MEAN THEORETIC OPERATOR FUNCTIONS FOR EXTENSIONS OF THE GRAND FURUTA INEQUALITY

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Abstract. The Furuta inequality is known as an exquisite extension of the Löwner-Heinz inequality. The grand Furuta inequality is its further extension including Ando-Hiai inequality on log-majorization. We introduce two mean theoretic operator functions defined for positive invertible operators $X \leq A$, for a given positive invertible operator A and given $t \in [0,1]$:

$$\Phi(X)(=\Phi(u,p;X)) = A^{-u}\sharp_{\frac{u+1}{u+p}}X^p(\leq X) \quad \text{for} \quad u \geq 0,\ p \geq 1$$

and

$$\Psi(X)(=\Psi(q,s;X)) = (A^t\sharp_s X^q)^{\frac{1}{(1-s)t+sq}}(\leq X) \quad \text{for} \quad q \geq 1,\ s \geq 1.$$

We have the grand Furuta inequality by using their composition and also have its further extension presented recently by Furuta, by using a successive composition of them.

1. Introduction

Throughout this note, we consider bounded linear operators on a Hilbert space H. An operator A is positive, denoted by $A \geq 0$, if $(Ax,x) \geq 0$ for all $x \in H$ and strictly positive, denoted by $A > 0$ if A is positive invertible.

The Löwner-Heinz inequality is well-known as one of the most important inequalities:

(LH) $\quad A \geq B \quad \text{implies} \quad A^p \geq B^p \quad \text{for} \quad 0 \leq p \leq 1.$

The Furuta inequality is an exquisite extension of the Löwner-Heinz inequality:

Theorem F (The Furuta inequality [6], [8]**).** *If $A \geq B \geq 0$, then the following inequality holds:*

(F) $\quad (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{r+1}{r+p}} \leq A^{r+1}$ *for $p \geq 1$ and $r \geq 0$.*

More precisely, define

$f(r,p) := A^{\frac{-r}{2}}(A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{r+1}{r+p}}A^{\frac{-r}{2}}.$

Then $f(r,p)$ is a decreasing function for $r \geq 0$ and $p \geq 1$.

Combining the above inequality with Ando-Hiai inequality on majorization [1], Furuta presented the following generalization, which is called the grand Furuta inequality:

Theorem G (The Grand Furuta inequality [7]**).** *If $A \geq B \geq 0$ with $A > 0$, then for each $t \in [0,1]$ and $p \geq 1$,*

(G) $\quad \{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{r-t+1}{r+(p-t)s}} \leq A^{r-t+1}$

for $r \geq t, s \geq 1$.

More precisely, define

$$g(r,s) := A^{\frac{-r}{2}}\{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{r-t+1}{r+(p-t)s}}A^{\frac{-r}{2}}.$$

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Then $g(r,s)$ is a decreasing function for $r \geq t$ and $s \geq 1$.

Recently, Furuta presented a further extension of the grand Furuta inequality as follows:
Theorem FG (Further extension of the grand Furuta inequality [9]**).** *If $A \geq B \geq 0$ with $A > 0$, then for each $t \in [0,1]$, and for $p_1, p_2, ..., p_{2n-1} \geq 1$,*

(FG) $[A^{\frac{r}{2}}(A^{\frac{-t}{2}}\{A^{\frac{t}{2}}...(A^{\frac{-t}{2}}\{A^{\frac{t}{2}}(A^{\frac{-t}{2}}B^{p_1}A^{\frac{-t}{2}})^{p_2}A^{\frac{t}{2}}\}^{p_3}A^{\frac{-t}{2}})^{p_4}...A^{\frac{t}{2}}\}^{p_{2n-1}}A^{\frac{-t}{2}})^{p_{2n}}A^{\frac{r}{2}}]^{\frac{r-t+1}{r-t+q(2n)}} \leq A^{r-t+1}$

for $r \geq t, p_{2n} \geq 1$ with $q(0) = 1$, $q(2k) = \{q(2k-2) \cdot p_{2k-1} - t\}p_{2k} + t$ $(k \geq 1)$.

More precisely, if we define

$h(r,p_{2n}) := A^{\frac{-r}{2}}[A^{\frac{r}{2}}(A^{\frac{-t}{2}}\{A^{\frac{t}{2}}...(A^{\frac{-t}{2}}\{A^{\frac{t}{2}}(A^{\frac{-t}{2}}B^{p_1}A^{\frac{-t}{2}})^{p_2}A^{\frac{t}{2}}\}^{p_3}A^{\frac{-t}{2}})^{p_4}...$
$\times A^{\frac{t}{2}}\}^{p_{2n-1}}A^{\frac{-t}{2}})^{p_{2n}}A^{\frac{r}{2}}]^{\frac{r-t+1}{r-t+q(2n)}}A^{\frac{-r}{2}}.$

Then $h(r,p_{2n})$ is a decreasing function for $r \geq t, p_{2n} \geq 1$.

Mean theoretic approach to the Furuta or the grand Furuta inequality was given by several authors [2], [3], [4], [5], [10], [12], etc. The same approach to the above further extension of the grand Furuta inequality has been presented by Ito and Kamei [11].

For operators $A, B > 0$, α-power mean $A\sharp_\alpha B$ for $\alpha \in [0,1]$ is introduced [13] as $A\sharp_\alpha B := A^{\frac{1}{2}}(A^{\frac{-1}{2}}BA^{\frac{-1}{2}})^\alpha A^{\frac{1}{2}}$. Then in terms of a power mean and extended power mean, that is, α-power mean for $\alpha \geq 0$, the Furuta inequality (F) and the grand Furuta inequality (G) can be reformed as their *satellite forms* (F*) and (G*), respectively [12], [4], [5]:

Satellite forms of (F) and (G) ([12]**,** [4]**).** *If $A \geq B \geq 0$, then*

(F*) $\quad A^{-r}\sharp_{\frac{r+1}{r+p}}B^p \leq B \ (\leq A)$ *for $r \geq 0$, $p \geq 1$.*

(G*) $\quad A^{-r+t}\sharp_{\frac{r-t+1}{r+(p-t)s}}(A^t\sharp_s B^p) \leq B \ (\leq A)$ *for $r \geq t$, $s \geq 1$.*

The left-hand side of (F*) *is $f(r,p)$. For the left-hand side of* (G*), *if we denote it by $\tilde{g}(r,s)$, then*

$\tilde{g}(r,s) := A^{-r+t}\sharp_{\frac{r-t+1}{r+(p-t)s}}(A^t\sharp_s B^p) = A^{\frac{t}{2}}g(r,s)A^{\frac{t}{2}},$

so that we can see $\tilde{g}(r,s)$ is, similarly to $g(r,s)$, a decreasing function for $r \geq t$ and $s \geq 1$.

Now we notice that with the same assumption on A, B, t above and for $p, s \geq 1$, the following *key* inequality due to Fujii and Kamei [4, Theorem 2] holds:

(FK) $\quad (A^t\sharp_s B^p)^{\frac{1}{(1-s)t+sp}} \leq B.$

Hence, related to (G*), if we put $r' = r-t$, $p' = t \nabla_s p \ (= (1-s)t+sp)$ and $B_1 = (A^t\sharp_s B^p)^{\frac{1}{p'}}$, then since $r' \geq 0, p' \geq 1$, and $B_1 \leq B$ as above, we obtain, from (F*),

(G**) $\quad A^{-r'}\sharp_{\frac{r'+1}{r'+p'}}B_1^{p'} \leq B_1 \leq B \leq A,$

which is of same form as (F*) (and is, at the same time, more precise than (G*)).

Under the circumstances, we introduce, in Section 2, two mean theoretic operator functions related to (F*) (or (G**)) and (FK), and by using their composition and successive composition we express the grand Furuta inequality and its further extension.

2. Mean theoretic functions for the grand Furuta inequality

Definition 2.1. Let $A > 0$, $t \in [0,1]$. Then for $0 < X \leq A$, define

$$\Phi(X)(= \Phi(u,p;X)) = A^{-u}\sharp_{\frac{u+1}{u+p}}X^p \quad \text{for} \quad u \geq 0,\ p \geq 1$$

and

$$\Psi(X)(= \Psi(q,s;X)) = (A^t\sharp_s X^q)^{\frac{1}{t\nabla_s q}} \quad \text{for} \quad q \geq 1,\ s \geq 1.$$

Then as their composition we have

(H) $$\Phi \circ \Psi(X) = \Phi(u,p;\Psi(q,s;X)) = A^{-u}\natural_{\frac{u+1}{u+p}}(A^t\natural_s X^q)^{\frac{p}{t\nabla_s q}}.$$

Both of $\Phi(X)$ and $\Psi(X)$ are contractive, so that $\Phi \circ \Psi(X) \leq \Psi(X) \leq X$. If we put $X = B(\leq A)$ and $p = t \nabla_s q$ in (H), then from $\Phi \circ \Psi(B) \leq \Psi(B) \leq B$, we have

$$A^{-u}\natural_{\frac{u+1}{u+t\nabla_s q}}(A^t\natural_s B^q) \ (= \hat{g}(u,s)) \leq (A^t\natural_s B^q)^{\frac{1}{t\nabla_s q}} \leq B,$$

an equivalent inequality with (G**).

Let

$$\hat{g}(u,s) = \Phi \circ \Psi(B)$$

as above. Putting $u = r - t$ and replacing q by p in $\hat{g}(u,s)$, we obtain the function $\tilde{g}(r,s)$ defined before (as the left-hand side of (G*)), a decreasing function for $r \geq t, \ s \geq 1$. Thus $\hat{g}(u,s)$ is also a *decreasing* function for $u \geq 0, \ s \geq 1$.

Now we show as an equivalent inequality with (FG) by using operator means:

Lemma 2.2. *With the same assumption as in Theorem FG, let $C_0 = B$ and for $k \geq 1$*
$C_{2k} = A^{\frac{-t}{2}}\{A^{\frac{t}{2}}...(A^{\frac{-t}{2}}\{A^{\frac{t}{2}}(A^{\frac{-t}{2}}B^{p_1}A^{\frac{-t}{2}})^{p_2}A^{\frac{t}{2}}\}^{p_3}A^{\frac{-t}{2}})^{p_4}...A^{\frac{t}{2}}\}^{p_{2k-1}}A^{\frac{-t}{2}}$.
Then as an equivalent inequality with (FG), we have

(FG*) $$A^{-r+t}\natural_{\frac{r-t+1}{r-t+q(2n)}}(A^{\frac{t}{2}}C_{2n}^{p_{2n}}A^{\frac{t}{2}}) \leq A.$$

The function $h(r,p_{2n})$ is expressed as follows:

(1) $$h(r,p_{2n}) = A^{-r}\natural_{\frac{r-t+1}{r-t+q(2n)}}C_{2n}^{p_{2n}} = A^{-\frac{t}{2}}\left\{A^{-r+t}\natural_{\frac{r-t+1}{r-t+q(2n)}}(A^{\frac{t}{2}}C_{2n}^{p_{2n}}A^{\frac{t}{2}})\right\}A^{-\frac{t}{2}}.$$

Here

(2) $$A^{\frac{t}{2}}C_{2n}^{p_{2n}}A^{\frac{t}{2}} = A^t\natural_{p_{2n}}\left(A^t\natural_{p_{2n-2}}\cdots(A^t\natural_{p_4}(A^t\natural_{p_2}B^{p_1})^{p_3}\cdots)^{p_{2n-3}}\right)^{p_{2n-1}}.$$

Proof. First we can see $C_{2k+2} = A^{\frac{-t}{2}}\{A^{\frac{t}{2}}C_{2k}^{p_{2k}}A^{\frac{t}{2}}\}^{p_{2k+1}}A^{\frac{-t}{2}}$, or (for $k \geq 1$)

$$C_{2k} = A^{\frac{-t}{2}}\{A^{\frac{t}{2}}C_{2k-2}^{p_{2k-2}}A^{\frac{t}{2}}\}^{p_{2k-1}}A^{\frac{-t}{2}}.$$

Hence

(3) $$A^{\frac{t}{2}}C_{2k}^{p_{2k}}A^{\frac{t}{2}} = A^{\frac{t}{2}}(A^{\frac{-t}{2}}\{A^{\frac{t}{2}}C_{2k-2}^{p_{2k-2}}A^{\frac{t}{2}}\}^{p_{2k-1}}A^{\frac{-t}{2}})^{p_{2k}}A^{\frac{t}{2}} = A^t\natural_{p_{2k}}\{A^{\frac{t}{2}}C_{2k-2}^{p_{2k-2}}A^{\frac{t}{2}}\}^{p_{2k-1}}.$$

Hence for (2), we see

$$\begin{aligned} A^{\frac{t}{2}}C_{2n}^{p_{2n}}A^{\frac{t}{2}} &= A^t\natural_{p_{2n}}\{A^{\frac{t}{2}}C_{2n-2}^{p_{2n-2}}A^{\frac{t}{2}}\}^{p_{2n-1}} = A^t\natural_{p_{2n}}\{A^t\natural_{p_{2n-2}}(A^{\frac{t}{2}}C_{2n-4}^{p_{2n-4}}A^{\frac{t}{2}})^{p_{2n-3}}A^{\frac{t}{2}}\}^{p_{2n-1}} \\ &= A^t\natural_{p_{2n}}\left(A^t\natural_{p_{2n-2}}(\cdots(A^t\natural_{p_2}B^{p_1})^{p_3}\cdots)^{p_{2n-3}}\right)^{p_{2n-1}}. \end{aligned}$$

Now it is easy to see that (FG) can be rewritten as

(4) $$\left\{A^{\frac{r}{2}}C_{2n}^{p_{2n}}A^{\frac{r}{2}}\right\}^{\frac{r-t+1}{r-t+q(2n)}} \leq A^{r-t+1}.$$

Multiplying $A^{\frac{-r}{2}}$ from the both sides of (4), we have

$$A^{\frac{-r}{2}}[A^{\frac{r}{2}}C_{2n}^{p_{2n}}A^{\frac{r}{2}}]^{\frac{r-t+1}{r-t+q(2n)}}A^{\frac{-r}{2}} \leq A^{-t+1}.$$

Hence

$$A^{-r}\natural_{\frac{r-t+1}{r-t+q(2n)}}C_{2n}^{p_{2n}} \leq A^{-t+1}.$$

Again, multiplying $A^{\frac{t}{2}}$ from the both sides, we have (by the transformer identity of the geometric mean [13]) the desired inequality (FG*)

$$A^{-r+t}\sharp_{\frac{r-t+1}{r-t+q(2n)}}(A^{\frac{t}{2}}C_{2n}^{p_{2n}}A^{\frac{t}{2}}) \leq A.$$

For $h(r,p_{2n})$, we can see that

$h(r,p_{2n}) = A^{\frac{-r}{2}}(A^{\frac{r}{2}}C_{2n}^{p_{2n}}A^{\frac{r}{2}})^{\frac{r-t+1}{r-t+q(2n)}}A^{\frac{-r}{2}} = A^{-r}\sharp_{\frac{r-t+1}{r-t+q(2n)}}C_{2n}^{p_{2n}}$

$= A^{-\frac{t}{2}}\left\{A^{-r+t}\sharp_{\frac{r-t+1}{r-t+q(2n)}}(A^{\frac{t}{2}}C_{2n}^{p_{2n}}A^{\frac{t}{2}})\right\}A^{-\frac{t}{2}}$ (by the transformer identity).

Now in order to approach to (FG) or (FG*) with the functions Φ and Ψ, we need the following:

Definition 2.3. With the assumption as in (FG), let for $0 < X \leq A$, define $\Psi_k(X) = \Psi(q_k, s_k; X)$ $\left(= (A^t\sharp_{s_k}X^{q_k})^{\frac{1}{t\nabla_{s_k}q_k}}\right)$ with $q_k = p_{2k-1}\cdot q(2k-2), s_k = p_{2k}$ $(k \geq 1)$, that is,

$$\Psi_k(X) = \Psi(p_{2k-1}\cdot q(2k-2), p_{2k}; X) \ \left(= (A^t\sharp_{p_{2k}}X^{p_{2k-1}q(2k-2)})^{\frac{1}{q(2k)}}\right)$$

(recall $q(2k) = \ t\ \nabla_{p_{2k}}\ p_{2k-1}q(2k-2)$).

Further, if we put $\Psi^{(0)}(X) = X$, $\Psi^{(2k)}(X) = \Psi_k \circ \Psi^{(2k-2)}(X)$, then we obtain

$$\begin{aligned}\Psi^{(2k)}(X) &= \Psi(p_{2k-1}\cdot q(2k-2), p_{2k}; \Psi^{(2k-2)}(X)) \\ &= \left\{A^t\sharp_{p_{2k}}(\Psi^{(2k-2)}(X))^{p_{2k-1}\cdot q(2k-2)}\right\}^{\frac{1}{q(2k)}},\end{aligned}$$

or,

$\Psi^{(2k)}(X)^{q(2k)} = A^t\sharp_{p_{2k}}\left(\Psi^{(2k-2)}(X)^{q(2k-2)}\right)^{p_{2k-1}}.$

Then, inductively,

$$\begin{aligned}\Psi^{(2n)}(X)^{q(2n)} &= A^t\sharp_{p_{2n}}\left(\Psi^{(2n-2)}(X)^{q(2n-2)}\right)^{p_{2n-1}} \\ &= A^t\sharp_{p_{2n}}(A^t\sharp_{p_{2n-2}}(\cdots A^t\sharp_{p_4}(A^t\sharp_{p_2}X^{p_1})^{p_3}\cdots)^{p_{2n-3}})^{p_{2n-1}}.\end{aligned}$$

Hence if we put $X = B \leq A$, we have, from (2),

$$\Psi^{(2n)}(B)^{q(2n)} = A^{\frac{t}{2}}C_{2n}^{p_{2n}}A^{\frac{t}{2}} \tag{5}$$

Remark 2.4 ([11]). To define $\Psi_k(X)$ above, it suffices to assume that $q_k \geq 1 \Leftrightarrow p_{2k-1} \geq 1/q(2k-2)$. Hence the condition $p_1, ..., p_{2n} \geq 1$ in Theorem FG can be weaken such as

$$p_{2k-1} \geq 1/q(2k-2) \text{ and } p_{2k} \geq 1 \quad \text{for} \quad k = 1, ..., n. \tag{6}$$

Now we have:

Proposition 2.5. *Assume that A, X, t are the same as before and $p_1, ..., p_{2n}$ satisfy (6). Define*

$$\Phi_n(X) = \Phi(u, q(2n); X) = A^{-u}\sharp_{\frac{u+1}{u+q(2n)}}X^{q(2n)}.$$

Then

$$\begin{aligned}\Phi_n \circ \Psi^{(2n)}(X) &= A^{-u}\sharp_{\frac{u+1}{u+q(2n)}}(\Psi^{(2n)}(X))^{q(2n)} \\ &= A^{-u}\sharp_{\frac{u+1}{u+t\nabla_{p_{2n}}p_{2n-1}\cdot q(2n-2)}}\left(A^t\sharp_{p_{2n}}\Psi^{(2n-2)}(X)\right)^{p_{2n-1}\cdot q(2n-2)}.\end{aligned}$$

In particular, if we put $X = B$, *then*

$$\tilde{h}(u,p_{2n}) := \Phi_n \circ \Psi^{(2n)}(B) = A^{-u}\sharp_{\frac{u+1}{u+q(2n)}}(\Psi^{(2n)}(B))^{q(2n)} = A^{-u}\sharp_{\frac{u+1}{u+t\nabla_{p_{2n}}Q_n}}\left(A^t\sharp_{p_{2n}}B_n\right)^{Q_n}$$

with $Q_n = p_{2n-1}\cdot q(2n-2)$, $B_n = \Psi^{(2n-2)}(B)$, *both of them not depending on* u *and* p_{2n}. *Hence* $\tilde{h}(u,p_{2n})$ *possessing the same form with* $\hat{g}(u,s)$ *is a decreasing function for* $u \geq 0$ *and* $p_{2n} \geq 1$.

Corollary 2.6. *Notice that from (5)*

$$\tilde{h}(u,p_{2n}) = A^{-u}\sharp_{\frac{u+1}{u+q(2n)}}(\Psi^{(2n)}(B))^{q(2n)} = A^{-u}\sharp_{\frac{u+1}{u+q(2n)}}A^{\frac{t}{2}}C_{2n}^{p_{2n}}A^{\frac{t}{2}}.$$

If we put $u = r - t$ *in* $\tilde{h}(u,p_{2n})$, *then we obtain the left-hand side of* (FG*) :

$$\tilde{h}(r-t,p_{2n}) = A^{-r+t}\sharp_{\frac{r-t+1}{r-t+q(2n)}}(\Psi^{(2n)}(B))^{q(2n)} = A^{-r+t}\sharp_{\frac{r-t+1}{r-t+q(2n)}}(A^{\frac{t}{2}}C_{2n}^{p_{2n}}A^{\frac{t}{2}}).$$

Hence by (1), $\tilde{h}(r-t,p_{2n}) = A^{\frac{t}{2}}h(r,p_{2n})A^{\frac{t}{2}}$, *or* $h(r,p_{2n}) = A^{-\frac{t}{2}}\tilde{h}(r-t,p_{2n})A^{-\frac{t}{2}}$, *so that monotone decrease of* $h(r,p_{2n})$ *stated in Theorem FG can be obtained from that of* $\tilde{h}(r,p_{2n})$ *given in Proposition 2.5.*

Remark 2.7. We constructed the further extension (FG*) of the grand Furuta inequality by using a successive composition of type $\Phi\circ\overbrace{\Psi\circ\cdots\circ\Psi}^{n}$. Similarly, we could construct various extensions of the Furuta or the grand Furuta inequality by using successive compositions of Φ and Ψ.

3. Monotonicity of a generalized Furuta-type operator function

By composing the contractive functions $\Phi(X)$ and $\Psi(X)$, we introduce a parameterized operator function:

Definition 3.1. In (H), put $X = B \leq A$, $p = \lambda(\ t\ \nabla_s\ q\ ) \geq 1$ (for $\lambda > 0$), and define

$$\hat{g}_\lambda(u,s) := \Phi\circ\Psi(B) = A^{-u}\sharp_{\frac{u+1}{u+\lambda(t\nabla_s q)}}(A^t\sharp_s B^q)^\lambda.$$

We then see $\hat{g}_\lambda(u,s) \leq B$, since $\Phi\circ\Psi$ is contractive for $u \geq 0, s \geq 1$. The grand Furuta inequality is understood as the case $\lambda = 1$.

Now as an extension of Ito-Kamei's result [10] on monotonicity of the operator function $\hat{g}_\lambda(u,s)$, we have the following result:

Theorem 3.2. *Let* $q \geq 1, t \in [0,1]$ *and* $\hat{g}_\lambda(u,s)$ *be defined as above. Then*

(i) *If* $\frac{1}{q} \leq \lambda$, *then* $\hat{g}_\lambda(u,s)$ *is monotone decreasing for* $u \geq 0$.
(ii) *If* $\frac{1}{q} \leq \lambda \leq 1$, *then* $(B =)\ \hat{g}_\lambda(0,1) \leq \hat{g}_\lambda(u,s)$ *for* $-1 \leq u \leq 0$, $\frac{1-\lambda t}{\lambda(q-t)} \leq s \leq 1$.

(iii) *If* $\frac{1}{q} \leq \lambda \leq 1$, *then* $\hat{g}_\lambda(u,s)$ *is monotone decreasing for* $u \geq 0, s \geq 1$.

Proof. First note that $\lambda(\ t\ \nabla_s\ q\ )\ (= p) \geq 1$ since $\lambda \geq \frac{1}{q}$ and $t\ \nabla_s\ q\ \geq\ q$.

Now for (i), it is easy, since $\Phi(u,p;\Psi(B))$ is monotone decreasing for $u \geq 0$.

For (ii), we can see that $0 \le \frac{1-\lambda t}{\lambda(q-t)} \le 1$ for $\frac{1}{q} \le \lambda \le 1$. By $A \ge B > 0$ and the Löwner-Heinz inequality, we have

$$\hat{g}_\lambda(u,s) = A^{-u}\sharp_{\frac{u+1}{u+\lambda(t\nabla_s q)}}(A^t\natural_s B^q)^\lambda \ge B^{-u}\sharp_{\frac{u+1}{u+\lambda(t\nabla_s q)}}(B^t\natural_s B^q)^\lambda = B \ = \hat{g}_\lambda(0,1).$$

For (iii), it suffices to show that $\hat{g}_\lambda(u, ss_1) \le \hat{g}_\lambda(u,s)$ for $1 \le s_1 \le 2$. Since $A^t\natural_{ss_1}B^q = A^t\natural_{s_1}(A^t\natural_s B^q)$, we see that

$$A^t\natural_{ss_1}B^q = A^t\natural_{s_1}B_1^{t\nabla_s q} \text{ for } B_1 = (A^t\natural_s B^q)^{\frac{1}{t\nabla_s q}}\ (\le B)\ \le A.$$

Now from [4, Lemma 1],

$$A^t\natural_{s_1}B_1^{t\nabla_s q} \le B_1^{t\nabla_{s_1}(t\nabla_s q)} = B_1^{t\nabla_{ss_1}q},$$

or $(A^t\natural_{s_1}B_1^{t\nabla_s q})^\lambda \le B_1^{\lambda(t\nabla_{ss_1}q)}$. Hence, we have

$$\begin{aligned}
\hat{g}_\lambda(u,ss_1) &= A^{-u}\sharp_{\frac{u+1}{u+\lambda(t\nabla_{ss_1}q)}}(A^t\natural_{ss_1}B^q)^\lambda \\
&= A^{-u}\sharp_{\frac{u+1}{u+\lambda(t\nabla_{ss_1}q)}}\left(A^t\natural_{s_1}B_1^{t\nabla_s q}\right)^\lambda \\
&\le A^{-u}\sharp_{\frac{u+1}{u+\lambda(t\nabla_{ss_1}q)}}B_1^{\lambda(t\nabla_{ss_1}q)}\ (= \hat{g}(u,\lambda(\ t\ \nabla_{ss_1}\ q\ ))) \\
&\le A^{-u}\sharp_{\frac{u+1}{u+\lambda(t\nabla_s q)}}B_1^{\lambda(t\nabla_s q)}\ (= \hat{g}(u,\lambda(\ t\ \nabla_s\ q\ ))) \\
&= \hat{g}_\lambda(u,s).
\end{aligned}$$

The last inequality above is obtained from monotone decrease of $\hat{g}(u,s)$ (with respect s).

Remark 3.3. Is $\hat{g}_\lambda(u,s)$ monotone decreasing, if $\lambda \ge 1$? To this question, we give a negative answer from the following numerical computation. Let
$A = \begin{bmatrix} 2.1 & 1.1 \\ 1.1 & 1.1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ and $\lambda = 1.3,\ t = 0.7,\ q = 2,\ u = 8.$

Define $D_{i,j} = \hat{g}_\lambda(8,i) - \hat{g}_\lambda(8,j)$. Then we obtain (discarded less than 10^{-5}):

$D_{1,3} = \hat{g}_{1.3}(8,1) - \hat{g}_{1.3}(8,3) = \begin{bmatrix} 0.3142 & 0.1805 \\ 0.1805 & 0.1389 \end{bmatrix}$ and $\det D_{1,3} = 0.0110 > 0$.

$D_{3,5} = \hat{g}_{1.3}(8,3) - \hat{g}_{1.3}(8,5) = \begin{bmatrix} 0.1653 & 0.0931 \\ 0.0931 & 0.0709 \end{bmatrix}$ and $\det D_{3,5} = 0.0030 > 0$.

$D_{5,7} = \hat{g}_{1.3}(8,5) - \hat{g}_{1.3}(8,7) = \begin{bmatrix} -0.0965 & 0.3633 \\ 0.3633 & -0.4328 \end{bmatrix}$ and $\det D_{5,7} = -0.0902 < 0$.

$D_{7,9} = \hat{g}_{1.3}(8,7) - \hat{g}_{1.3}(8,9) = \begin{bmatrix} -1.0401 & 1.7586 \\ 1.7586 & -2.6428 \end{bmatrix}$ and $\det D_{7,9} = -0.3438 < 0$.

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