EQUIVALENTS TO EKELAND'S VARIATIONAL PRINCIPLE IN LOCALLY COMPLETE SPACES.

C. Bosch[1], A. Garcia[2], C. Gómez-Wulschner[1] and S. Hernández-Linares[3]

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Abstract. In this paper we prove an extension of Ekeland's variational principle in the setting of locally complete spaces. We also present an Equilibrium version of the Ekeland-type variational principle, a Caristi-Kirk type fixed point theorem for multivalued maps and a Takahashi minimization theorem, we then prove that they are equivalent.

1 Introduction

The fundamental idea of the variational principle due to I. Ekeland [4] is to assign to a minimization problem a slightly perturbed problem having a solution which is at the same time an approximate solution to the original problem. This localization property is very useful and explains the importance of the result. It is one of the most important tools to solve problems in optimization, optimal control, game theory, nonlinear equations, dynamical systems, etc. [1,4,6,7]. Since the discovery of the Ekeland's variational principle there have also appeared many extensions or equivalent formulations of the principle [2,3,5,6,7,10,11,12]. In [2] C.Bosch, A.Garcia, C.L.Garcia, established the variational principle in the setting of locally complete spaces with the perturbations introduced by J.Qiu [11]. In this paper we will improve the previous result by showing that the second inequality in the main theorem can be strict. We introduce also, in the setting of locally complete spaces, the equilibrium version of the Ekeland-type variational principle, a Caristi-Kirk type fixed point theorem for multivalued maps and a Takahashi type minimization theorem, then we prove that they are all equivalent to our Ekeland-type variational principle. The results of this paper extend and generalize many results appearing recently in the literature and are of a different kind as was noted by S.Al-Homidan, Q.H. Ansari, J.-C. Yao [7] in remark 3.3.

2 Preliminaries

Throughout this paper (E, τ) will denote a Hausdorff locally convex space (briefly locally convex space) with topology τ, generated by a family of seminorms $\{\rho_\alpha : \alpha \in \Lambda\}$ with Λ a set of indices. A disk B in E is a closed, bounded and absolutely convex set. We denote by (E_B, ρ_B) the linear span of B endowed with the topology defined by the Minkowski functional associated with B. When B is bounded ρ_B is a norm, and the norm topology is finer than the topology inherited from E. If (E_B, ρ_B) is a Banach space we say that B is a Banach disk. We say that E is a locally complete locally convex space (briefly locally complete space) if each closed, bounded disk is a Banach disk. There are many examples of locally complete spaces, in fact every sequential complete space is locally complete. Typical examples of locally complete spaces arise in the following way. Let $(E, \|.\|)$ be a Banach space and $\sigma(E, E')$ be the weak topology in E then $(E, \sigma(E, E'))$ is a locally complete space which is not sequentially complete. For metrizable locally convex spaces these concepts are equivalent. The class Φ of perturbations we will use is defined as the family of functions

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$\varphi : [0, \infty) \to [0, \infty)$ which are subadditive, strictly increasing, continuous, $\varphi(0) = 0$, and $\lim_{x\to\infty} \varphi(x) = \infty$. Clearly the inverse of φ exists and is superadditive, strictly increasing and continuous, $\varphi^{-1}(0) = 0$. Here φ is said to be subadditive if $\varphi(s + t) \leq \varphi(s) + \varphi(t)$, for every $s, t \in [0, \infty)$, and φ^{-1} is said to be superadditive if $\varphi^{-1}(s + t) \geq \varphi^{-1}(s) + \varphi^{-1}(t)$, for every $s, t \in [0, \infty)$. Functions like $\varphi(t) = t$, $\varphi(t) = \sqrt[n]{t}$, $\varphi(t) = \ln(1 + t)$, are examples of elements in Φ.

3 Ekeland-Type variational principle

In this section we present a generalization of Ekeland-Type variational principle for locally complete spaces. This theorem is more precise than the one in [2] since here we will have in (b) a strict inequality, the proof is similar to that of the cited theorem but for completeness we will write down the details here.

Theorem 1 *Let* (E,τ) *be a locally complete space and* $f : E \to \mathbb{R} \cup \{\infty\}$ *be a proper, lower semicontinuous and bounded below function. Let* φ *be in* Φ *and* x_0 *be a point in* $Dom(f)$*, that is* $f(x_0) < \infty$*. Then for any Banach disk* B *in* E *such that* $x_0 \in E_B$ *there exists* $x^* \in E_B$ *such that :*

(a) $f(x^*) + \varphi(\rho_B(x^* - x_0)) \leq f(x_0)$ and

(b) $f(x^*) < f(x) + \varphi(\rho_B(x^* - x))$ *for all* $x \in E \setminus \{x^*\}$.

Proof Let B be a Banach disk in E such that $x_0 \in (E_B, \rho_B)$ and let

$$S(x_0) = \{x \in E_B : f(x) + \varphi(\rho_B(x - x_0)) \leq f(x_0)\}.$$

Observe that $S(x_0)$ is nonempty and ρ_B-closed in E_B and that if x_1 is in $S(x_0)$ then $S(x_1) \subset S(x_0)$, since for $x_2 \in S(x_1)$, we have

$$\begin{aligned}\varphi(\rho_B(x_2 - x_0) &\leq \varphi(\rho_B(x_2 - x_1) + \varphi(\rho_B(x_1 - x_0) \\ &\leq f(x_1) - f(x_2) + f(x_0) - f(x_1) = f(x_0) - f(x_2)\end{aligned}$$

Let $g : E_B \to \mathbb{R} \cup \{\infty\}$ be the function defined by,

$$g(x) = \begin{cases} f(x), & S(x_0) \\ \infty, & E_B \setminus S(x_0). \end{cases}$$

Note that g is both ρ_B-lower semicontinuous and bounded below.

Now, starting from any x_1 in $S(x_0)$ construct, inductively, x_k in $S(x_{k-1})$ such that,

$$g(x_k) \leq \inf\{g(x) : x \in S(x_{k-1})\} + \frac{1}{k}$$

since $x_k \in S(x_{k-1})$, we obtain from the definition of $S(x_{k-1})$,

$$g(x_k) + \varphi(\rho_B(x_k - x_{k-1})) \leq g(x_{k-1})$$

then $0 \leq \varphi(\rho_B(x_k - x_{k-1})) \leq g(x_{k-1}) - g(x_k)$.

From which we have that the bounded below sequence $(g(x_k))$ is decreasing, that is,

$$g(x_k) \leq g(x_{k-1}).$$

So $g(x_k) \downarrow r$. Now let us prove that the sequence (x_k) is contained in $S(x_0)$ and is ρ_B-Cauchy. By using the triangle inequality and the subadditivity of φ we have that

$$\begin{aligned} \varphi(\rho_B(x_k - x_l)) &\leq \varphi(\rho_B(x_k - x_{k+1}) + \rho_B(x_{k+1} - x_{k+2}) + ... + \rho_B(x_{l-1} - x_l)) \\ &\leq \varphi(\rho_B(x_k - x_{k+1})) + \varphi(\rho_B(x_{k+1} - x_{k+2})) + ... + \\ &\quad +... + \varphi(\rho_B(x_{l-1} - x_l)) \\ &\leq g(x_k) - g(x_{k+1}) + g(x_{k+1}) - g(x_{k+2}) + ... + g(x_{l-1}) - g(x_l) \\ &= g(x_k) - g(x_l) < \delta \text{ if } \quad l \geq k \geq N(\delta) \text{ for some } N(\delta). \end{aligned}$$

Since φ^{-1} is continuous then for every $\epsilon > 0$ there is a $\delta > 0$ and therefore $N(\delta) \in \mathbb{N}$, such that if $l \geq k \geq N(\delta)$ then $g(x_k) - g(x_l) < \delta$ and $\varphi^{-1}(g(x_k) - g(x_l)) < \epsilon$. So $\rho_B(x_k - x_l) < \varphi^{-1}(g(x_k) - g(x_l)) < \epsilon$ means that (x_k) is ρ_B-Cauchy. So E Locally complete implies that there is $x^* \in E_B$ such that (x_k) is ρ_B-convergent to x^* . Note that since, x_1 is in $S(x_0)$, $S(x_1) \subset S(x_0)$ and in an analogous way $S(x_{k+1}) \subset S(x_k) \subset \cdot\cdot \subset S(x_0)$. Now (x_k) is ρ_B-Cauchy in $S(x_0)$ which is ρ_B-closed and therefore $x_k \to x^*$and $x^* \in S(x_0)$. Then we have

$$g(x^*) + \varphi(\rho_B(x^* - x_0)) \leq g(x_0).$$

Now we will prove that $S(x^*) = \{x^*\}$. Suppose that $x \in S(x^*)$, then $x \in E_B$ and $g(x) + \varphi(\rho_B(x - x^*)) \leq g(x^*)$ so $g(x) \leq g(x^*)$. Furthermore using again the sequence (x_n) we have $g(x^*) \leq g(x_n) \leq g(x) + \frac{1}{n}$ for every n in $\mathbb{N}$, and since g is lower semicontinuous and $g(x_k) \downarrow r$ we get $g(x^*) \leq r \leq g(x) \leq g(x^*)$, then $g(x^*) = r = g(x)$. We have that $x \in S(x^*) \subset S(x_n)$ so

$$0 \leq \varphi(\rho_B(x - x_n)) \leq g(x_n) - g(x) = g(x_n) - r \to 0.$$

And since $x^* \in S(x^*) \subset S(x_n)$, we have

$$0 \leq \varphi(\rho_B(x^* - x_n)) \leq g(x_n) - g(x^*) = g(x_n) - r \to 0.$$

By the continuity of φ^{-1} , and the fact that $\varphi(0) = 0$, we have that

$$\begin{aligned} \rho_B(x - x_n) &\to 0, \\ \rho_B(x^* - x_n) &\to 0. \end{aligned}$$

We can then conclude that $x^* = x$ then $S(x^*) = \{x^*\}$ and if $x \in E_B$, $x \neq x^*$we get inequality (b) in the theorem. If $x \in E \setminus E_B$, it is clear that the inequality holds, since $\rho_B(x) = \infty$.

Remark 1 The main result in Bosch, Garcia, Garcia [2], is slightly different from the theorem in this section and this one is not comparable to Corollary 3.1 in Al-Homidan, Ansari, Yao [7] as themselves pointed it out.

4 Some equivalences

Now we will give the equivalences between a Caristi-Kirk type fixed point theorem, Takahashi type minimization theorem, an equilibrium version of Ekeland-type Variational Principle and Theorem 1 for Qiu's perturbations in the setting of locally complete spaces.

Theorem 2 *Let (E,τ) be a locally complete space, $f : E \to \mathbb{R}$ be a lower semicontinuous bounded below function and let φ be in Φ. Then the following statements are equivalent to Theorem 1:*

*(i) (**Caristi-Kirk type fixed point theorem**). Let 2^E be the set of all subsets of E and $T : E \to 2^E$, be a multivalued map with nonempty values. If there exists a Banach disk B in E such that for all $x \in E_B$ and $y \in Tx$ we have that*

$$\varphi(\rho_B(x-y)) \leq f(x) - f(y) \qquad (1)$$

holds, then T has a stationary point in E_B, that is, there exists $x^* \in E_B$ such that $\{x^*\} = Tx^*$.

*(ii) (**Takahashi type minimization theorem**). Assume that for each $x' \in E$ with $\inf_{z\in E} f(z) < f(x')$ there exists a Banach disk B in E such that $x' \in E_B$ and there exists $x \in E_B - \{x'\}$ such that $\varphi(\rho_B(x'-x)) \leq f(x') - f(x)$. Then there exists $x^* \in E_B$ such that $f(x^*) = \inf_{y\in E_B} f(y)$*

*(iii)(**Equilibrium version of Ekeland-type Variational Principle**). Let $F : E \times E \to \mathbb{R}$ such that*

1. *For all $x, y, z \in E$, $F(x,z) \leq F(x,y) + F(y,z)$.*
2. *For each fixed $x \in E$, $F(x,\cdot) : E \to \mathbb{R}$ is lower semi continuous.*
3. *There exists $x' \in E$ such that $\inf_{x\in E} F(x',x) > -\infty$.*

Then there exists a Banach disk B in E, $x' \in E_B$ and $x^* \in E_B$ such that

(a) $F(x',x^*) + \varphi(\rho_B(x'-x^*)) \leq 0$

(b) $F(x^*,x) + \varphi(\rho_B(x^*-x)) > 0$, for all $x \in E_B - \{x^*\}$.

Proof We will prove $Theorem\ 1 \Longrightarrow (i) \Longrightarrow (ii) \Rightarrow (iii) \Longrightarrow Theorem\ 1$.

From Theorem 1 part (b) there exists $x^* \in E_B$ such that

$$f(x^*) - f(x) < \varphi(\rho_B(x^*-x)) \quad \text{for all} \quad x \in E_B - \{x^*\}.$$

We claim that $\{x^*\} = Tx^*$. Otherwise if $y \in Tx^* - \{x^*\}$ from $(i-1)$ we have

$$\varphi(\rho_B(x^*-y)) \leq f(x^*) - f(y)$$

which contradicts the previous inequality. Note that from inequality (1), for all $x \in E_B$ and $y \in Tx$, we must have $\rho_B(x-y) < \infty$, this means $y \in E_B$, i.e. $Tx \subset E_B$, for all $x \in E_B$.

$(i) \Longrightarrow (ii)$. Define $T : E \to 2^E$ as

$$T(x) = \begin{cases} \{y \in E_B : \varphi(\rho_B(x-y)) \leq f(x) - f(y)\} & if \ \ x \in E_B. \\ E & if \ \ x \in E - E_B. \end{cases}$$

Note that T satisfies inequality (1). Then by (i) there exists $x^* \in E_B$ such that $\{x^*\} = Tx^*$. Now by assumption for each $x' \in E_B$ there exists $x \in E_B - \{x'\}$, we have $x \in Tx'$ and then $Tx' - \{x'\} \neq \varnothing$ whenever $\inf_{z\in E} f(z) < f(x')$ hence we must have $\inf_{z\in E_B} f(z) = f(x^*)$.

$(ii) \Longrightarrow (iii)$ Define a function $f : E \to \mathbb{R}$ by $f(x) = F(x',x)$ for all $x \in E$ where x' is the element given in condition $(iii-3)$. Then we have $\inf_{x\in E} F(x',x) > -\infty$ which means that f is bounded below and by $(iii-2)$, f is proper lower semicontinuous. Let $B \subset E$ be a Banach disk such that $x' \in E_B$. Now suppose that in (iii) (b) does not hold. Then for all $x \in E_B$ there exists $y \in E_B - \{x\}$ and $F(x,y) + \varphi(\rho_B(x-y)) \leq 0$. By condition $(iii-1)$, we have $F(x',y) - F(x',x) \leq F(x,y)$ so using this and the previous inequality :

$$F(x',y) - F(x',x) + \varphi(\rho_B(x-y)) \leq F(x,y) + \varphi(\rho_B(x-y)) \leq 0 \qquad (2)$$

that is, for all $x \in E_B$ there exists $y \in E_B - \{x\}$ and

$$f(y) - f(x) + \varphi(\rho_B(x-y)) \leq 0, \text{ or equivalently } \varphi(\rho_B(x-y)) \leq f(x) - f(y).$$

Then by (ii), there exists $x^* \in E_B$ such that $f(x^*) \leq f(z)$ for all $z \in E_B$. By substituting x by x^* in inequality (2), we obtain that there exists $y \in E_B - \{x^*\}$ and $\varphi(\rho_B(x^* - y)) \leq f(x^*) - f(y)$. Now since ρ_B is a norm in E_B, $\rho_B(x^* - y) > 0$, and then $f(y) < f(x^*)$ which is a contradiction.

Finally let us prove that $(iii) \Rightarrow Theorem$ 1. Define $F : E \times E \to \mathbb{R}$ as $F(x, y) = f(y) - f(x)$ for all $x, y \in E$, with $x' \in dom(f)$.Then by hypothesis, F satisfies all the conditions of (iii).Then (iii) implies the existence of $x^* \in E_B$ such that (a) and (b) hold.

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[1]Instituto Tecnológico Autónomo de México : bosch@itam.mx, claudiag@itam.mx

[2] Universidad Autónoma de San Luis Potosí : agarciamartinez@lycos.com

[3]Universidad Autónoma Metropolitana, Cuajimalpa : sergioheranaya@luam.cuj.mx