tau_\gamma \) (cf. Theorem 2.10(ii)). □

**Definition 3.6.** Let \( \gamma_s: SO(X, \tau) \rightarrow P(X) \) be an operation and a subset \( B \) of \( (X, \tau) \). The following closure of \( B \), say \( \gamma_s \tau^\alpha - Cl(B) \), is defined:

- \( \gamma_s \tau^\alpha - Cl(B) := \bigcap \{ F | B \subset F, F \setminus B \in \gamma_s \tau^\alpha \} \).
Theorem 3.7. Let \( \gamma : SO(X, \tau) \to P(X) \) be an operation and \( A \) a subset of \((X, \tau)\). Then, the following properties hold.

(i) The closure \( \gamma_\alpha^{\tau^\alpha \text{-Cl}(A)} \) is \( \gamma_\alpha \)-\( \alpha \)-closed in \((X, \tau)\).

(ii) \( A \cup s\text{Int}_{\gamma_\alpha}(\text{Cl}_{\gamma_\alpha}(A)) \subset \gamma_\alpha^{\tau^\alpha \text{-Cl}(A)} \).

(iii) If \( \gamma : SO(X, \tau) \to P(X) \) is open on \( SO(X, \tau) \) (cf. Definition 2.5(ii)), then \( \gamma_\alpha^{\tau^\alpha \text{-Cl}(A)} \subset A \cup s\text{Int}_{\gamma_\alpha}(\text{Cl}_{\gamma_\alpha}(A)) \) holds and so \( \gamma_\alpha^{\tau^\alpha \text{-Cl}(A)} = A \cup s\text{Int}_{\gamma_\alpha}(\text{Cl}_{\gamma_\alpha}(A)) \).

Proof. (i) Using Proposition 3.5, the intersection of every family of \( \gamma_\alpha \)-\( \alpha \)-closed sets is \( \gamma_\alpha \)-\( \alpha \)-closed. Thus, \( \gamma_\alpha^{\tau^\alpha \text{-Cl}(B)} \) is \( \gamma_\alpha \)-\( \alpha \)-closed in \((X, \tau)\).

(ii) Since \( \gamma_\alpha^{\tau^\alpha \text{-Cl}(A)} \) is a \( \gamma_\alpha \)-\( \alpha \)-closed set, \( s\text{Int}_{\gamma_\alpha}(\text{Cl}_{\gamma_\alpha}(\gamma_\alpha^{\tau^\alpha \text{-Cl}(A)})) \subset \gamma_\alpha^{\tau^\alpha \text{-Cl}(A)} \) holds by Theorem 3.4(i), Definition 3.1 and Remark 3.3. Since \( A \subset \gamma_\alpha^{\tau^\alpha \text{-Cl}(A)} \), we have that \( A \subset s\text{Int}_{\gamma_\alpha}(\text{Cl}_{\gamma_\alpha}(A)) \subset \gamma_\alpha^{\tau^\alpha \text{-Cl}(A)} \).

(iii) We first claim that a subset \( A \cup (s\text{Int}_{\gamma_\alpha}(\text{Cl}_{\gamma_\alpha}(A))) \) is \( \gamma_\alpha \)-\( \alpha \)-closed. We put \( B := A \cup (s\text{Int}_{\gamma_\alpha}(\text{Cl}_{\gamma_\alpha}(A))) \), then, we have \( s\text{Int}_{\gamma_\alpha}(\text{Cl}_{\gamma_\alpha}(B)) \subset s\text{Int}_{\gamma_\alpha}(\text{Cl}_{\gamma_\alpha}(\gamma_\alpha^{\tau^\alpha \text{-Cl}(A)})) = s\text{Int}_{\gamma_\alpha}(\text{Cl}_{\gamma_\alpha}(\gamma_\alpha^{\tau^\alpha \text{-Cl}(A)})) \subset A \cup s\text{Int}_{\gamma_\alpha}(\text{Cl}_{\gamma_\alpha}(A)) \) and so \( s\text{Int}_{\gamma_\alpha}(\text{Cl}_{\gamma_\alpha}(A)) \subset B \). Using Theorem 3.4(iii), the set \( B \) is \( \gamma_\alpha \)-\( \alpha \)-closed. Thus we prove that \( \gamma_\alpha^{\tau^\alpha \text{-Cl}(A)} \subset B \) holds, because \( \gamma_\alpha^{\tau^\alpha \text{-Cl}(A)} \) is the smallest \( \gamma_\alpha \)-\( \alpha \)-closed set including the set \( A \) (cf. Theorem 3.4); and so we have the required equality using (ii).

By Theorem 3.7(iii), the explicit form of the operation-closure \( \gamma_\alpha^{\tau^\alpha \text{-Cl}(\bullet)} \) is obtained, under assumption that \( \gamma \) is open on \( SO(X, \tau) \). Let \( \gamma_\alpha = \text{"id"} : SO(X, \tau) \to P(X) \) in the theorem above. Then, we have the following well known formula.

Corollary 3.8. ([1, Theorems 2.2,2.3, Corollary 2.4], [2, Theorem 1.5(c)]) For any subset \( E \) of \((X, \tau)\), \( \tau^\alpha \text{-Cl}(E) = E \cup \text{Cl}(\text{Int}(E)) \) holds.

Proof. Put \( \gamma_\alpha = \text{"id"} : SO(X, \tau) \to P(X) \) in Theorem 3.7(iii) and put \( \xi = SO(X, \tau) \) in Remark 2.11(i) and Remark 3.3. It is shown that \( \tau^\alpha \text{-Cl}(E) = E \cup s\text{Int}(E) = E \cup \{\text{Cl}(E) \cap \text{Cl}(\text{Int}(E))\} = E \cup \text{Cl}(\text{Int}(E)) \) hold, because the identity operation is open on \( SO(X, \tau) \) and \( s\text{Int}(A) = A \cap \text{Cl}(\text{Int}(A)) \) and \( \text{Cl}(\text{Int}(C(A))) \subset C(A) \) hold for any subset \( A \) of \((X, \tau)\).

4. REMARKS ON OPERATION-SEMI-CLOSURES

In the end of the present paper, for an operation \( \gamma : SO(X, \tau) \to P(X) \) on \( SO(X, \tau) \), we define the concept of \( \gamma_\alpha \)-semi-open sets (cf. Definition 4.1) and an operation-semi-closure, say \( \gamma_\alpha SO(X, \tau) \text{-Cl}(E) \), (cf. Definition 4.6) of a subset \( E \) of a topological space \((X, \tau)\). In 2006, Krishnan and Balachandran [8] introduced and investigated the concept of \( \gamma \text{-semi-open sets} \) for an operation \( \gamma : \tau \to P(X) \); by [8, Definition 3.1], a subset \( A \) of \((X, \tau)\) is said to be \( \gamma \text{-semi-open} \) in \((X, \tau)\), if there exists a \( \gamma \)-open set \( U \) such that \( U \subset A \subset \tau_\gamma \text{-Cl}(U) \), where \( \tau_\gamma \) denotes the set of all \( \gamma \)-open sets in \((X, \tau)\).

In the present paper, we attempt to define analogously \( \gamma_\alpha \text{-semi-open sets} \) for an operation \( \gamma : SO(X, \tau) \to P(X) \) as follows.

Definition 4.1. Let \( \gamma : SO(X, \tau) \to P(X) \) be an operation on \( SO(X, \tau) \). A subset \( A \) of \((X, \tau)\) is called \( \gamma_\alpha \text{-semi-open} \) in \((X, \tau)\), if there exists a \( \gamma_\alpha \)-open set \( U \) such that \( U \subset A \subset \tau_\gamma \text{-Cl}(U) \) (cf. Definition 3.1(i), Definition 3.2). The complement of a \( \gamma_\alpha \)-semi-open set is called \( \gamma_\alpha \text{-semi-closed in} \) \((X, \tau)\).

- The collection of all \( \gamma_\alpha \)-semi-open sets (resp. \( \gamma_\alpha \)-semi-closed sets) in \((X, \tau)\) is denoted by \( \gamma_\alpha SO(X, \tau) \) (resp. \( \gamma_\alpha SC(X, \tau) \)).
Remark 4.2. For a subset $A$ of $(X, \tau)$, $A$ is "id" $SO(X, \tau)$ if and only if $A \in SO(X, \tau)$. Indeed, for the identity operation $\gamma_s = "id" : SO(X, \tau) \to P(X)$, a subset $A$ is "id"-semi-open if and only if there exists a set $U \in \tau$ such that $U \subset A \subset Cl(U)$ (i.e., $A$ is semi-open in $(X, \tau)$).

Theorem 4.3. Let $A$ be a subset of $(X, \tau)$.

(i) (a) If $A \in \gamma_s SO(X, \tau)$, then $A \subset Cl_{\gamma_s}(Int_{\gamma_s}(A))$ holds.

(b) If $A \in \gamma_s SC(X, \tau)$, then $Int_{\gamma_s}(Cl_{\gamma_s}(A)) \subset A$ holds.

(ii) Suppose that $\gamma_s : SO(X, \tau) \to P(X)$ is open on $SO(X, \tau)$. If $A \in Cl_{\gamma_s}(Int_{\gamma_s}(A))$ holds, then $A \in \gamma_s SO(X, \tau)$.

(iii) Suppose that $\gamma_s : SO(X, \tau) \to P(X)$ is open on $SO(X, \tau)$ (cf. Definition 2.5(ii)). For a subset $A$ of $(X, \tau)$, $A \in \gamma_s SO(X, \tau)$ if and only if $A \in Cl_{\gamma_s}(Int_{\gamma_s}(A))$ holds. For a subset $B$ of $(X, \tau)$, $B \in \gamma_s SC(X, \tau)$ if and only if $Int_{\gamma_s}(Cl_{\gamma_s}(B)) \subset B$ holds.

Proof. (i) (a) It follows from assumption that there exists a set $V \in \tau$, such that $V \subset A \subset Cl_{\gamma_s}(V)$ holds. Using Theorem 2.10(viii) for $\xi = SO(X, \tau)$, we have that $V = Int_{\gamma_s}(V) \subset Int_{\gamma_s}(A)$ and so $A \subset Cl_{\gamma_s}(Int_{\gamma_s}(A))$ holds.

(b) It is obtained by (a) above and Theorem 2.10(vii) for $\xi = SO(X, \tau)$.

(ii) We have $Int_{\gamma_s}(A) \subset A \subset Cl_{\gamma_s}(Int_{\gamma_s}(A))$ hold. By Theorem 2.10(ix), it is shown that $Int_{\gamma_s}(A)$ is $\gamma_s$-semi-open and so $A \in \gamma_s SO(X, \tau)$.

(iii) It is obvious from (i), (ii) and Theorem 2.10(vii) for $\xi = SO(X, \tau)$.

Theorem 4.4. Suppose that $\gamma_s : SO(X, \tau) \to P(X)$ is open on $SO(X, \tau)$ (cf. Definition 2.5(ii)). If $A \in \gamma_s SO(X, \tau)$ and $Int_{\gamma_s}(A) \subset B \subset Cl_{\gamma_s}(A)$, then $B \in \gamma_s SO(X, \tau)$ and $Cl_{\gamma_s}(Int_{\gamma_s}(A)) = Cl_{\gamma_s}(Int_{\gamma_s}(B))$ holds.

Proof. Since $A$ is $\gamma_s$-semi-open in $(X, \tau)$, there exists a $\gamma_s$-open set $U$ such that $U \subset A \subset Cl_{\gamma_s}(U)$. Then, we have $Int_{\gamma_s}(U) \subset Int_{\gamma_s}(A) \subset B \subset Cl_{\gamma_s}(A) \subset Cl_{\gamma_s}(Cl_{\gamma_s}(U))$ and so $Int_{\gamma_s}(U) \subset B \subset Cl_{\gamma_s}(Cl_{\gamma_s}(U))$.

By using Theorem 2.10 (viii) and (ix) for $\xi = SO(X, \tau)$, it is shown that $U = Int_{\gamma_s}(U) \subset B \subset Cl_{\gamma_s}(Cl_{\gamma_s}(U)) = Cl_{\gamma_s}(U)$ hold. Thus, $B$ is $\gamma_s$-semi-open in $(X, \tau)$. It follows from assumption, Theorem 4.3(ii) and Theorem 2.10(ix) and Remark 2.7 for $\xi = SO(X, \tau)$ that $Cl_{\gamma_s}(Int_{\gamma_s}(B)) \subset Cl_{\gamma_s}(Int_{\gamma_s}(Cl_{\gamma_s}(A))) \subset Cl_{\gamma_s}(Cl_{\gamma_s}(A)) \subset Cl_{\gamma_s}(Cl_{\gamma_s}(Cl_{\gamma_s}(Int_{\gamma_s}(A)))) = Cl_{\gamma_s}(Int_{\gamma_s}(A))$ hold and so $Cl_{\gamma_s}(Int_{\gamma_s}(B)) \subset Cl_{\gamma_s}(Int_{\gamma_s}(A))$ holds. The converse implication is similarly proved as follows: $Cl_{\gamma_s}(Int_{\gamma_s}(A)) \subset Cl_{\gamma_s}(B) \subset Cl_{\gamma_s}(Cl_{\gamma_s}(Int_{\gamma_s}(B))) = Cl_{\gamma_s}(Int_{\gamma_s}(B))$ hold. Therefore we have the required equality: $Cl_{\gamma_s}(Int_{\gamma_s}(A)) = Cl_{\gamma_s}(Int_{\gamma_s}(B))$.

Proposition 4.5. If $A_i \in \gamma_s SO(X, \tau)$ for each $i \in \Sigma$, then $\bigcup_{i \in \Sigma} A_i \in \gamma_s SO(X, \tau)$, where $\Sigma$ is any index set.

(ii) If $F_i \in \gamma_s SC(X, \tau)$ for each $i \in \Omega$, then $\bigcap_{i \in \Omega} F_i \in \gamma_s SC(X, \tau)$, where $\Omega$ is any index set.

Proof. (i) For each $i \in \Sigma$, there exists a set $U_i \in \tau$, such that $U_i \subset A_i \subset Cl_{\gamma_s}(U_i)$.

Then, $\bigcup_{i \in \Sigma} A_i \subset \bigcup_{i \in \Sigma} Cl_{\gamma_s}(U_i) \subset Cl_{\gamma_s}(\bigcup_{i \in \Sigma} A_i)$ holds and so $\bigcup_{i \in \Sigma} F_i \in \gamma_s$-semi-open set, because $\bigcup_{i \in \Sigma} F_i \in \tau$ (cf. Theorem 2.10 for $\xi = SO(X, \tau)$).

(ii) It is obvious from (i).

Definition 4.6. For an operation $\gamma_s : SO(X, \tau) \to P(X)$ and a subset $B$ of $(X, \tau)$, the following operation-closure of $B$, say $\gamma_s SO(X, \tau)-Cl(B)$ (shortly $\gamma_s sCl(B)$), is defined as follows:

- $\gamma_s SO(X, \tau)-Cl(B) := \bigcap \{F \mid B \subset F \subset X \setminus F \in \gamma_s SO(X, \tau)\}$. 

It is evident that, for $\gamma = "id" : SO(X, \tau) \to P(X)$ and a subset $B$ of $(X, \tau)$, "id" $SO(X, \tau)$-Cl$(B) = "id" sCl(B) = sCl(B)$ hold.

**Theorem 4.7.** Let $\gamma : SO(X, \tau) \to P(X)$ be an operation and $A$ a subset of $(X, \tau)$.

(i) The closure $\gamma_s Cl(A)$ is $\gamma_s$-semi-closed in $(X, \tau)$.

(ii) If the following implication $A \cup Int_\gamma(Cl_{\gamma}(A)) \subset \gamma_s sCl(A)$ holds.

(iii) If $\gamma : SO(X, \tau) \to P(X)$ is open on $SO(X, \tau)$ (cf. Definition 2.5(ii)), then $\gamma_s Cl(A) \subset A \cup Int_\gamma(Cl_{\gamma}(A))$ holds and so $\gamma_s sCl(A) = A \cup Int_\gamma(Cl_{\gamma}(A))$ holds.

**Proof.**

(i) By Proposition 4.5, it is shown that the intersection of every family of $\gamma_s$-semi-closed sets is $\gamma_s$-semi-closed. Thus, $\gamma_s sCl(B)$ is $\gamma_s$-semi-closed in $(X, \tau)$.

(ii) Since $\gamma_s sCl(A)$ is a $\gamma_s$-semi-closed set, $Int_{\gamma_s}(Cl_{\gamma_s}(A)) \subset \gamma_s sCl(A)$ holds by Theorem 4.3(i)(b). Since $A \subset \gamma_s sCl(A)$, we have that $A \cup Int_{\gamma_s}(Cl_{\gamma_s}(A)) \subset \gamma_s sCl(A)$ holds.

(iii) We first claim that a subset $A \cup (Int_{\gamma_s}(Cl_{\gamma_s}(A)))$ is $\gamma_s$-semi-closed. Let $B := A \cup (Int_{\gamma_s}(Cl_{\gamma_s}(A)))$. Then, we have that $Int_{\gamma_s}(Cl_{\gamma_s}(B)) \subset Int_{\gamma_s}(Cl_{\gamma_s}(A)) = Int_{\gamma_s}(Cl_{\gamma_s}(A)) \subset A \cup Int_{\gamma_s}(Cl_{\gamma_s}(A)) = B$ and so $Int_{\gamma_s}(Cl_{\gamma_s}(B)) \subset B$. Using Theorem 4.3(iii), the set $B$ is $\gamma_s$-semi-closed. Thus we prove the implication $\gamma_s sCl(A) \subset B$, because $\gamma_s sCl(A)$ is the smallest $\gamma_s$-semi-closed set including the set $A$ and $A \subset B$. We have the required equality using (ii).

In the above theorem, let $\gamma_s = "id" : SO(X, \tau) \to P(X)$. Then, we have the following well known formula:

**Corollary 4.8.** ([6]) For any subset $E$ of $(X, \tau)$, $sCl(E) = E \cup Int(Cl(E))$ holds. \hfill $\square$

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