A NOTE ON f-DERIVATIONS OF SUBTRACTION ALGEBRAS

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Abstract. In this paper, we introduced the concept of f-derivation which is a generalization of derivation in subtraction algebra, and some related properties are investigated.

1. Introduction

B. M. Schein [2] considered systems of the form $(\Phi; \circ, \backslash)$, where Φ is a set of functions closed under the composition "$\circ$" of functions (and hence $(\Phi; \circ)$ is a function semigroup) and the set theoretic subtraction "$\backslash$" (and hence $(\Phi; \backslash)$ is a subtraction algebra in the sense of [1]). He proved that every subtraction semigroup is isomorphic to a difference semigroup of invertible functions. B. Zelinka [4] discussed a problem proposed by B. M. Schein concerning the structure of multiplication in a subtraction semigroup. He solved the problem for subtraction algebras of a special type, called the atomic subtraction algebras. In this paper, we introduced the concept of f-derivation which is a generalization of derivation in subtraction algebra, and some related properties are investigated.

2. Preliminaries.

We first recall some basic concepts which are used to present the paper.

By a *subtraction algebra* we mean an algebra $(X; -)$ with a single binary operation "$-$" that satisfies the following identities: for any $x, y, z \in X$,

(S1) $x - (y - x) = x$;
(S2) $x - (x - y) = y - (y - x)$;
(S3) $(x - y) - z = (x - z) - y$.

The last identity permits us to omit parentheses in expressions of the form $(x - y) - z$. The subtraction determines an order relation on X: $a \leq b \Leftrightarrow a - b = 0$, where $0 = a - a$ is an element that does not depend on the choice of $a \in X$. The ordered set $(X; \leq)$ is a semi-Boolean algebra in the sense of [1], that is, it is a meet semilattice with zero 0 in which every interval $[0, a]$ is a Boolean algebra with respect to the induced order. Here $a \wedge b = a - (a - b)$; the complement of an element $b \in [0, a]$ is $a - b$; and if $b, c \in [0, a]$, then

$$\begin{aligned} b \vee c &= (b' \wedge c')' = a - ((a - b) \wedge (a - c)) \\ &= a - ((a - b) - ((a - b) - (a - c))). \end{aligned}$$

In a subtraction algebra, the following are true:

(p1) $(x - y) - y = x - y$.
(p2) $x - 0 = x$ and $0 - x = 0$.
(p3) $(x - y) - x = 0$.
(p4) $x - (x - y) \leq y$.
(p5) $(x - y) - (y - x) = x - y$.

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(p6) $x-(x-(x-y))=x-y$.
(p7) $(x-y)-(z-y)\leq x-z$.
(p8) $x\leq y$ if and only if $x=y-w$ for some $w\in X$.
(p9) $x\leq y$ implies $x-z\leq y-z$ and $z-y\leq z-x$ for all $z\in X$.
(p10) $x,y\leq z$ implies $x-y=x\wedge(z-y)$.
(p11) $(x\wedge y)-(x\wedge z)\leq x\wedge(y-z)$.
(p12) $(x-y)-z=(x-z)-(y-z)$.

A mapping d from a subtraction algebra X to a subtraction algebra Y is called a *morphism* if $d(x-y)=d(x)-d(y)$ for all $x,y\in X$. A self map d of a subtraction algebra X which is a morphism is called an *endomorphism*.

Lemma 2.1 Let X be a subtraction algebra. Then the following properties hold:
(1) $x\wedge y=y\wedge x$, for every $x,y\in X$.
(2) $x-y\leq x$ for all $x,y\in X$.

Lemma 2.2 Every subtraction algebra X satisfies the following property.

$$(x-y)-(x-z)\leq z-y$$

for all $x,y,z\in X$.

Proof. Using (S3) and (p7), we have

$$\begin{aligned}((x-y)-(x-z))-(z-y)&=((x-(x-z))-y)-(z-y)\\&\leq(x-(x-z))-z\\&(x-z)-(x-z)=0\end{aligned}$$

for all $x,y,z\in X$.

Definition 2.3 Let X be a subtraction algebra and Y a non-empty set of X. Then Y is called a *subalgebra* if $x-y\in Y$ whenever $x,y\in Y$.

3. f-derivations of subtraction algebras.

Definition 3.1. ([3]) Let X be a subtraction algebra. By a *derivation* of X, a self-map d of X satisfying the identity $d(x-y)=(d(x)-y)\wedge(x-d(y))$ for all $x,y\in X$ is meant.

Example 3.2. Let $X=\{0,a,b\}$ be a subtraction algebra with the following Cayley table

$-$	0	a	b
0	0	0	0
a	a	0	a
b	b	b	0

Define a map $d:X\to X$ by

$$d(x)=\begin{cases}0 & \text{if } x=0,b\\ b & \text{if } x=a\end{cases}$$

Then it is easily checked that d is a derivation of subtraction algebra X.

Definition 3.3 Let X be a subtraction algebra. A function $d:X\to X$ is called an *f-derivation* on X if there exists a function $f:X\to X$ such that

$$d(x-y)=(d(x)-f(y))\wedge(f(x)-d(y))$$

for all $x,y\in X$.

Example 3.4. Let $X=\{0,1,2,3\}$ in which "$-$" is defined by

$-$	0	1	2	3
0	0	0	0	0
1	1	0	1	0
2	2	2	0	0
3	3	2	1	0

It is easy to check that $(X; -)$ is a subtraction algebra. Define a map $d : X \to X$ by

$$d(x) = \begin{cases} 0 & \text{if } x = 0, 3 \\ 1 & \text{if } x = 2 \\ 2 & \text{if } x = 1 \end{cases}$$

and define a map $f : X \to X$ by

$$f(x) = \begin{cases} 0 & \text{if } x = 0, 2 \\ 2 & \text{if } x = 1, 3 \end{cases}$$

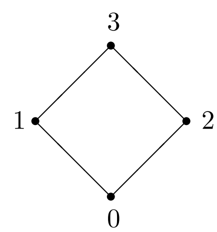

FIGURE 1. Hesse diagram of Example 3.4

Then it is easily checked that d is an f-derivation of a subtraction algebra X.

Example 3.5. Let $X = \{0, a, b\}$ be a subtraction algebra with the following Cayley table

$-$	0	a	b
0	0	0	0
a	a	0	a
b	b	b	0

Define a map $d : X \to X$ by

$$d(x) = \begin{cases} 0 & \text{if } x = 0, a \\ b & \text{if } x = b \end{cases}$$

and define a map $f : X \to X$ by $f : X \to X$ by

$$f(x) = \begin{cases} 0 & \text{if } x = 0, a \\ b & \text{if } x = b \end{cases}$$

Then it is easily checked that d is an f-derivation of subtraction algebra X.

Example 3.6. In Example 3.4, f-derivation d is not a derivation of X since $2 = d(1) = d(1 - 2) \neq (d(1) - 2) \wedge (1 - d(2)) = (2 - 2) \wedge (1 - 1) = 0 \wedge 0 = 0$.

Proposition 3.7. Let X be a subtraction algebra and d an f-derivation. Then the following identities hold:
(1) $d(x) \leq f(x)$ for all $x, y \in X$,
(2) $d(x-y) \leq f(x)$ for all $x, y \in X$.

Proof. (1). Since $x - 0 = x$, we get

$$\begin{aligned} d(x) = d(x-0) &= (d(x) - f(0)) \wedge (f(x) - d(0)) \\ &= (f(x) - d(0)) \wedge (d(x) - f(0)) \\ &\leq f(x) - d(0) \leq f(x) \end{aligned}$$

from (p4) and Lemma 2.1 (2).

(2). By definition of f-derivation, we have

$$\begin{aligned} d(x-y) &= (d(x) - f(y)) \wedge (f(x) - d(y)) \\ &\leq f(x) - d(y) \\ &\leq f(x) \end{aligned}$$

for all $x, y \in X$.

Proposition 3.8. Let X be a subtraction algebra and d an f-derivation. Then $d(0) = 0$.

Proof. By definition of f-derivation, we have

$$\begin{aligned} d(0) = d(0-0) &= (d(0) - f(0)) \wedge (f(0) - d(0)) \\ &= (d(0) - f(0)) - ((d(0) - f(0)) - (f(0) - d(0))) \\ &= (d(0) - f(0)) - (d(0) - f(0)) = 0 \end{aligned}$$

from (p5).

Proposition 3.9. Let X be a subtraction algebra and d an f-derivation. Then the following identities hold:
(1) $d(x-y) \leq d(x) - d(y)$ for all $x, y \in X$,
(2) $d(x) - f(y) \leq f(x) - d(y)$ for all $x, y \in X$.

Proof. (1) By definition of f-derivation and Proposition 3.7 (1), we have $d(x-y) \leq d(x) - f(y) \leq d(x) - d(y)$ for all $x, y \in X$.

(2) Since $d(x) \leq f(x)$ for all $x \in X$, we have $d(x) - f(y) \leq f(x) - f(y) \leq f(x) - d(y)$.

Theorem 3.10. Let X be a subtraction algebra. If d is an f-derivation of X, $d(x-y) = d(x) - f(y)$ for all $x, y \in X$.

Proof. Suppose that d is an f-derivation of X. Then for any $x, y \in X$, we have $d(x) - f(y) \leq f(x) - d(y)$ by Proposition 3.9 (2) and

$$d(x-y) = (d(x) - f(y)) \wedge (f(x) - d(y)) = d(x) - f(y).$$

Definition 3.11. Let X be a subtraction algebra and d a derivation on X. If $x \leq y$ implies $d(x) \leq d(y)$, d is called an *isotone derivation*.

Theorem 3.12. Let d be an f-derivation of X. Then d is an isotone derivation.

Proof. Let $x \leq y$ for all $x, y \in X$. Then by (p8), $x = y - w$ for some $w \in X$. Hence we have

$$d(x) = d(y-w) = (d(y) - f(w)) \wedge (f(y) - d(w)) \leq d(y) - f(w) \leq d(y)$$

by Lemma 2.3 (2).

Let d be a f-derivation of X. Define a set by

$$F := \{x \mid d(x) = f(x)\}$$

for all $x \in X$.

Proposition 3.13. Let d be an f-derivation and f an endomorphism. Then F is a subalgebra of X.

Proof. Let $x, y \in F$. Then we get $d(x) = f(x)$ and $d(y) = f(y)$, and so $d(x-y) = d(x) - f(y) \wedge f(x) - d(y) = f(x) - f(y) \wedge f(x) - f(y) = f(x) - f(y) = f(x-y)$. Hence $x - y \in F$. This completes the proof.

Theorem 3.14. Let d be an f-derivation and f an increasing endomorphism. If $x \leq y$ and $y \in F$, then we have $x \in F$.

Proof. Let $x \leq y$ and $y \in F$. Then we obtain $f(x) \leq f(y)$ and $f(y) = d(y)$, and so we have

$$\begin{aligned} d(x) &= d(x \wedge y) = d(x - (x - y)) = d(y - (y - x)) \\ &= d(y) - f(y - x) = d(y) - (f(y) - f(x)) \quad \text{(by Theorem 3.10)} \\ &= f(y) - (f(y) - f(x)) = f(x) - (f(x) - f(y)) \\ &= f(x) - 0 = f(x). \end{aligned}$$

This completes the proof.

Definition 3.15. Let X be a subtraction algebra and d an f-derivation. Define a $Kerd$ by

$$Kerd = \{x \in X \mid d(x) = 0\}.$$

Proposition 3.16. Let X be a subtraction algebra and d an f-derivation. Then $Kerd$ is a subalgebra of X.

Proof. Let $x, y \in Kerd$. Then $d(x) = d(y) = 0$, and so $d(x-y) \leq d(x) - d(y) = 0 - 0 = 0$ by Proposition 3.9 (1). Thus $d(x-y) = 0$ that is, $x - y \in Kerd$. Hence $Kerd$ is a subalgebra of X.

Proposition 3.17. Let X be a subtraction algebra and d an f-derivation. If $x \in Kerd$ and $y \in X$, then $x \wedge y \in Kerd$.

Proof. Let $x \in Kerd$. Then we get $d(x) = 0$, and so

$$\begin{aligned} d(x \wedge y) &= d(x - (x - y)) = d(x) - f(x - y) \\ &= 0 - f(x - y) \\ &= 0. \end{aligned}$$

This completes the proof.

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