REGULARITY IN ORDERED SEMIGROUPS

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Abstract. It is known that for a semigroup S and any two idempotent elements e, f of S, the following assertions are satisfied: (1) $Reg(eSf) = Reg(eS) \cap Reg(Sf)$; (2) $Gr(eSf) = gr(eSf) = Gr(eS) \cap Gr(Sf)$; (3) $E(eSf) = E(eS) \cap E(Sf)$; (4) $Gr(Se) = gr(Se)$ and $Gr(eS) = gr(eS)$. Also that for a semigroup S and an idempotent element e of S, the following conditions are equivalent: (i) $Reg(eSe) = Reg(Se)$; (ii) $Reg(Se) \subseteq Reg(eS)$; (iii) $E(eSe) = E(Se)$; (iv) $E(Se) \subseteq E(eS)$. We extend these results in ordered semigroups. As an application of the result of the present paper, the above mentioned results hold for any elements a, b of a semigroup S and not only for idempotent elements of S. Some additional information are also obtained.

1. Introduction and prerequisites.

The Lemma 2.1.1, Lemma 2.1.2 and the Theorem 2.1.1 below are from [5]. The Lemma 2.1.1 has been used in Lemma 2.1.2 and Lemma 2.1.2 in Theorem 2.1.1.

Lemma 2.1.1. Let e, f be idempotent elements of a semigroup S. Then the following hold:
(1) $Reg(eSf) = Reg(eS) \cap Reg(Sf)$
(2) $Gr(eSf) = gr(eSf) = Gr(eS) \cap Gr(Sf)$
(3) $E(eSf) = E(eS) \cap E(Sf)$.

Lemma 2.1.2. Let e be an idempotent element of a semigroup S. Then the following hold:
(1) $Reg(eSe) = reg(eSe) = Reg(Se) \cap Reg(eS)$
(2) $Gr(eSe) = gr(eSe)$
(3) $Gr(Se) = gr(Se)$ and $Gr(eS) = gr(eS)$
(4) $E(eSe) = E(Se) \cap E(eS)$.

Theorem 2.1.1. Let e be an idempotent element of a semigroup S. Then the following conditions are equivalent:
(i) $Reg(eSe) = Reg(Se)$
(ii) $Reg(Se) \subseteq Reg(eS)$
(iii) $E(eSe) = E(Se)$
(iv) $E(Se) \subseteq E(eS)$.

The new results in Lemma 2.1.2 above are the following: The part of (1) referring to $Reg(eSe) = reg(eSe)$ and condition (3), that is, that $Gr(Se) = gr(Se)$ and $Gr(eS) = gr(eS)$, the rest being immediate consequences of Lemma 2.1.1. So Lemma 2.1.2 can be formulated as follows:

Lemma 2.1.2. Let e be an idempotent element of a semigroup S. Then the following hold:
(1) $Reg(eSe) = reg(eSe)$

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(2) $Gr(Se) = gr(Se)$ and $Gr(eS) = gr(eS)$.

In the present paper we extend and generalize these results for an ordered semigroup S and for arbitrary elements of S. The results in [5] mentioned above can be also obtained as application of the results of the present paper as every semigroup S with the equality relation $\{(x, y) \in S \times S \mid x = y\}$ is an ordered semigroup. As a consequence, the results of Lemma 2.1.1 and Theorem 2.1.1 hold for arbitrary elements and not only for idempotent elements of S. As far as the Lemma 2.1.2 is concerned, for $a, b \in S$, the property $Reg(aSb) = reg(aSb)$ does not hold, in general, while we have $Reg(eSe) = reg(eSe)$ for every idempotent element e of S; the rest of the lemma being true for arbitrary elements a, b and not only for idempotent elements of S. Some additional information are also obtained.

An element a of a semigroup S is called *regular* [1] if there exists $x \in S$ such that $a = axa$; it is called *completely regular* [6] if there exists $x \in S$ such that $a = a^2xa^2$. An element a of an ordered semigroup $(S, ., \leq)$ is called *regular* if $a \leq axa$ for some $x \in S$ [3]; it is called *completely regular* $a \leq a^2xa^2$ for some $x \in S$ [2]. For ordered semigroups, we keep the same notations as in semigroups in [5]. Thus, for an ordered semigroup S, we denote by $Reg(S)$ the set of regular elements of S, by $Gr(S)$ the set of completely regular elements of S, for a subsemigroup T of S we denote by $reg(T)$ the intersection $T \cap Reg(S)$, that is, the elements of T which are regular in S, and by $gr(T)$ the intersection $T \cap Gr(S)$. On the other hand, while $E(S)$ denotes the set of idempotent elements in [5], for an ordered semigroup $(S, ., \leq)$ we denote by $E(S)$ the set of the elements t of S such that $t \leq t^2$.

2. Main results.

If $(S, ., \leq)$ is an ordered semigroup and H a subset of S, we denote by $(H]$ the subset of S defined by $(H] = \{t \in S \mid t \leq h \text{ for some } h \in H\}$.

Definition 1. Let $(S, ., \leq)$ be an ordered semigroup and T a subsemigroup of S. We define
$Reg(T) := \{a \in T \mid a \leq axa \text{ for some } x \in T\}$
$Gr(T) := \{a \in T \mid a \leq a^2xa^2 \text{ for some } x \in T\}$
$reg(T) := T \cap Reg(S)$
$gr(T) := T \cap Gr(S)$
$E(T) := \{e \in T \mid e \leq e^2\}$.

Throughout the paper, we use the fact that the sets $(aSb]$, $(aS]$ and $(Sb]$ $(a, b \in S)$ are subsemigroups of S.

Proposition 2. *Let $(S, ., \leq)$ be an ordered semigroup and $a, b \in S$. Then*

$$Reg(aSb] = Reg(aS] \cap Reg(Sb].$$

Proof. Since $aSb \subseteq aS, Sb$, we have $Reg(aSb] \subseteq Reg(aS] \cap Reg(Sb]$. Let now $c \in Reg(aS] \cap Reg(Sb]$. Since $c \in Reg(aS]$, we have $c \in (aS]$ and $c \leq cxc$ for some $x \in (aS]$. Since $c \in Reg(Sb]$, we have $c \in (Sb]$ and $c \leq cyc$ for some $y \in (Sb]$. Since $c \in (aS]$, $c \leq as$ for some $s \in S$. Since $c \in (Sb]$, $c \leq tb$ for some $t \in S$. Thus we have

$$c \leq cxc \leq (as)x(tb) = a(sxt)b \in aSb,$$

so $c \in (aSb]$. Since $x \in (aS]$, $x \leq aw$ for some $w \in S$. Since $y \in (Sb]$, $y \leq ub$ for some $u \in S$. Hence we obtain

$$c \leq cxc \leq cx(cyc) \leq c(aw)c(ub)c = c(awcub)c,$$

with $awcub \in aSb \subseteq (aSb]$. Therefore we have $c \in Reg(aSb]$. □

Proposition 3. *Let $(S,.,\leq)$ be an ordered semigroup $a,b\in S$. Then we have*

$$Gr(aSb] = gr(aSb].$$

Proof. Let $c\in Gr(aSb]$. Then $c\in(aSb]$ and $c\leq c^2yc^2$ for some $y\in(aSb]$. Since $c,y\in S$ and $c\leq c^2yc^2$, we have $c\in Gr(S)$, so $c\in(aSb]\cap Gr(S):=gr(aSb]$.
Let now $d\in gr(aSb]:=(aSb]\cap Gr(S)$. Since $d\in Gr(S)$, $d\leq d^2xd^2$ for some $x\in S$. Then

$$d\leq d^2xd^2\leq d(d^2xd^2)x(d^2xd^2)d=d^2(dxd^2xd^2xd)d^2.$$

Since $d\in(aSb]$, $d\leq asb$ for some $s\in S$. We put $t:=dxd^2xd^2xd$, and we have $t\leq(asb)xd^2xd^2x(asb)\in aSb$, so $t\in(aSb]$. Since $d\in(aSb]$ and $d\leq d^2td^2$, where $t\in(aSb]$, we have $d\in Gr(aSb]$. $\square$

Proposition 4. *Let $(S,.,\leq)$ be an ordered semigroup and $a,b\in S$. Then we have*

$$Gr(aSb]=Gr(aS]\cap(Sb]=(aS]\cap Gr(Sb].$$

Proof. Let $c\in Gr(aSb]$. Then $c\in(aSb]$ and $c\leq c^2yc^2$ for some $y\in(aSb]$. Since $aSb\subseteq aS,Sb$, we have $(aSb]\subseteq(aS],(Sb]$. Since $c,y\in(aS]$ and $c\leq c^2yc^2$, we have $c\in Gr(aS]$, so $c\in Gr(aS]\cap(Sb]$. Let now $d\in Gr(aS]\cap(Sb]$. Since $d\in Gr(aS]$, $d\in(aS]$ and $d\leq d^2xd^2$ for some $x\in(aS]$. Since $d\in(aS]$, $d\leq as$ for some $s\in S$. Since $d\in(Sb]$, $d\leq tb$ for some $t\in S$. Then we have $d\leq d^2xd^2\leq(as)dxd(tb)\in aSb$, so $d\in(aSb]$. Moreover, we have

$$d\leq d^2xd^2\leq d(d^2xd^2)x(d^2xd^2)d=d^2(dxd^2xd^2xd)d^2.$$

Putting $t:=dxd^2xd^2xd$, we have $t\leq(as)xd^2xd^2x(tb)\in aSb$, so $t\in(aSb]$. Since $d\in(aSb]$ and $d\leq d^2td^2$, where $t\in(aSb]$, we have $d\in Gr(aSb]$. In a similar way we prove that $Gr(aSb]=(aS]\cap Gr(Sb]$. $\square$

Proposition 5. *Let S be an ordered semigroup $a,b\in S$. Then we have*

$$Gr(aSb]=Gr(aS]\cap Gr(Sb].$$

Proof. By Proposition 4, we have

$$Gr(aSb]=Gr(aSb]\cap Gr(aSb]=Gr(aS]\cap(Sb]\cap(aS]\cap Gr(Sb].$$

Since $Gr(aS]\subseteq(aS]$ and $Gr(Sb]\subseteq(Sb]$, we have $Gr(aSb]=Gr(aS]\cap Gr(Sb]$. $\square$

Proposition 6. *Let $(S,.,\leq)$ be an ordered semigroup and $a,b\in S$. Then*

$$E(aSb]=E(aS]\cap(Sb]=(aS]\cap E(Sb].$$

Proof. Let $c\in E(aSb]$. Then $c\in(aSb]$ and $c\leq c^2$. Since $c\in(aSb]$, $c\leq aub$ for some $u\in S$. Since $c\leq aub\in aS$, we have $c\in(aS]$. Since $c\in(aS]$ and $c\leq c^2$, we have $c\in E(aS]$. Besides, since $c\leq aub\in Sb$, we have $c\in(Sb]$. Thus $c\in E(aS]\cap(Sb]$. Let now $d\in E(aS]\cap(Sb]$. Since $d\in E(aS]$, we have $d\in(aS]$ and $d\leq d^2$. Since $d\in(aS]$, $d\leq as$ for some $s\in S$. Since $d\in(Sb]$, $d\leq tb$ for some $t\in S$. Thus we have $d\leq d^2\leq(as)(tb)=a(st)b\in aSb$, so $d\in(aSb]$. Since $d\in(aSb]$ and $d\leq d^2$, we have $d\in E(aSb]$. In a similar way we get $E(aSb]=(aS]\cap E(Sb]$. $\square$

Proposition 7. *If S is an ordered semigroup $a,b\in S$, then we have*

$$E(aSb]=E(aS]\cap E(Sb].$$

Proof. By Proposition 6 , we have

$$E(aSb] = E(aSb] \cap E(aSb] = E(aS] \cap (Sb] \cap (aS] \cap E(Sb].$$

Since $E(aS] \subseteq (aS]$ and $E(Sb] \subseteq (Sb]$, we have $E(aSb] = E(aS] \cap E(Sb]$. □

Proposition 8. *If $(S, ., \leq)$ is an ordered semigroup, then we have*

$$Gr(Sa] = gr(Sa] \text{ and } Gr(aS] = gr(aS].$$

Proof. First of all, for any subsemigroup T of S, we have $Gr(T) \subseteq gr(T)$. In fact: if $d \in Gr(T)$, then $d \in T$ and $d \leq d^2yd^2$ for some $y \in T$. Since $d, y \in S$ and $d \leq d^2yd^2$, we have $d \in Gr(S)$, thus $d \in T \cap Gr(S) = gr(T)$. Since $(Sa]$ is a subsemigroup of S, we have $Gr(Sa] \subseteq gr(Sa]$. Let now $x \in gr(Sa] := (Sa] \cap Gr(S)$. Since $x \in (Sa]$, $x \leq sa$ for some $s \in S$. Since $x \in Gr(S)$, we get $x \leq x^2tx^2$ for some $t \in S$. Then

$$x \leq x^2tx^2 \leq x^2t(x^2tx^2)x = x^2(tx^2tx)x^2.$$

On the other hand, $tx^2tx \leq tx^2tsa \in Sa$, so $tx^2tx \in (Sa]$. Since $x \in (Sa]$ and $x \leq x^2(tx^2tx)x^2$, where $tx^2tx \in (Sa]$, we have $x \in Gr(Sa]$. Similarly we get $Gr(aS] = gr(aS]$. □

For an ordered semigroup S and an element e of S such that $e \leq e^2$, we have $Reg(eSe) = reg(eSe)$ [4], while for arbitrary elements $a, b \in S$, condition $Reg(aSb) = reg(aSb)$ does not hold in general. We prove it by the following

Example. Let S be the ordered semigroup defined by the multiplication and the order below:

.	a	b	c	d	e
a	a	a	a	a	a
b	a	b	a	d	a
c	a	e	c	c	e
d	a	b	d	d	b
e	a	e	a	c	a

$$\leq := \{(a,a),(a,b),(a,c),(a,d),(a,e),(b,b),(c,c),(d,d),(e,e)\}.$$

The set $cSb = \{a, e\}$ is a subsemigroup of S,
$Reg(cSb) = \{t \in cSb \mid t \leq txt \text{ for some } x \in cSb\} = \{a\}$,
$reg(S) = \{t \in S \mid t \leq txt \text{ for some } x \in S\} = S$, and
$reg(cSb) = cSb \cap Reg(S) = \{a, e\}$.
Thus $Reg(cSb) \neq reg(cSb)$.

We notice that if S is the semigroup defined in the table above, then $Reg(cSb) \neq reg(cSb)$. Which means that for a semigroup S, as well, the relation $Reg(aSb) = reg(aSb)$ which holds for $a = b = e$, where e is an idempotent element of S, does not hold for arbitrary elements a, b of S, in general.

Theorem 9. *Let S be an ordered semigroup and $a, b \in S$. The following are equivalent:*
(1) $Reg(aS] \subseteq Reg(aSb]$
(2) $Reg(aS] = Reg(aSb]$
(3) $Reg(aS] \subseteq Reg(Sb]$
(4) $Reg(aS] \subseteq (Sb]$
(5) $E(aS] = E(aSb]$

(6) $E(aS] \subseteq E(aSb]$
(7) $E(aS] \subseteq E(Sb]$
(8) $E(aS] \subseteq (Sb]$.

Proof. (1) $\Longrightarrow$ (2). Since $aSb \subseteq aS$, we have $(aSb] \subseteq (aS]$. Then, since $(aSb]$ and $(aS]$ are subsemigroups of S, the sets $Reg(aSb]$ and $Reg(aS]$ are defined, and we have $Reg(aSb] \subseteq Reg(aS]$. Then, by (1), $Reg(aS] = Reg(aSb]$.
(2) $\Longrightarrow$ (3). Since $aSb \subseteq Sb$, we have $(aSb] \subseteq (Sb]$. Then, since $(aSb]$ and $(Sb]$ are subsemigroups of S, we have $Reg(aSb] \subseteq Reg(Sb]$. Then, by (2), $Reg(aS] \subseteq Reg(Sb]$.
(3) $\Longrightarrow$ (4). Since $(Sb]$ is a subsemigroup of S, we have $Reg(Sb] \subseteq (Sb]$ then, by (3), $Reg(aS] \subseteq (Sb]$.
(4) $\Longrightarrow$ (5). For every subsemigroup T of S, we have $E(T) \subseteq Reg(T)$. Indeed, if $x \in E(T)$, then $x \in T$ and $x \leq x^2 \leq xxx$, so $x \in Reg(T)$. Since $(aS]$ is a subsemigroup of S, we have $E(aS] \subseteq Reg(aS]$. Then, by (4), $E(aS] \subseteq (Sb]$, so $E(aS] \cap (Sb] = E(aS]$. By Proposition 6, $E(aSb] = E(aS] \cap (Sb]$. Thus we have $E(aS] = E(aSb]$.
The implication (5) $\Longrightarrow$ (6) is obvious.
(6) $\Longrightarrow$ (7). If T_1, T_2 are subsemigroups of S such that $T_1 \subseteq T_2$, then $E(T_1) \subseteq E(T_2)$. Since $(aSb]$ and $(Sb]$ are subsemigroups of S such that $(aSb] \subseteq (Sb]$, we have $E(aSb] \subseteq E(Sb]$. Then, by (6), $E(aS] \subseteq E(Sb]$.
(7) $\Longrightarrow$ (8). Since $(Sb]$ is a subsemigroup of S, $E(Sb] \subseteq (Sb]$ and, by (7), we get $E(aS] \subseteq (Sb]$.
(8) $\Longrightarrow$ (1). Let $c \in Reg(aS]$. Then $c \in (aS]$ and $c \leq cxc$ for some $x \in (aS]$. Then $xc \leq xcxc = (xc)^2$, $x \leq as$ for some $s \in S$, $xc \leq asc \in aS$, $xc \in (aS]$, and $xc \in E(aS]$. Then, by (8), $xc \in (Sb]$, that is, $xc \leq tb$ for some $t \in S$. On the other hand, since $c \in (aS]$, $c \leq az$ for some $z \in S$. Hence we have $c \leq c(xc) \leq (az)(tb) \in aSb$, so $c \in (aSb]$. Furthermore, since $c \leq cxc$, we have $cx \leq (cx)^2$ and, since $c \leq az$, we have $cx \leq azx \in aS$, that is, $cx \in (aS]$. Hence we have $cx \in E(aS]$ and, by (8), $cx \in (Sb]$, then $cx \leq tb$ for some $t \in S$. Now $c \leq cxc \leq cx(cxc) = c(xcx)c$ and $xcx \leq (as)(tb) \in aSb$, so $xcx \in (aSb]$. Since $c \in (aSb]$ and $c \leq c(xcx)c$, where $xcx \in (aSb]$, we have $c \in Reg(aSb]$. □
In a similar way we prove the following:

Theorem 10. *Let S be an ordered semigroup and $a, b \in S$. The following conditions are equivalent:*
(1) $Reg(Sb] \subseteq Reg(aSb]$
(2) $Reg(Sb] = Reg(aSb]$
(3) $Reg(Sb] \subseteq Reg(aS]$
(4) $Reg(Sb] \subseteq (aS]$
(5) $E(Sb] = E(aSb]$
(6) $E(Sb] \subseteq E(aSb]$
(7) $E(Sb] \subseteq E(aS]$
(8) $E(Sb] \subseteq (aS]$.
Putting $a = b$ in Theorem 9, we have the following:

Theorem 11. *Let S be an ordered semigroup and $a \in S$. The following are equivalent:*
(1) $Reg(aS] \subseteq Reg(aSa]$
(2) $Reg(aS] = Reg(aSa]$
(3) $Reg(aS] \subseteq Reg(Sa]$
(4) $Reg(aS] \subseteq (Sa]$
(5) $E(aS] = E(aSa]$
(6) $E(aS] \subseteq E(aSa]$
(7) $E(aS] \subseteq E(Sa]$

(8) $E(aS] \subseteq (Sa]$.

Putting $a = b$ in Theorem 10, we have the following:

Theorem 12. *Let S be an ordered semigroup and $a \in S$. The following are equivalent:*

(1) $Reg(Sa] \subseteq Reg(aSa]$
(2) $Reg(Sa] = Reg(aSa]$
(3) $Reg(Sa] \subseteq Reg(aS]$
(4) $Reg(Sa] \subseteq (aS]$
(5) $E(Sa] = E(aSa]$
(6) $E(Sa] \subseteq E(aSa]$
(7) $E(Sa] \subseteq E(aS]$
(8) $E(Sa] \subseteq (aS]$.

Applying Propositions 2, 3, 5, 7 and 8 to a semigroup (without order), we obtain the following

Corollary 13. (Cf. also Lemma 2.1.1 and Lemma 2.1.2) For a semigroup S and arbitrary elements $a, b \in S$, the following conditions are satisfied:

(1) $Reg(aSb) = Reg(aS) \cap Reg(Sb)$
(2) $Gr(aSb) = gr(aSb) = Gr(aS) \cap Gr(Sb)$
(3) $E(aSa) = E(aS) \cap E(Sb)$
(4) $Gr(Sa) = gr(Sa)$ and $Gr(aS) = gr(aS)$.

We have already seen that if S is an ordered semigroup and e an element of S such that $e \leq e^2$, then $Reg(eSe) = reg(eSe)$. As a consequence, if S is a semigroup and e an idempotent element of S, then $Reg(eSe) = reg(eSe)$ (the part of (1) in Lemma 2.1.2 mentioned above).

By conditions (2),(3),(5),(7) of Theorem 10, we have the following

Corollary 14. (Cf. also Theorem 2.1.1) For a semigroup S and arbitrary elements $a, b \in S$, the following are equivalent:

(1) $Reg(aSb) = Reg(Sb)$
(2) $Reg(Sb) \subseteq Reg(aS)$
(3) $E(aSb) = E(Sb)$
(4) $E(Sb) \subseteq E(aS)$.

Besides, applying conditions (2),(3),(5) and (7) of Theorem 12 to semigroups (without order) or just putting $a = b$ in Corollary 14 we get the Theorem 2.1.1 in [5] for an arbitrary element of S.

So the Lemma 2.1.1, Lemma 2.1.2 and the Theorem 2.1.1 in [5] except the part of Lemma 2.1.2 referring to $Reg(eSe) = reg(eSe)$ hold for arbitrary elements a, b and not only for idempotent elements e, f of a semigroup S.

References

[1] A.H. Clifford, G.B. Preston, *The Algebraic Theory of Semigroups* I. Mathematical Surveys, no. 7, American Mathematical Society, Providence, Rhode Island 1961. xv+224 pp.

[2] N. Kehayopulu, *On completely regular poe-semigroups*, Math. Japon. **37**(1) (1992), 123–130.

[3] N. Kehayopulu, *On regular duo ordered semigroups*, Math. Japon. **37**(3) (1992), 535–540.

[4] N. Kehayopulu, M. Tsingelis, *The set of regular elements in ordered semigroups*, Math. Japon. **72**(1) (2010), 61–66.

[5] M. Mitrović, *Semilattices of Archimedean Semigroups*, University of Nis, Faculty of Mechanical Engineering, Nis, 2003. xiv+160 pp. ISBN: 86-80587-32-X

[6] M. Petrich, *Introduction to Semigroups*, Merrill Research and Lecture Series, Charles E. Merrill Publishing Co., Columbus, Ohio, 1973. viii+198 pp.

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