TOTALLY η-UMBILIC HYPERSURFACES IN A NONFLAT COMPLEX SPACE FORM AND THEIR ALMOST CONTACT METRIC STRUCTURES

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Abstract. We bridge between submanifold theory and contact geometry. We give a geometric meaning of some fundamental notions such as Sasakian, nearly Sasakian, K-contact from the viewpoint of the result in [3] (Theorem 1). Moreover, motivated by the notion of totally η-umbilic hypersurfaces in a nonflat complex space form, we give a new notion in contact geometry (Theorem 2).

1. Introduction

We denote by $\widetilde{M}_n(c)$ a complex n-dimensional connected and simply connected Kähler manifold of constant holomorphic sectional curvature $c(\neq 0)$, namely it is holomorphically isometric to either an n-dimensional complex projective space $\mathbb{C}P^n(c)$ of constant holomorphic sectional curvature c or an n-dimensional complex hyperbolic space of constant holomorphic sectional curvature c according as c is positive or negative. $\widetilde{M}_n(c)$ is so-called a *nonflat complex space form* of constant holomorphic sectional curvature c.

In this paper we consider real hypersurfaces M^{2n-1} of $\widetilde{M}_n(c)$ furnished with the standard Kähler structure J and Riemannian metric g through an isometric immersion. In the following, we recall some fundamental notions in contact geometry such as Sasakian, nearly Sasakian, K-contact for the real hypersurface M. We take and fix a unit normal vector $\mathcal{N}$ on M. On M it is well-known that an almost contact metric structure (ϕ, ξ, η, g) associated with $\mathcal{N}$ is canonically induced from the structure (J, g) of the ambient space $\widetilde{M}_n(c)$, which is defined by

$$g(\phi X, Y) = g(JX, Y), \xi = -J\mathcal{N} \quad \text{and} \quad \eta(X) = g(\xi, X) = g(JX, \mathcal{N}).$$

On the other hand, $(\phi, -\xi, -\eta, g)$ is clearly also an almost contact metric structure by taking a unit normal $-\mathcal{N}$ on M. Hence it is natural to say that a real hypersurface M is Sasakian if one of these induced structures is a Sasakian structure. That is, if we fix a unit normal $\mathcal{N}$ of M, this real hypersurface is a *Sasakian manifold* if and only if the structure tensor ϕ of M satisfies either the equation $(\nabla_X\phi)Y = g(X, Y)\xi - \eta(Y)X$ for all vectors $X, Y \in TM$ or $(\nabla_X\phi)Y = -g(X, Y)\xi + \eta(Y)X$ for all vectors $X, Y \in TM$. A Sasakian manifold M is called a *Sasakian space form* if every ϕ-sectional curvature $K(u, \phi u) := g(R(u, \phi u)\phi u, u)$ associated to a unit vector $u(\in TM)$ orthogonal to ξ does not depend on the choice of u, where R is the curvature tensor of M. We next review the notion of nearly Sasakian for a real hypersurface M in $\widetilde{M}_n(c)$. M is called a *nearly Sasakian manifold* if M satisfies either the equation $(\nabla_X\phi)Y + (\nabla_Y\phi)X = 2g(X, Y)\xi - \eta(X)Y - \eta(Y)X$ for all $X, Y \in TM$ or

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$(\nabla_X \phi)Y + (\nabla_Y \phi)X = -2g(X,Y)\xi + \eta(X)Y + \eta(Y)X$ for all $X, Y \in TM$. A real hypersurface M of $\widetilde{M}$ is called a *contact manifold* if the exterior differentiation $d\eta$ of the contact form η on M satisfies either $d\eta(X,Y) = g(X, \phi Y)$ for all $X, Y \in TM$ or $d\eta(X,Y) = -g(X, \phi Y)$ for all $X, Y \in TM$, where $d\eta$ is defined by $d\eta(X,Y) = (1/2)\{X(\eta(Y)) - Y(\eta(X)) - \eta([X,Y])\}$. When M is contact and $L_\xi g = 0$, M is called a *K-contact manifold*. Here, L is the Lie derivative on M.

In contact geometry, it is known that Sasakian always implies nearly Sasakian and K-contact. But, in general the converses do not hold (cf. [2]). However, in the theory of real hypersurfaces M in $\widetilde{M}_n(c)$ we emphasize that these notions are equivalent (see Theorem 1). Moreover, in Theorem 1 we give a geometric characterization of such notions by observing some geodesics on M.

In order to describe Theorem 2, we present a new notion. A real hypersurface M is called an *α-nearly Sasakian manifold* if M satisfies either the equation $(\nabla_X \phi)Y + (\nabla_Y \phi)X = \alpha(2g(X,Y)\xi - \eta(X)Y - \eta(Y)X)$ for all $X, Y \in TM$ or $(\nabla_X \phi)Y + (\nabla_Y \phi)X = \alpha(-2g(X,Y)\xi + \eta(X)Y + \eta(Y)X)$ for all $X, Y \in TM$, where α is a positive constant. Using this notion, we obtain a characterization of all totally η-umbilic hypersurfaces in $\widetilde{M}_n(c)$ which are the simplest examples in the theory of real hypersurfaces (see Theorem 2).

2. Fundamental theory of real hypersurfaces in $\widetilde{M}_n(c)$

Let M^{2n-1} be a real hypersurface with a unit normal local vector field $\mathcal{N}$ of an n-dimensional nonflat complex space form $\widetilde{M}_n(c)$ with the standard Riemannian metric g and the canonical Kähler structure J. The Riemannian connections $\widetilde{\nabla}$ of $\widetilde{M}_n(c)$ and ∇ of M are related by the following formulas of Gauss and Weingarten:

$$\widetilde{\nabla}_X Y = \nabla_X Y + g(AX, Y)\mathcal{N}, \tag{2.1}$$

$$\widetilde{\nabla}_X \mathcal{N} = -AX \tag{2.2}$$

for arbitrary vector fields X and Y on M, where g is the Riemannian metric of M induced from the ambient space $\widetilde{M}_n(c)$ and A is the shape operator of M in $\widetilde{M}_n(c)$. An eigenvector of the shape operator A is called a *principal curvature vector* of M in $\widetilde{M}_n(c)$ and an eigenvalue of A is called a *principal curvature* of M in $\widetilde{M}_n(c)$. We set $V_\lambda = \{v \in TM \mid Av = \lambda v\}$ which is called the principal foliation associated to the principal curvature λ.

M has an almost contact metric structure (ϕ, ξ, η, g) (see Introduction). It follows from (2.1), (2.2) and $\widetilde{\nabla} J = 0$ that

$$(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi, \tag{2.3}$$

$$\nabla_X \xi = \phi AX. \tag{2.4}$$

Denoting the curvature tensor of M by R, we have the equation of Gauss given by

$$\begin{aligned} &g(R(X,Y)Z,W) \\ &\quad = (c/4)\{g(Y,Z)g(X,W) - g(X,Z)g(Y,W) + g(\phi Y, Z)g(\phi X, W) \\ &\qquad\quad - g(\phi X, Z)g(\phi Y, W) - 2g(\phi X, Y)g(\phi Z, W)\} \\ &\qquad + g(AY,Z)g(AX,W) - g(AX,Z)g(AY,W). \end{aligned} \tag{2.5}$$

We usually call M a *Hopf hypersurface* if the characteristic vector ξ of M is a principal curvature vector at each point of M. We remark that the principal curvature δ associated with ξ is automatically constant on M in local.

Furthermore, every tube of sufficiently small constant radius around each Kähler submanifold of a nonflat complex space form $\widetilde{M}_n(c)$ is a Hopf hypersurface. This fact means that the notion of Hopf hypersurfaces is natural in the theory of real hypersurfaces in a nonflat complex space form.

In $\mathbb{C}P^n(c)$ ($n \geqq 2$), a Hopf hypersurface all of whose principal curvatures are constant is locally congruent to one of the following (cf. [4]):

(A_1) A geodesic sphere of radius r, where $0 < r < \pi/\sqrt{c}$;
(A_2) A tube of radius r around totally geodesic $\mathbb{C}P^\ell(c)$ ($1 \leqq \ell \leqq n-2$), where $0 < r < \pi/\sqrt{c}$;
(B) A tube of radius r around complex hyperquadric $\mathbb{C}Q^{n-1}$, where $0 < r < \pi/(2\sqrt{c})$;
(C) A tube of radius r around $\mathbb{C}P^1(c) \times \mathbb{C}P^{(n-1)/2}(c)$, where $0 < r < \pi/(2\sqrt{c})$ and n ($\geqq 5$) is odd;
(D) A tube of radius r around complex Grassmann $\mathbb{C}G_{2,5}$, where $0 < r < \pi/(2\sqrt{c})$ and $n = 9$;
(E) A tube of radius r around Hermitian symmetric space $SO(10)/U(5)$, where $0 < r < \pi/(2\sqrt{c})$ and $n = 15$.

These real hypersurfaces are said to be of types (A_1), (A_2), (B), (C), (D) and (E). Summing up real hypersurfaces of types (A_1) and (A_2), we call them hypersurfaces of type (A). The numbers of distinct principal curvatures of these real hypersurfaces are $2, 3, 3, 5, 5, 5$, respectively. The principal curvatures of these real hypersurfaces in $\mathbb{C}P^n(c)$ are given as follows:

	(A_1)	(A_2)	(B)	(C, D, E)
λ_1	$\frac{\sqrt{c}}{2}\cot(\frac{\sqrt{c}}{2}r)$	$\frac{\sqrt{c}}{2}\cot(\frac{\sqrt{c}}{2}r)$	$\frac{\sqrt{c}}{2}\cot\big(\frac{\sqrt{c}}{2}r - \frac{\pi}{4}\big)$	$\frac{\sqrt{c}}{2}\cot\big(\frac{\sqrt{c}}{2}r - \frac{\pi}{4}\big)$
λ_2	—	$-\frac{\sqrt{c}}{2}\tan(\frac{\sqrt{c}}{2}r)$	$\frac{\sqrt{c}}{2}\cot\big(\frac{\sqrt{c}}{2}r + \frac{\pi}{4}\big)$	$\frac{\sqrt{c}}{2}\cot\big(\frac{\sqrt{c}}{2}r + \frac{\pi}{4}\big)$
λ_3	—	—	—	$\frac{\sqrt{c}}{2}\cot(\frac{\sqrt{c}}{2}r)$
λ_4	—	—	—	$-\frac{\sqrt{c}}{2}\tan(\frac{\sqrt{c}}{2}r)$
δ	$\sqrt{c}\cot(\sqrt{c}\,r)$	$\sqrt{c}\cot(\sqrt{c}\,r)$	$\sqrt{c}\cot(\sqrt{c}\,r)$	$\sqrt{c}\cot(\sqrt{c}\,r)$

One should notice that in $\mathbb{C}P^n(c)$ a tube of radius r ($0 < r < \pi/\sqrt{c}$) around totally geodesic $\mathbb{C}P^\ell(c)$ ($0 \leqq \ell \leqq n-1$) is congruent to a tube of radius $\big((\pi/\sqrt{c}) - r\big)$ around totally geodesic $\mathbb{C}P^{n-\ell-1}(c)$.

In $\mathbb{C}H^n(c)$ ($n \geqq 2$), a Hopf hypersurface all of whose principal curvatures are constant is locally congruent to one of the following (cf. [4]):

(A_0) A horosphere in $\mathbb{C}H^n(c)$;
($A_{1,0}$) A geodesic sphere of radius r ($0 < r < \infty$);
($A_{1,1}$) A tube of radius r around totally geodesic $\mathbb{C}H^{n-1}(c)$, where $0 < r < \infty$;
(A_2) A tube of radius r around totally geodesic $\mathbb{C}H^\ell(c)$ ($1 \leqq \ell \leqq n-2$), where $0 < r < \infty$;
(B) A tube of radius r around totally real totally geodesic $\mathbb{R}H^n(c/4)$, where $0 < r < \infty$.

These real hypersurfaces are said to be of types (A_0), (A_1), (A_1), (A_2) and (B). Here, type (A_1) means either type ($A_{1,0}$) or type ($A_{1,1}$). Summing up real hypersurfaces of types (A_0), (A_1) and (A_2), we call them hypersurfaces of type (A). A real hypersurface of type (B) with radius $r = (1/\sqrt{|c|})\log_e(2+\sqrt{3})$ has two distinct constant principal curvatures $\lambda_1 = \delta = \sqrt{3|c|}/2$ and $\lambda_2 = \sqrt{c}/(2\sqrt{3}\,)$. Except for this real hypersurface, the numbers of distinct principal curvatures of Hopf hypersurfaces with constant principal curvatures are $2, 2, 2, 3, 3$, respectively. The principal curvatures of these real hypersurfaces in $\mathbb{C}H^n(c)$ are given as follows:

	(A_0)	$(A_{1,0})$	$(A_{1,1})$	(A_2)	(B)
λ_1	$\frac{\sqrt{\lvert c\rvert}}{2}$	$\frac{\sqrt{\lvert c\rvert}}{2}\coth(\frac{\sqrt{\lvert c\rvert}}{2}r)$	$\frac{\sqrt{\lvert c\rvert}}{2}\tanh(\frac{\sqrt{\lvert c\rvert}}{2}r)$	$\frac{\sqrt{\lvert c\rvert}}{2}\coth(\frac{\sqrt{\lvert c\rvert}}{2}r)$	$\frac{\sqrt{\lvert c\rvert}}{2}\coth(\frac{\sqrt{\lvert c\rvert}}{2}r)$
λ_2	—	—	—	$\frac{\sqrt{\lvert c\rvert}}{2}\tanh(\frac{\sqrt{\lvert c\rvert}}{2}r)$	$\frac{\sqrt{\lvert c\rvert}}{2}\tanh(\frac{\sqrt{\lvert c\rvert}}{2}r)$
δ	$\sqrt{\lvert c\rvert}$	$\sqrt{\lvert c\rvert}\coth(\sqrt{\lvert c\rvert}r)$	$\sqrt{\lvert c\rvert}\coth(\sqrt{\lvert c\rvert}r)$	$\sqrt{\lvert c\rvert}\coth(\sqrt{\lvert c\rvert}r)$	$\sqrt{\lvert c\rvert}\tanh(\sqrt{\lvert c\rvert}r)$

In this paper, real hypersurfaces of types (A), (B), (C), (D) and (E) in $\widetilde{M}_n(c)$ are said to be *standard real hypersurfaces*. It is well-known that every standard real hypersurface M is a homogeneous real hypersurface of $\widetilde{M}_n(c)$, namely M is an orbit of some subgroup of the full isometry group $\mathrm{I}(\widetilde{M}_n(c))$ of $\widetilde{M}_n(c)$.

It is well-known that our ambient manifold $\widetilde{M}_n(c)$ admits no totally umbilic real hypersurfaces. In this context, we recall the notion of totally η-umbilic. A real hypersurface M of a nonflat complex space form $\widetilde{M}_n(c)$, $n \geqq 2$ is called *totally η-umbilic* if its shape operator A is of the form $A = \alpha I + \beta\eta\otimes\xi$ for some smooth functions α and β on M. This definition is equivalent to saying that $Au = \alpha u$ for each vector u on M which is orthogonal to the characteristic vector ξ of M, where α is a smooth function on M. It is known that every totally η-umbilic hypersurface is a member of Hopf hypersurfaces with two distinct constant principal curvatures α and $\alpha + \beta$.

A totally η-umbilic hypersurface M^{2n-1}, $n \geqq 2$ with shape operator $A = \alpha I + \beta\eta\otimes\xi$ of a nonflat complex space form $\widetilde{M}_n(c)$ is locally congruent to one of the following:

(P) A geodesic sphere of radius r $(0 < r < \pi/\sqrt{c})$ in $\mathbb{C}P^n(c)$, where
$\alpha = (\sqrt{c}/2)\cot(\sqrt{c}\,r/2)$ and $\beta = -(\sqrt{c}/2)\tan(\sqrt{c}\,r/2)$;

($\mathrm{H_i}$) A horosphere in $\mathbb{C}H^n(c)$, where $\alpha = \beta = \sqrt{|c|}/2$;

($\mathrm{H_{ii}}$) A geodesic sphere of radius r $(0 < r < \infty)$ in $\mathbb{C}H^n(c)$, where
$\alpha = (\sqrt{|c|}/2)\coth(\sqrt{|c|}\,r/2)$ and $\beta = (\sqrt{|c|}/2)\tanh(\sqrt{|c|}\,r/2)$;

($\mathrm{H_{iii}}$) A tube of radius r $(0 < r < \infty)$ around totally geodesic complex hyperplane $\mathbb{C}H^{n-1}(c)$ in $\mathbb{C}H^n(c)$, where
$\alpha = (\sqrt{|c|}/2)\tanh(\sqrt{|c|}\,r/2)$ and $\beta = (\sqrt{|c|}/2)\coth(\sqrt{|c|}\,r/2)$.

Totally η-umbilic hypersurfaces are interesting examples of Riemannian manifolds. The length spectrum of such a hypersurface was studied in detail (see [1]). Moreover, it is well-known that every geodesic sphere $G(r)$ of radius r $(0 < r < \pi/\sqrt{c}\,)$ with $\tan^2(\sqrt{c}\,r/2) > 2$ in $\mathbb{C}P^n(c)$ is a Berger sphere ([5]).

For later use we prepare the following two lemmas ([4]).

Lemma 1. *For a real hypersurface M in a nonflat complex space form $\widetilde{M}_n(c)$ $(n \geqq 2)$, the following conditions are mutually equivalent.*

1. *M is of type (A).*
2. *$\phi A = A\phi$ holds on M.*
3. *$g((\nabla_X A)Y, Z) = (c/4)(-\eta(Y)g(\phi X, Z) - \eta(Z)g(\phi X, Y))$ for all X, Y and $Z \in TM$.*

Lemma 2. *A real hypersurface M in a nonflat complex space form $\widetilde{M}_n(c)$ $(n \geqq 2)$ is of either type (A_0), type (A_1) or type (B) if and only if M satisfies $\phi A + A\phi = k\phi$, where k is a nonzero constant.*

At the end of this section we review the definition of circles in Riemannian geometry. A real smooth curve $\gamma = \gamma(s)$ parametrized by its arclength s in a Riemannian manifold M

with Riemannian connection ∇ is called a *circle* of curvature k if it satisfies the ordinary differential equations $\nabla_{\dot\gamma}\dot\gamma = kY_s$ and $\nabla_{\dot\gamma}Y_s = -k\dot\gamma$, where k is a nonnegative constant and Y_s is the unit normal vector of γ. A circle of null curvature is nothing but a geodesic. The definition of circles is equivalent to the equation

$$\nabla_{\dot\gamma}(\nabla_{\dot\gamma}\dot\gamma) + g(\nabla_{\dot\gamma}\dot\gamma, \nabla_{\dot\gamma}\dot\gamma)\dot\gamma = 0. \tag{2.6}$$

3. Statements of results

Theorem 1. *For a real hyperaurface M in a nonflat complex space form $\widetilde{M}_n(c)$ $(n \geqq 2)$, the following six conditions are mutually equivalent.*

1. *M is a Sasakian space form of constant ϕ-sectional curvature. Here, it has automatically constant ϕ-sectional curvature $c+1$.*
2. *M is a Sasakian manifold.*
3. *M is a K-contact manifold.*
4. *M is a nearly Sasakian manifold.*
5. *M is locally congruent to one of the following totally η-umbilic hypersurfaces in the ambient space $\widetilde{M}_n(c)$:*
   - 5i) *A geodesic sphere $G(r)$ of radius r with $\tan(\sqrt{c}\,r/2) = \sqrt{c}/2$ $(0 < r < \pi/\sqrt{c}\,)$ in $\mathbb{C}P^n(c)$;*
   - 5ii) *A horosphere in $\mathbb{C}H^n(-4)$;*
   - 5iii) *A geodesic sphere $G(r)$ of radius r with $\tanh(\sqrt{|c|}\,r/2) = \sqrt{|c|}/2$ $(0 < r < \infty)$ in $\mathbb{C}H^n(c)$ $(-4 < c < 0)$;*
   - 5iv) *A tube of radius r around totally gedesic $\mathbb{C}H^{n-1}(c)$ with $\tanh(\sqrt{|c|}\,r/2) = 2/\sqrt{|c|}$ $(0 < r < \infty)$ in $\mathbb{C}H^n(c)$ $(c < -4)$.*

   In these cases, the shape operator A of M is of the form $A = I - (c/4)\eta\otimes\xi$.
6. *There exist orthonormal vectors $v_1, v_2, \ldots, v_{2n-2}$ orthogonal to ξ at each point p of M satisfying the following two conditions*:
   - 6i) *All geodesics $\gamma = \gamma_i(s)$ $(1 \leqq i \leqq 2n-2)$ on M with $\gamma_i(0) = p$ and $\dot\gamma_i(0) = v_i$ are mapped to circles of the same curvature 1 in the ambient space $\widetilde{M}_n(c)$;*
   - 6ii) *All geodesics $\gamma_{ij} = \gamma_{ij}(s)$ $(1 \leqq i < j \leqq 2n-2)$ on M with $\gamma_{ij}(0) = p$ and $\dot\gamma_{ij}(0) = (v_i+v_j)/\sqrt{2}$ are mapped to circles of the same curvature 1 in the ambient space $\widetilde{M}_n(c)$.*

Proof. We first show that Condition (3) implies Condition (5). It follows from $d\eta(X,Y) = \pm g(X,\phi Y)$ that

$$X(g(\xi,Y)) - Y(g(\xi,X)) - g(\nabla_X Y - \nabla_Y X, \xi) \mp 2g(X,\phi Y) = 0.$$

This, together with (2.4), shows that

$$\begin{aligned} 0 &= g(\phi AX, Y) - g(\phi AY, X) \mp 2g(X,\phi Y) \\ &= g((\phi A + A\phi \pm 2\phi)X, Y). \end{aligned}$$

Hence we can see that

$$\phi A + A\phi = \mp 2\phi. \tag{3.1}$$

So M is one of types $(A_0), (A_1)$ and (B) (see Lemma 2). On the other hand, from $(L_\xi g)(X,Y) = 0$ and (2.4) we have

$$\begin{aligned} 0 &= g(\nabla_X\xi, Y) + g(X, \nabla_Y\xi) = g(\phi AX, Y) + g(X, \phi AY) \\ &= g((\phi A - A\phi)X, Y), \end{aligned}$$

so that

$$\phi A = A\phi. \tag{3.2}$$

Then M is of type (A) (see Lemma 1). Therefore from (3.1) and (3.2) we find that M is one of types (A_0) and (A_1) satisfying $A\phi = \mp\phi$. This implies that M is a totally η-umbilic hypersurface with coefficient $\alpha = \mp 1$. Thus, from the classification theorem of totally η-umbilic hypersurfaces in a nonflat complex space form we see that the shape operator A of M satisfies either $A = -I + (c/4)\eta \otimes \xi$ or $A = I - (c/4)\eta \otimes \xi$. Hence we can obtain Condition (5).

We next show that Condition (4) implies Condition (5). In consideration of $(\nabla_X \phi)Y + (\nabla_Y \phi)X = \pm(2g(X,Y)\xi - \eta(X)Y - \eta(Y)X)$ and (2.4) we find that

$$\eta(X)AY + \eta(Y)AX - 2g(AX,Y)\xi = \pm(2g(X,Y)\xi - \eta(X)Y - \eta(Y)X). \tag{3.3}$$

Setting $X = Y = \xi$ in (3.3), we get $A\xi = g(A\xi,\xi)$, so that ξ is principal. Next, putting X as an arbitrary vector orthogonal to ξ and $Y = \xi$ in (3.3), we know that either $AX = -X$ for all $X(\perp \xi) \in TM$ or $AX = X$ for all $X(\perp \xi) \in TM$. Hence we get Condition (5).

We here compute ϕ-sectional curvatures of each real hypersurface M in the list of Condition (5). It follows from (2.5) and the expression of the shape operator A of M with either $A = -I + (c/4)\eta \otimes \xi$ or $A = I - (c/4)\eta \otimes \xi$ that $K(u,\phi u) = g(R(u,\phi u)\phi u, u) = c + 1$ for each unit vector u perpendicular to ξ.

Therefore we find that each of Conditions (1), (2), (3) and (4) is equivalent to Condition (5).

In the following, we shall verify that Condition (5) is equivalent to Condition (6). We suppose Condition (5). Without loss of generality we assume that $A = I - (c/4)\eta \otimes \xi$. We take a point $p \in M$ and a unit vector $u(\perp \xi_p)$. Let $\gamma = \gamma(s)$ denote a geodesic parametrized by its arclength s on M satisfying the initial condition that $\gamma(0) = p$ and $\dot\gamma(0) = u$. In view of (2.4), Lemma 1, the symmetry of A and the skew-symmetry of ϕ we have

$$\begin{aligned}\nabla_{\dot\gamma}\big(g(\dot\gamma(s),\xi)\big) &= g(\dot\gamma(s), \nabla_{\dot\gamma}\xi) = g(\dot\gamma(s), \phi A\dot\gamma(s)) \\ &= g(\dot\gamma(s), A\phi\dot\gamma(s)) = -g(\phi A\dot\gamma(s), \dot\gamma(s)) = 0,\end{aligned}$$

which, together with $g(\dot\gamma(0),\xi_p) = 0$, yields $g(\dot\gamma(s),\xi) = 0$ for each $s \in (-\infty,\infty)$. That is, we see that $A\dot\gamma(s) = \dot\gamma(s)$ for every s. This, combined with (2.1) and (2.2), shows that $\widetilde\nabla_{\dot\gamma}\dot\gamma = \mathcal{N}$ and $\widetilde\nabla_{\dot\gamma}\mathcal{N} = -\dot\gamma$. Thus we get Condition (6).

We finally suppose Condition (6). We take the orthonormal vectors $v_1, \dots, v_{2n-2}$ orthogonal to ξ_p at an arbitrary fixed point p of M. Then, from Condition 6i) and (2.6) they satisfy

$$\widetilde\nabla_{\dot\gamma_i}\widetilde\nabla_{\dot\gamma_i}\dot\gamma_i = -\dot\gamma_i. \tag{3.4}$$

On the other hand, from (2.1) and (2.2) we have

$$\widetilde\nabla_{\dot\gamma_i}\widetilde\nabla_{\dot\gamma_i}\dot\gamma_i = g((\nabla_{\dot\gamma_i}A)\dot\gamma_i,\dot\gamma_i)\mathcal{N} - g(A\dot\gamma_i,\dot\gamma_i)A\dot\gamma_i. \tag{3.5}$$

Comparing the tangential components of (3.4) and (3.5), we see that

$$g(A\dot\gamma_i,\dot\gamma_i)A\dot\gamma_i = \dot\gamma_i,$$

so that at $s = 0$ we get

$$g(Av_i,v_i)Av_i = v_i \quad \text{for each } i = 1,\dots,2n-2,$$

which yields that

$$Av_i = v_i \quad \text{or} \quad Av_i = -v_i \quad \text{for each } i = 1,\dots,2n-2. \tag{3.6}$$

This implies that ξ is a principal curvature vector, because $\langle A\xi, v_i\rangle = \langle \xi, Av_i\rangle = 0$ for each $i = 1, \dots, 2n-2$. Therefore M is a Hopf hypersurface with at most three distinct constant principal curvatures $1, -1$ and $\delta = g(A\xi, \xi)$ at its each point. On the other hand, applying the same discussion as above to Condition 6ii), we get the following corresponding to Equation (3.6):

$$A((v_i + v_j)/\sqrt{2}\,) = (v_i + v_j)/\sqrt{2} \ \text{ or } \ A((v_i + v_j)/\sqrt{2}\,) = -(v_i + v_j)/\sqrt{2} \tag{3.7}$$

for $1 \leqq i < j \leqq 2n-2$. Thus, from (3.6) and (3.7) we can see that either $Av_i = v_i$ $(1 \leqq i \leqq 2n-2)$ or $Av_i = -v_i$ $(1 \leqq i \leqq 2n-2)$ holds. This implies that our real hypersurface M is totally η-umbilic with coefficient $\alpha = \pm 1$ in the ambient space $\widetilde{M}_n(c)$. We hence get Condition (5). ☐

As an immediate consequence of the proof of Theorem 1, we get the following.

Theorem 2. *For a real hypersurface M in a nonflat complex space form $\widetilde{M}_n(c)$ $(n \geqq 2)$, the following three conditions are mutually equivalent.*

1. *M is an α-nearly Sasakian manifold.*
2. *M is totally η-umbilic in $\widetilde{M}_n(c)$ satisfying $Au = \alpha u$ for all $u(\perp \xi) \in TM$.*
3. *There exist orthonormal vectors $v_1, v_2, \dots, v_{2n-2}$ orthogonal to ξ at each point p of M satisfying the following two conditions:*
   - 3*i*) *All geodesics $\gamma = \gamma_i(s)$ $(1 \leqq i \leqq 2n-2)$ on M with $\gamma_i(0) = p$ and $\dot\gamma_i(0) = v_i$ are mapped to circles of the same curvature α in the ambient space $\widetilde{M}_n(c)$;*
   - 3*ii*) *All geodesics $\gamma_{ij} = \gamma_{ij}(s)$ $(1 \leqq i < j \leqq 2n-2)$ on M with $\gamma_{ij}(0) = p$ and $\dot\gamma_{ij}(0) = (v_i + v_j)/\sqrt{2}$ are mapped to circles of the same curvature α in the ambient space $\widetilde{M}_n(c)$.*

If we remove Conditions 6ii) and 3ii), then Theorems 1 and 2 are no longer true. The following example is worth mentioning.

Example. Let M^{2n-1} be a tube of radius $\pi/(2\sqrt{c}\,)$ around totally geodesic Kähler submanifold $\mathbb{C}P^\ell(c)$ $(1 \leqq \ell \leqq n-2)$ in the ambient space $\mathbb{C}P^n(c), n \geqq 3$. We remark that this hypersurface is of type (A_2). The tangent bundle TM is decomposed as: $TM = \{\xi\}_{\mathbb{R}} \oplus V_{\sqrt{c}/2} \oplus V_{-\sqrt{c}/2}$ with $A\xi = 0, \dim V_{\sqrt{c}/2} = 2n-2\ell-2, \dim V_{-\sqrt{c}/2} = 2\ell, \phi V_{\sqrt{c}/2} = V_{\sqrt{c}/2}$ and $\phi V_{-\sqrt{c}/2} = V_{-\sqrt{c}/2}$. We emphasize that there exist orthonormal vectors $v_1, v_2, \dots, v_{n-2}$ orthonormal to ξ at each point p of M satisfying Condition 3i) in Theorem 2. In fact, if we take orthonormal vectors $v_1, \dots, v_{2\ell}$ and $v_{2\ell+1}, \dots, v_{2n-2}$ in $V_{-\sqrt{c}/2}$ and $V_{\sqrt{c}/2}$, respectively, then all geodesics $\gamma_i = \gamma_i(s)$ $(1 \leqq i \leqq 2n-2)$ on M with $\gamma_i(0) = p$ and $\dot\gamma_i(0) = v_i$ are mapped to circles of the same positive curvature $\sqrt{c}/2$ in $\mathbb{C}P^n(c)$ (for details, see [3]). However these vectors do not satisfy Condition 3ii) in Theorem 2. Indeed, for unit vectors $v_i \in V_{-\sqrt{c}/2}$ and $v_j \in V_{\sqrt{c}/2}$ if we take a geodesic $\gamma_{ij} = \gamma_{ij}(s)$ on M with $\dot\gamma_{ij}(0) = (v_i + v_j)/\sqrt{2}$, then the geodesic γ_{ij} is mapped to a geodesic in the ambient space $\mathbb{C}P^n(c)$ (see Lemma 1).

Needless to say, when $c = 4$, our real hypersurfaces show that Theorem 1 is not true without Condition 6ii).

Finally we present the following remarks.

Remark.
1. Theorems 1 and 2 are local results. If we add the condition that M is complete and simply connected to the hypothesis, then these theorems are global results.
2. In Conditions (6) of Theorem 1 and (3) of Theorem 2 we do need to take vectors $v_1, \dots, v_{2n-2}$ orthogonal to ξ as a smooth local field of orthonormal frames on M.

We only to take such vectors at each point of M in the hypothesis of Theorems 1 and 2.

3. As a matter of course our example is *not* a K-contact manifold. However the proof of Theorem 1 shows that its characteristic vector ξ is a Killing vector field.

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