MATHEMATICS EXPERIMENT IN UNIVERSITY MATHEMATICS BY USING CAS

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Abstract. As an effective use of the CAS (computer algebra system) in the mathematics education, we can perform mathematics experiment and can do heuristic learning. One of the roles expected to use of CAS in mathematics learning is to be able to carry out experimental mathematics to make heuristically by myself. We introduce a trial (mathematical search) to practice such an activity in university mathematics.

We can obtain degeneracy conditions of singularities by using Gröbner basis. In classification of singularities, the defining equations of certain types are polynomials with parametric coefficients. In general, the Gröbner basis generated by such polynomials depends on the values of the parameters, and those calculations are not easy. A structure of zero set defined by polynomials with parameters changes by the values of the parameters. This is the learning problem that is important in the university mathematics. As one of such teaching materials, the defining equation of singularity with parameters is suitable. We show various examples of the calculations to students and teach these calculations. When students learn those calculations as algorithm, an interesting subject is preferable, mathematically. The singularity on the algebraic curve can arouse the interest of the students. We can make the learning of the algebraic curve with parameters more interesting in university mathematics.

This trial is significant to show utility of the mathematics by using computer and to clarify the pleasure of the calculations to students. Then, the calculation based on Gröbner basis theory is necessary. If they calculate thoughtlessly, they will not obtain significant results. The mathematical result to show in this paper is still an experiment stage on the way. We show these search activities as the material of the experiment in university mathematics in this paper.

1 Introduction

We will recall the following definitions and theorems. Let $f(x,\ y,\ z)$ be a polynomial with variables $x,\ y,\ z$ in $\mathbf{C}^3$. Then the analytic set defined by $f(x,\ y,\ z)\ =\ 0$ has a singularity at O in $\mathbf{C}^3$ if

$$f(0,\ 0,\ 0)\ =\ \frac{\partial f(0,0,0)}{\partial x}\ =\ \frac{\partial f(0,0,0)}{\partial y}\ =\ \frac{\partial f(0,0,0)}{\partial z}\ =\ 0.$$

Definition 1.1
A quasihomogeneous function f is said to be non-degenerate if O is an isolated singularity.

Definition 1.2
Let $\mathbf{N}\ \subset\ \mathbf{R}_+\ \subset\ \mathbf{R}$ be the sets of all non-negative integers, all non-negative real numbers, all real numbers respectively. Let $\mathbf{H}\ \subset\ \mathbf{N}^K$ be a subset. Newton polyhedron of a set $\mathbf{H}$ is defined by the convex hull in $\mathbf{R}_+^K$ of the set

$$\bigcup_{n\in\mathbf{H}}(\ n\ +\ \mathbf{R}_+^{\mathbf{H}}\ ).$$

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Newton boundary of a set $\mathbf{H}$ is defined by the union of all compact faces of Newton polyhedron of $\mathbf{H}$. Newton polyhedron is defined by $\Gamma_+(\mathbf{H})$ and Newton boundary by $\Gamma(\mathbf{H})$.

Definition 1.3
Let $f = \sum_{n\in\mathbf{H}} a_n x^n, \quad a_n \in \mathbf{C}$. *Let us write* $\mathrm{supp} f = \{ n \in \mathbf{N}^K \mid a_n \neq 0\}$. *Newton polyhedron (or Newton boundary) of a series* f *is defined by Newton polyhedron of the* $\mathrm{supp} f$. *Newton polyhedron of the series* f *is denoted by* $\Gamma_+(f)$ *(and* $\Gamma(f)$ *respectively).*

Definition 1.4
The principal part of a series f *is defined by the polynomial*

$$f_0 = \sum_{n\in\Gamma(f)} a_n x^n.$$

For any closed face $\triangle \subset \Gamma(f)$ *we shall denote by* $f_\triangle$ *the polynomial*

$$\sum_{n\in\triangle} a_n x^n.$$

We say that f *is non-degenerate on* $\triangle$ *if the equation*

$$\frac{\partial f_\triangle}{\partial x_1} = \frac{\partial f_\triangle}{\partial x_2} = \dots = \frac{\partial f_\triangle}{\partial x_n} = 0$$

has no solution in $(\mathbf{C}^*)^{\mathrm{n}}$. *When* f *has a non-degenerate on every face* $\triangle$ *of* $\Gamma(f)$, *we say that* f *has a non-degenerate principal part.*

The following theorem holds about the Newton boundary and the topological type of singularity.

Theorem 1.5 ([5])
Suppose that $f(x)$ *has an isolated singularity at* O *and* $f(x)$ *has a non-degenerate principal part. Then the Milnor fibration at* O *is determined by the Newton boundary* $\Gamma(f(x))$.

Corollary 1.6 ([4])
The topological type of singularity and the multiplicity μ *are independent of the particular choice of* $f(x)$ *for a fixed* $\Gamma(f(x))$.

2 Defining Equations of Simple $K3$ Singularities In the theory of two-dimensional singularities, the defining equation of certain type singularity is a polynomial that has parametric coefficients. Arnol'd showed the non-degenerate conditions(restrictions). The normal forms(defining equations of singularities) are given in [1].

What are natural generalizations in three-dimensional case of those singularities? They are purely elliptic singularities. And we regard simple $K3$ singularities as natural generalizations of simple elliptic singularities in three-dimensional case. We define the simple $K3$ singularities. The notion of a simple $K3$ singularity was defined by Ishii and Watanabe [3] as a three-dimensional Gorenstein purely elliptic singularity of $(0, 2)$-type, whereas a simple elliptic singularity is two-dimensional purely elliptic singularity of $(0, 1)$-type.

Definition 2.1 ([9])
Let $(X,\ x)$ be a normal isolated singularity. For any positive integer m,

$$\delta_m(X,\ x) \ = \ \frac{dim_c \Gamma(X - \{x\}, \vartheta(mK))}{L^{2/m}(X - \{x\})},$$

where K is the canonical line bundle on $X - \{x\}$, and $L^{2/m}(X - \{x\})$ is the set of all $L^{2/m}$-integrable (at x) holomorphic m-tuple n-forms on $X - \{x\}$.

Then δ_m is finite and does not depend on the choice of a Stein neighborhood on X.

Definition 2.2 ([9])
A singularity $(X,\ x)$ is said to be purely elliptic if $\delta_m \ = \ 1$ for every positive integer m.

When X is a two-dimensional analytic space, purely elliptic singularities are quasi-Gorenstein singularities, i.e., there exists a non-vanishing holomorphic 2-form on $X - \{x\}$.

Definition-Proposition 2.3 ([3])
A three-dimensional singularity $(X,\ x)$ is a simple $K3$ singularity if the following two equivalent conditions are satisfied:
(1) $(X,\ x)$ is Gorenstein purely elliptic of $(0, 2)$-type.
(2) $(X,\ x)$ is quasi-Gorenstein and the exceptional divisor E is a normal K3 surface for any minimal resolution $\pi \ : \ (\tilde{X},\ E) \ \to \ (X,\ x)$.

Simple elliptic singularities and cusp singularities are characterized as two-dimensional purely elliptic singularities of $(0, 1)$-type and of $(0, 0)$-type, respectively. The notion of a simple $K3$ singularity is defined as a three-dimensional isolated Gorenstein purely elliptic singularity of (0,2)-type.

Let $f \ \in \ \mathbf{C}[\mathrm{z}_0,\ \mathrm{z}_1,\ \mathrm{z}_2,\ \mathrm{z}_3]$ be a polynomial which is nondegenerate with respect to its Newton boundary $\Gamma(f)$ in the sense of [8], and whose zero locus $X \ = \ \{f \ = \ 0\}$ in $\mathbf{C^4}$ has an isolated singularity at the origin $0 \ \in \ \mathbf{C^4}$. Then the condition for $(X,\ 0)$ to be a simple $K3$ singularity is given by a property of the Newton boundary $\Gamma(f)$ of f.

Next we consider the case where $(X,\ x)$ is a hypersurface singularity defined by a nondegenerate polynomial $f \ = \ \sum a_\nu z^\nu \ \in \ \mathbf{C}[\mathrm{z}_0,\ \mathrm{z}_1,\ ...\ ,\ \mathrm{z}_\mathrm{n}]$, and $x \ = \ 0 \ \in \ \mathbf{C^{n+1}}$. We denote by $\mathbf{R_0}$ the set of all nonnegative real numbers. Recall that the Newton boundary $\Gamma(f)$ of f is the union of the compact faces of $\Gamma_+(f)$, where $\Gamma_+(f)$ is the convex hull of $\bigcup_{a_\nu \neq 0}(\nu \ + \ \mathbf{R_0^{n+1}})$ in $\mathbf{R^{n+1}}$.

For any face Δ of $\Gamma_+(f)$, set $f_\Delta \ : = \ \sum_{\nu \in \Delta} a_\nu z^\nu$. We say f to be nondegenerate, if

$$\frac{\partial f_\Delta}{\partial z_0} \ = \ \frac{\partial f_\Delta}{\partial z_1} \ = \ ... \ = \ \frac{\partial f_\Delta}{\partial z_n} \ = \ 0$$

has no solution in $(\mathbf{C}^*)^{\mathbf{n+1}}$ for any face Δ.

When f is nondegenerate, the condition for $(X,\ x)$ to be a purely elliptic singularity is given as follows:

Theorem 2.4 ([10])
Let f be a nondegenerate polynomial and suppose $X \ = \ \{f \ = \ 0\}$ has an isolated singularity at $x \ = \ 0 \ \in \ \mathbf{C^{n+1}}$.

(1) *$(X,\ x)$ is purely elliptic if and only if* $(1,\ 1,\ \dots,\ 1)\ \in\ \Gamma(f)$.
(2) *Let* $n\ =\ 3$ *and let* Δ_0 *be the face of* $\Gamma(f)$ *containing* $(1,\ 1,\ 1,\ 1)$ *in the relative interior of* Δ_0.
Then $(X,\ x)$ is a simple $K3$ singularity if and only if $dim_R \Delta_0\ =\ 3$.

Thus if f is nondegenerate and defines a simple $K3$ singularity, then f_{Δ_0} is a quasi-homogeneous polynomial with a uniquely determined weights α , which called the weights of f and denoted $\alpha(f)$. We denote by $\mathbf{Q}_+$ the set of all positive rational numbers. Then $\alpha\ =\ (\alpha_1,\ \alpha_2,\ \alpha_3,\ \alpha_4)\ \in\ \mathbf{Q}_+^4$ and $deg_\alpha(\nu)\ :=\ \sum_{i=1}^{4} \alpha_i \nu_i\ =\ 1$ for any $\nu\ \in\ \Delta_0$. In particular, $\sum_{i=1}^{4} \alpha_i\ =\ 1$, since $(1,\ 1,\ 1,\ 1)$ is always contained in Δ_0.

We denote by $\mathbf{Z_0}$ the set of all nonnegative integer numbers.

Let $W'\ :=\ \{\alpha\ =\ (\alpha_1,\ \alpha_2,\ \alpha_3,\ \alpha_4)\ \in\ \mathbf{Q}_+^4\ |\ \alpha_1+\alpha_2+\alpha_3+\alpha_4\ =\ 1\}$ and for an element α of W', set

$$T(\alpha)\ :=\ \{\nu\ \in\ \mathbf{Z_0^4}\ |\ \alpha\cdot\nu\ =\ \mathbf{1}\}$$

and

$$< T(\alpha) >\ :=\ \{\sum_{\nu\in T(\alpha)} t_\nu\cdot\nu\ \in\ \mathbf{R^4}\ |\ \mathrm{t}_\nu\ \in\ \mathbf{R_0}\}.$$

Then the set $< T(\alpha) >$ is a closed cone in $\mathbf{R^4}$ spanned by $T(\alpha)$.

Let $W_4\ := \{\alpha\ \in\ W'\ |\ (1,\ 1,\ 1,\ 1)\ \in\ \mathrm{Int} < \mathrm{T}(\alpha) >,\ \alpha_1\ \geq\ \alpha_2\ \geq\ \alpha_3\ \geq\ \alpha_4\}$. Then W_4 is the set of weights of simple $K3$ singularities. W_4 is classified, there are ninety five classes in terms of the weights of f([12]). The defining equations of simple $K3$ singularities are polynomials that have parametric coefficients.

Yonemura listed the weights of hypersurface simple $K3$ singularities by nondegenerate polynomials and obtained the examples such that the polynomial f is quasi-homogeneous and that $\{f\ =\ 0\}\ \subset\ \mathbf{C^4}$ has a simple $K3$ singularity at the origin([12]). The minimum number of parameters in the polynomial is less than or equal to 19 and is associated with the moduli of the $K3$ surface with singularities.

3 Application of Gröbner Bases In the elimination theory, one of basic strategy is Elimination Theorem. The calculation of Gröbner basis([2]) for such polynomials is not easy. In calculation process, we need to classify conditions of parameters for the leading term. By a study of Comprehensive Gröbner bases([11]), the calculation algorithm for a certain type is obtained. The following theorem holds.

Theorem 3.1 ([2])
Let $I\ \subset\ k[x_1,\ \dots,x_n]$ be an ideal and let G be a Gröbner basis of I with respect to lex order where $x_1\ >\ x_2\ >\ \dots\ >\ x_n$. Then, for every $0\ \leq\ l\ \leq\ n$, the set

$$G_l\ =\ G\ \cap\ k[x_{l+1},\ \dots,x_n]$$

is a Gröbner basis of the lth elimination ideal I_l.

Let f be a defining equation, $I\ :=\ < f,\ \frac{\partial f}{\partial x_1},\ \frac{\partial f}{\partial x_2},\ \dots,\ \frac{\partial f}{\partial x_n} >$. And let G be a Gröbner basis of I with respect to *lex* order where $x_1\ >\ x_2\ >\ \dots\ >\ x_n$. Then, for every $0\ \leq\ l\ \leq\ n$, the set

$$G_l\ =\ G\ \cap\ k[x_{l+1},\ \dots,\ x_n]$$

is a Gröbner basis of the lth elimination ideal I_l. We can obtain the non-degeneracy condition of singularity at the origin from the Gröbner basis of the lth elimination ideal I_l. (The degeneracy condition of singularity at the origin means the singularity is non-isolated singularity at the origin.) In the process, we need to classify the conditions of parameters for the leading term.

We consider the ideal

$$I = \{f_i(t_1, \dots, t_m, x_1, \dots, x_n) : 1 \le i \le s\}$$

in $k(t_1, \dots, t_m)[x_1, \dots, x_n]$ and fix a monomial order. We thought of $t_1, \dots, t_m$ as symbolic parameters appearing in the coefficients of $f_1, \dots, f_s$. By dividing each f_i by its leading coefficient which lies in $k(t_1, \dots, t_m)$, we assumed that the leading coefficients of the f_i are all equal to 1. Then let $g_1, \dots, g_s$ be a reduced Gröbner basis for I. Thus the leading coefficients of the g_i were also 1.

4 Degeneracy Conditions The defining equations of singularities are polynomials with parameter coefficients. We can obtain degeneracy conditions of these singularities by using Gröbner basis. In general, the forms of Gröbner bases generated by polynomials depend on the values of parameters.

We obtain the degeneracy conditions of simple $K3$ singularities. Let W_4 be the set of defining equations which has a nondegenerate hypersurface simple $K3$ singularity at the origin and let $\#m(f)$ be the minimum number of parameters of the defining equation for any $f \in W_4$. We already obtained the following results for uni-modula case, bi-modulars case and tri-modulars case([6]). Here, the index number n of f_n denotes the number of the defining equation in the classification by Yonemura([12]). λ, μ, ν are parameteric coefficients.

For $\#m(f) = 1$,

No.	The defining equations
f_{52}	$x3 + 4\lambda xyzw + xz3 + y4 + zw4$
f_{56}	$x2y + y3z + 3\lambda yz2w2 + z5 + w6$

No.	The degeneracy conditions
f_{52}	$\lambda 4 - 1 = 0$
f_{56}	$\lambda 3 + 1 = 0$

For $\#m(f) = 2$,

No.	The defining equations
f_{46}	$x2 + y3 + 3\lambda yz4w4 + z^{11} + 2\mu z6w6 + zw^{12}$
f_{61}	$x2z + y4 + 2\sqrt{2}\lambda y2zw2 + z4w + 2\mu z2w4 + w7$

No.	The degeneracy conditions
f_{46}	$(\lambda 3 + \mu 2 + 1)2 - 4\mu 2 = 0$
f_{61}	$(\mu 2 - 1)((\lambda 2 - \mu)2 - 1) = 0$

For $\#m(f) = 3$,

f_{64} : $x2z + axy2 + by3w + \sqrt{2}\lambda y2zw2 + 2\sqrt{2}\mu yz2w3 + z6 - 2\nu z3w4 + w8$
where $a \neq 0$ or $b \neq 0$.

For this defining equation, we set as follows:
$f_{64(1)}$: $x2z + xy2 + 2\lambda y2zw2 + 2\sqrt{2}\mu yz2w3 + z6 + 2\nu z3w4 + w8$ (for the above $a \neq 0$)
$f_{64(2)}$: $x2z + y3w + 3\lambda yz2w3 + z6 + 2\mu z3w4 + w8$ (for the above $a = 0$)

No.	The defining equations
$f_{64(1)}$	$x2z + xy2 + \sqrt{2}\lambda y2zw2 + 2\sqrt{2}\mu yz2w3 + z6 - 2\nu z3w4 + w8$
$f_{64(2)}$	$x2z + y3w + 3\lambda yz2w3 + z6 + 2\mu z3w4 + w8$

No.	The degeneracy conditions
$f_{64(1)}$	$(-16\lambda 2 + 4\sqrt{2}\lambda 3\mu 2 - 27\mu 4 + 24\nu + 8\lambda 4\nu - 36\sqrt{2}\lambda\mu 2\nu - 16\lambda 2\nu 2 + 8\nu 3)2$ $-16(2 + 2\lambda 4 - 9\sqrt{2}\lambda\mu 2 - 8\lambda 2\nu + 6\nu 2)2 = 0$
$f_{64(2)}$	$(\lambda 3 + \mu 2 + 1)2 - 4\mu 2 = 0$

We show an example of the calculation in the following. We can calculate the other results by a similar method. (We use Mathematica)

Example
In [1] = $f64 = x2z + xy2 + py2zw2 + qyz2w3 + z6 + rz3w4 + w8;;$
Factor[GroebnerBasis [{f64, ∂_x (f64) , ∂_y (f64) , ∂_z (f64) , ∂_w (f64) } ,
{ x, y, z, w }, {x, y, z }]]

Out[1] = { $(-512 + 512p2 - 128p4 + 288pq2 - 16p3q2 + 27q4 + 768r - 512p2r + 64p4r - 144pq2r - 384r2 + 128p2r2 + 64r3)(512 + 512p2 + 128p4 - 288pq2 - 16p3q2 + 27q4 + 768r + 512p2r + 64p4r - 144pq2r + 384r2 + 128p2r2 + 64r3)\ w^{17}$ }

In [2] = $m1 = 512p2 - 16p3q2 + 27q4 + 768r + 64p4r - 144pq2r + 128p2r2 + 64r3;$
$m2 = 512 + 128p4 - 288pq2 + 512p2r + 384r2; p = \sqrt{2}\lambda; q = 2\sqrt{2}\mu; r = -2\nu;$

In [3] = Factor[m1]
Out[3] = $-64(-16\lambda 2 + 4\sqrt{2}\lambda 3\mu 2 - 27\mu 4 + 24\nu + 8\lambda 4\nu - 36\sqrt{2}\lambda\mu 2\nu - 16\lambda 2\nu 2 + 8\nu 3)$

In [4] = Factor[m2]
Out[4] = $256(2 + 2\lambda 4 - 9\sqrt{2}\lambda\mu 2 - 8\lambda 2\nu + 6\nu 2)$

We denote the degeneracy condition of f_i by $\mathfrak{M}(f_i)$. Then, for $\mu = 0$, $\mathfrak{M}(f_{46})$ transform to $\mathfrak{M}(f_{56})$. Similarly, for $\mu = 0$, $\mathfrak{M}(f_{61})$ transform to $\mathfrak{M}(f_{52})$. We denote them by $\mathfrak{M}(f_{46}) \to \mathfrak{M}(f_{56})$, $\mathfrak{M}(f_{61}) \to \mathfrak{M}(f_{52})$, respectively.

For $\mu = 0$, $\mathfrak{M}(f_{64(1)}) = (\nu 2 - 1)((\lambda 2 - \nu)2 - 1)$, then $\mathfrak{M}(f_{64(1)})$ is isomorphic to $\mathfrak{M}(f_{61})$ as the structure of parameter space. Similarly, $\mathfrak{M}(f_{64(2)})$ is isomorphic to $\mathfrak{M}(f_{46})$. We also denote them by $\mathfrak{M}(f_{46}) \leftarrow \mathfrak{M}(f_{64}) \to \mathfrak{M}(f_{61})$. By these results, we can show following relation.

$$\mathfrak{M}(f_{56}) \leftarrow \mathfrak{M}(f_{46}) \leftarrow \mathfrak{M}(f_{64}) \to \mathfrak{M}(f_{61}) \to \mathfrak{M}(f_{52})$$

We find the systematic moduli for simple $K3$ singularities.

5 Observation This trial is an application of the Gröbner basis and is the good teaching materials which can join calculations and theories together when we teach the theory of the polynomial ideal in the algebra. Till now, we had shown various examples of the calculations to students and taught these calculations.

When students learn those calculations as algorithm, an interesting subject is preferable, mathematically. As one of such teaching materials, the definition equation of singularity with parameters is suitable. The singularity on the algebra curve can arouse the interest of the students. This trial is significant to show utility of mathematics by using computer and to clarify the pleasure of the calculations to students. Then, the calculation based on Gröbner basis theory is necessary. If they calculate thoughtlessly, they will not obtain significant results.

The mathematical result to show in this paper is still an experiment stage on the way. We showed this trial as one way of the experiment material in the university mathematics.

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