**MEAN THEORETIC OPERATOR FUNCTIONS FOR EXTENSIONS OF THE GRAND FURUTA INEQUALITY**

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**Abstract.** The Furuta inequality is known as an exquisite extension of the Löwner-Heinz inequality. The grand Furuta inequality is its further extension including Ando-Hiai inequality on log-majorization. We introduce two mean theoretic operator functions defined for positive invertible operators $X \leq A$, for a given positive invertible operator $A$ and given $t \in [0, 1]$: $$\Phi(X)(= \Phi(u, p; X)) = A^{-u} X^{p} \left( \leq X \right) \quad \text{for} \quad u \geq 0, \quad p \geq 1$$ and $$\Psi(X)(= \Psi(q, s; X)) = (A_{\frac{t}{p}} X^{q})^{(1-t) \frac{1}{p} + t \frac{1}{q}} \left( \leq X \right) \quad \text{for} \quad q \geq 1, \quad s \geq 1.$$ We have the grand Furuta inequality by using their composition and also have its further extension presented recently by Furuta, by using a successive composition of them.

1. Introduction

Throughout this note, we consider bounded linear operators on a Hilbert space $H$. An operator $A$ is positive, denoted by $A \geq 0$, if $(Ax, x) \geq 0$ for all $x \in H$ and strictly positive, denoted by $A > 0$ if $A$ is positive invertible.

The Löwner-Heinz inequality is well-known as one of the most important inequalities:

(LH) \quad $A \geq B$ implies $A^p \geq B^p$ \quad for $0 \leq p \leq 1$.

The Furuta inequality is an exquisite extension of the Löwner-Heinz inequality:

**Theorem F (The Furuta inequality [6, 8]).** If $A \geq B \geq 0$, then the following inequality holds:

(F) \quad $(A^\frac{t}{p} B^p A^\frac{-t}{p})^{\frac{t}{p} = 1} \leq A^r$ for $p \geq 1$ and $r \geq 0$.

More precisely, define

$f(r, p) := A^\frac{r}{p} (A^\frac{t}{p} B^p A^\frac{-t}{p})^{\frac{r}{p} = 1} A^{-\frac{r}{p}}$.

Then $f(r, p)$ is a decreasing function for $r \geq 0$ and $p \geq 1$.

Combining the above inequality with Ando-Hiai inequality on majorization [1], Furuta presented the following generalization, which is called the grand Furuta inequality:

**Theorem G (The Grand Furuta inequality [7]).** If $A \geq B \geq 0$ with $A > 0$, then for each $t \in [0, 1]$ and $p \geq 1$,

(G) \quad $\{A^\frac{s}{p} (A^\frac{r}{p} B^p A^\frac{-r}{p})^* A^\frac{s}{p}\}^{\frac{r}{p} = 1} \leq A^{p-t+1}$

for $r \geq t, \quad s \geq 1$.

More precisely, define

$g(r, s) := A^\frac{r}{p} \{A^\frac{s}{p} (A^\frac{r}{p} B^p A^\frac{-r}{p})^* A^\frac{s}{p}\}^{\frac{r}{p} = 1} A^{-\frac{r}{p}}$.

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Then \( q(r,s) \) is a decreasing function for \( r \geq t \) and \( s \geq 1 \).

Recently, Furuta presented a further extension of the grand Furuta inequality as follows:

**Theorem FG (Further extension of the grand Furuta inequality [9])**. If \( A \geq B \geq 0 \) with \( A > B \), then for each \( t \in [0,1] \), and for \( p_1, p_2, ..., p_{2n-1} \geq 1 \),

\[
\begin{align*}
\text{(FG)} & \quad [A^\frac{r}{2}(A^\frac{r}{2}A^{\frac{p_1}{2}}B^{p_1}A^{\frac{p_2}{2}})A^{\frac{r}{2}}A^{\frac{p_3}{2}}A^{\frac{p_4}{2}}...A^{\frac{r}{2}}]^{p_{2n-1}} A^\frac{r}{2} A^\frac{r}{2} \leq A^{r - (t+1)} \\
& \quad \text{for } r \geq t, p_{2n} \geq 1 \text{ with } q(0) = 1, \quad g(2k) = \{q(2k-2) \cdot p_{2k-1} - t\}p_{2k} + t \quad (k \geq 1).
\end{align*}
\]

More precisely, if we define

\[
\begin{align*}
\text{then } h(r,p_{2n}) := A^\frac{r}{2} [A^\frac{r}{2}A^\frac{r}{2}A^{\prime}\{A^\frac{r}{2}(A^\frac{r}{2}A^{\frac{p_1}{2}}B^{p_1})A^{\frac{r}{2}}A^{\frac{p_3}{2}}A^{\frac{p_4}{2}}...A^{\frac{r}{2}}]^{p_{2n-1}} A^\frac{r}{2} A^\frac{r}{2}.
\end{align*}
\]

Then \( h(r,p_{2n}) \) is a decreasing function for \( r \geq t, p_{2n} \geq 1 \).

Mean theoretic approach to the Furuta or the grand Furuta inequality was given by several authors [2, 3, 4, 5, 10, 12], etc. The same approach to the above further extension of the grand Furuta inequality has been presented by Ito and Kamei [11].

For operators \( A, B > 0 \), \( \alpha \)-power mean \( A^\alpha_n B \) for \( \alpha \in [0,1] \) is introduced [13] as \( A^\alpha_n B := A^\frac{\alpha n}{2} (A^\frac{\alpha n}{2}BA^{-\frac{\alpha n}{2}})^\alpha A^\frac{\alpha n}{2} \). Then in terms of a power mean and extended power mean, that is, \( \alpha \)-power mean for \( \alpha \geq 0 \), the Furuta inequality (F) and the grand Furuta inequality (G) can be reformed as their satellite forms (F*) and (G*), respectively [12, 4, 5]:

**Satellite forms of (F) and (G) ([12, 4])**. If \( A \geq B \geq 0, \) then

**F**

\[
A^{-\frac{\alpha n}{2} p_{2n}} B^p \leq B (\leq A) \text{ for } r \geq 0, p \geq 1,
\]

**G**

\[
A^{-\frac{\alpha n}{2} p_{2n}} (A^\alpha_n B^p) \leq B (\leq A) \text{ for } r \geq t, s \geq 1.
\]

The left-hand side of (F*) is \( f(r,p) \). For the left-hand side of (G*), if we denote it by \( \tilde{g}(r,s) \), then

\[
\tilde{g}(r,s) := A^{-\frac{\alpha n}{2} p_{2n}} (A^\alpha_n B^p) = A^\frac{\alpha n}{2} g(r,s) A^\frac{\alpha n}{2},
\]

so that we can see \( \tilde{g}(r,s) \) is, similarly to \( g(r,s) \), a decreasing function for \( r \geq t \) and \( s \geq 1 \).

Now we notice that with the same assumption on \( A, B, t \) above and for \( p, s \geq 1 \), the following key inequality due to Fujii and Kamei [4, Theorem 2] holds:

**FK**

\[
(A^\alpha_n B^p)^{\frac{1}{\alpha n p + (1-s)p}} \leq B.
\]

Hence, related to (G*), if we put \( r' = r-t, p' = t \nabla p (= (1-s)t+sp) \) and \( B_1 = (A^\alpha_n B^p)^{\frac{1}{p'}} \), then since \( r' \geq 0, p' \geq 1, \) and \( B_1 \leq B \) as above, we obtain, from (F*),

**G**

\[
A^{-\frac{\alpha n}{2} p_{2n}} B_1^{p'} \leq B_1 \leq B \leq A,
\]

which is of same form as (F*) (and is, at the same time, more precise than (G*)).

Under the circumstances, we introduce, in Section 2, two mean theoretic operator functions related to (F*) (or (G**)) and (FK), and by using their composition and successive composition we express the grand Furuta inequality and its further extension.

2. **Mean theoretic functions for the grand Furuta inequality**

**Definition 2.1.** Let \( A > 0, \ t \in [0,1] \). Then for \( 0 < X \leq A \), define

\[
\Phi(X) (= \Phi(u,p;X)) = A^{-\frac{\alpha n}{2} p_{2n}} X^p \quad \text{for } u \geq 0, p \geq 1
\]

and

\[
\Psi(X) (= \Psi(q,s;X)) = (A^\alpha_n X^q)^{\frac{1}{\alpha n s}} \quad \text{for } q \geq 1, s \geq 1.
\]
Then as their composition we have
\[
\Phi \circ \Psi(X) = \Phi(u, p; \Psi(q, s; X)) = A^{-u \sharp \frac{t}{t + s}} (A^{t \sharp s} X^q) \frac{t}{t + s}.
\]
Both of \(\Phi(X)\) and \(\Psi(X)\) are contractive, so that \(\Phi \circ \Psi(X) \leq \Psi(X) \leq X\). If we put \(X = B(\leq A)\) and \(p = t \nabla s\) \(q\) in (H), then from \(\Phi \circ \Psi(B) \leq \Psi(B) \leq B\), we have
\[
A^{-u \sharp \frac{t}{t + s}} (A^{t \sharp s} B^q) \leq (A^{t \sharp s} B^q)^{1/t} \leq B,
\]
an equivalent inequality with \((G^{**})\).

Let
\[
\hat{g}(u, s) = \Phi \circ \Psi(B)
\]
as above. Putting \(u = r - t\) and replacing \(q\) by \(p\) in \(\hat{g}(u, s)\), we obtain the function \(\hat{g}(r, s)\) defined before (as the left-hand side of \((G^*)\)), a decreasing function for \(r \geq t, s \geq 1\). Thus \(\hat{g}(u, s)\) is also a decreasing function for \(u \geq 0, s \geq 1\).

Now we show as an equivalent inequality with \((FG)\) by using operator means:

**Lemma 2.2.** With the same assumption as in Theorem FG, let \(C_0 = B\) and for \(k \geq 1\)
\[
C_{2k} = A^{t \sharp (A \sharp \cdots (A \sharp \{A \sharp (A \sharp B^p, A \sharp)^p A \sharp)^p \cdots A \sharp)^p \cdots) A \sharp)}^{p_{2k - 1}} A \sharp.
\]
Then as an equivalent inequality with \((FG)\), we have
\[
(A \sharp C_{2n} A \sharp) \leq A.
\]
The function \(h(r, p_{2n})\) is expressed as follows:

1. \(h(r, p_{2n}) = A^{-r \sharp \frac{t}{t + s} + \frac{1}{r \nabla s(2n)}} (A \sharp C_{2n} A \sharp)\),

Here

2. \(A \sharp C_{2n} A \sharp = A^{t \sharp p_{2n}} \{A^{t \sharp p_{2n-2}} \cdots (A^{t \sharp p_{4}} A^{t \sharp p_{2}} p_{3 \cdots} p_{2n-3}) p_{2n-3} A \sharp\}^{p_{2n-1}}\).

**Proof.** First we can see \(C_{2k+2} = A^{t \sharp (A \sharp C_{2k+2} A \sharp)}^{p_{k+1}} A \sharp\), or (for \(k \geq 1\))
\[
C_{2k} = A^{t \sharp (A \sharp C_{2k+2} A \sharp)}^{p_{2k-1}} A \sharp.
\]

Hence

3. \(A \sharp C_{2k} A \sharp = A \sharp (A^{t \sharp} A \sharp C_{2k} A \sharp) A \sharp A \sharp = A^{t \sharp} A \sharp C_{2k} A \sharp A \sharp\),

Hence for (2), we see
\[
A \sharp C_{2n} A \sharp = A^{t \sharp p_{2n}} \{A^{t \sharp p_{2n-2}} A \sharp\} p_{2n-1} = A^{t \sharp p_{2n}} \{A^{t \sharp p_{2n-2}} (A \sharp C_{2n} A \sharp) p_{2n-3} A \sharp\} p_{2n-1}
\]
\[= A^{t \sharp p_{2n}} (A \sharp C_{2n} A \sharp) p_{2n-3} A \sharp.\]

Now it is easy to see that \((FG)\) can be rewritten as
\[
\{A \sharp C_{2n} A \sharp\}^{\frac{t}{t + s(2n)}} \leq A^{t \sharp 1}.
\]

Multiplying \(A \sharp A\) from the both sides of (4), we have
\[
A^{-t \sharp \frac{t}{t + s(2n)}} A \sharp C_{2n} A \sharp \leq A^{t \sharp 1}.
\]

Hence
\[
A^{-t \sharp \frac{t}{t + s(2n)}} C_{2n} \leq A^{t \sharp 1}.
\]
Again, multiplying \( A^{\frac{u}{q}} \) from the both sides, we have (by the transformer identity of the geometric mean [13]) the desired inequality (FG*):
\[
A^{-r+t} \left( \frac{r-t+1}{r-t+q(2n)} \right) A^{\frac{u}{q}} C_{2n}^{2n} A^{\frac{u}{q}} \leq A.
\]
For \( h(r, p_{2n}) \), we can see that
\[
h(r, p_{2n}) = A^{\frac{u}{q}} C_{2n}^{2n} A^{\frac{u}{q}}
\]
\[
= A^{-r} \left( \frac{r-t+1}{r-t+q(2n)} \right) C_{2n}^{2n}
\]
\[
= A^{-\Psi} \left( A^{-r} \left( \frac{r-t+1}{r-t+q(2n)} \right) C_{2n}^{2n} A^{\frac{u}{q}} \right) \text{ (by the transformer identity)}.
\]

Now in order to approach to (FG) or (FG*) with the functions \( \Phi \) and \( \Psi \), we need the following:

**Definition 2.3.** With the assumption as in (FG), let for \( 0 < X \leq A \), define
\[
\Psi_k(X) = \Psi(q_k, s_k; X) = (A^t \Psi_{q_k} X^{q_k})^{\frac{1}{t-q_k}} \text{ with } q_k = p_{2k-1} \cdot q(2k - 2), s_k = p_{2k} (k \geq 1),
\]
that is,
\[
\Psi_k(X) = \Psi(p_{2k-1} \cdot q(2k - 2), p_{2k}; X) = (A^t \Psi_{p_{2k}} X^{p_{2k-1} \cdot q(2k - 2)})^{\frac{1}{p_{2k}}}
\]
(recall \( q(2k) = t \nabla_{p_{2k}} p_{2k-1} q(2k - 2) \)).

Further, if we put \( \Psi^{(0)}(X) = X \), \( \Psi^{(2k)}(X) = \Psi_k \circ \Psi^{(2k-2)}(X) \), then we obtain
\[
\Psi^{(2k)}(X) = \Psi(p_{2k-1} \cdot q(2k - 2), p_{2k}; \Psi^{(2k-2)}(X))
\]
\[
= A^t \Psi_{p_{2k}} \left( \Psi^{(2k-2)}(X) \right)^{p_{2k-1} \cdot q(2k - 2)}^{\frac{1}{p_{2k}^2}},
\]
or,
\[
\Psi^{(2k)}(X)^{q(2k)} = A^t \Psi_{p_{2k}} \left( \Psi^{(2k-2)}(X)^{q(2k-2)} \right)^{p_{2k-1}}.
\]

Then, inductively,
\[
\Psi^{(2n)}(X)^{q(2n)} = A^t \Psi_{p_{2n}} \left( \Psi^{(2n-2)}(X)^{q(2n-2)} \right)^{p_{2n-1}}
\]
\[
= A^t \Psi_{p_{2n}} \left( A^t \Psi_{p_{2n-2}} \left( \cdots A^t \Psi_{p_2} \left( A^t \Psi_{p_4} \left( A^t \Psi_{p_2} X^{p_1} \right)^{p_3} \cdots \right)^{p_2} \right)^{p_3} \cdots \right)^{p_{2n-3}}.
\]
Hence if we put \( X = B \leq A \), we have, from (2),
\[
\Psi^{(2n)}(B)^{q(2n)} = A^t C_{2n}^{2n} A^t
\]

**Remark 2.4 ([11]).** To define \( \Psi_k(X) \) above, it suffices to assume that \( q_k \geq 1 \iff p_{2k-1} \geq 1/q(2k - 2) \). Hence the condition \( p_{2n} \geq 1 \) in Theorem FG can be weaken such as
\[
p_{2k-1} \geq 1/q(2k - 2) \text{ and } p_{2k} \geq 1 \text{ for } k = 1, ..., n.
\]

Now we have:

**Proposition 2.5.** Assume that \( A, X, t \) are the same as before and \( p_1, ..., p_{2n} \) satisfy (6). Define
\[
\Phi_n(X) = \Phi(u, q(2n); X) = A^{-u} \left( \frac{u+1}{u-q(2n)} \right) X^{q(2n)}.
\]
Then
\[
\Phi_n \circ \Psi^{(2n)}(X) = A^{-u} \left( \frac{u+1}{u-q(2n)} \right) \left( \Psi^{(2n)}(X) \right)^{q(2n)}
\]
\[
= A^{-u} \left( \frac{u+1}{u-q(2n)} \right)^{u+1} \left( A^t \Psi_{p_{2n}} \left( \Psi^{(2n-2)}(X) \right)^{p_{2n-1} \cdot q(2n-2)} \right)^{p_{2n-2}}.
\]
In particular, if we put $X = B$, then
\[
\hat{h}(u, p_{2n}) := \Phi_n \circ \Psi(2n)(B) = A^{-u_{\frac{p+1}{2}}} \frac{n+1}{n+q(2n)} (\Psi(2n)(B))^{q(2n)} = A^{-u_{\frac{p+1}{2}}} \frac{n+1}{n+q(2n)} (A^t p_{2n} B_n)^{Q_n}
\]
with $Q_n = p_{2n-1} \cdot q(2n - 2)$, $B_n = \Psi(2n-2)(B)$, both of them not depending on $u$ and $p_{2n}$. Hence $\hat{h}(u, p_{2n})$ possessing the same form with $\hat{g}(u, s)$ is a decreasing function for $u \geq 0$ and $p_{2n} \geq 1$.

**Corollary 2.6.** Notice that from (5)
\[
\hat{h}(u, p_{2n}) = A^{-u_{\frac{p+1}{2}}} \frac{n+1}{n+q(2n)} (\Psi(2n)(B))^{q(2n)} = A^{-u_{\frac{p+1}{2}}} \frac{n+1}{n+q(2n)} A^t C p_{2n} A^t.
\]
If we put $u = r - t$ in $\hat{h}(u, p_{2n})$, then we obtain the left-hand side of $(FG^+)$:
\[
\hat{h}(r - t, p_{2n}) = A^{-r+t_{\frac{p+1}{2}}} \frac{n+1}{n+q(2n)} (\Psi(2n)(B))^{q(2n)} = A^{-r+t_{\frac{p+1}{2}}} \frac{n+1}{n+q(2n)} (A^t C p_{2n} A^t).
\]
Hence by (1), $\hat{h}(r - t, p_{2n}) = A^t h(r, p_{2n})A^t$, or $h(r, p_{2n}) = A^{-\frac{t}{r}} \hat{h}(r - t, p_{2n})A^{-\frac{t}{r}}$, so that monotone decrease of $h(r, p_{2n})$ stated in Theorem FG can be obtained from that of $\hat{h}(r, p_{2n})$ given in Proposition 2.5.

**Remark 2.7.** We constructed the further extension $(FG^+)$ of the grand Furuta inequality by using a successive composition of type $\Phi \circ \Psi \circ \cdots \circ \Psi$. Similarly, we could construct various extensions of the Furuta or the grand Furuta inequality by using successive compositions of $\Phi$ and $\Psi$.

### 3. Monotonicity of a generalized Furuta-type operator function

By composing the contractive functions $\Phi(X)$ and $\Psi(X)$, we introduce a parameterized operator function:

**Definition 3.1.** In (H), put $X = B \leq A$, $p = \lambda\left(\begin{array}{c}t \nabla_s q \end{array}\right) \geq 1$ (for $\lambda > 0$), and define
\[
\hat{g}_\lambda(u, s) := \Phi \circ \Psi(B) = A^{-u_{\frac{p+1}{2}}} \frac{n+1}{n+q(2n)} (A^t p_{s} B^t)^{\lambda}.
\]
We then see $\hat{g}_\lambda(u, s) \leq B$, since $\Phi \circ \Psi$ is contractive for $u \geq 0, s \geq 1$. The grand Furuta inequality is understood as the case $\lambda = 1$.

Now as an extension of Ito-Kamei’s result [10] on monotonicity of the operator function $\hat{g}_\lambda(u, s)$, we have the following result:

**Theorem 3.2.** Let $q \geq 1$, $t \in [0, 1]$ and $\hat{g}_\lambda(u, s)$ be defined as above. Then

(i) If $\frac{1}{q} \leq \lambda$, then $\hat{g}_\lambda(u, s)$ is monotone decreasing for $u \geq 0$.

(ii) If $\frac{1}{q} \leq \lambda \leq 1$, then $(B =) \hat{g}_\lambda(0, 1) \leq \hat{g}_\lambda(u, s)$ for $-1 \leq u \leq 0$, $\frac{\lambda - 1}{\lambda(q - 1)} \leq s \leq 1$.

(iii) If $\frac{1}{q} \leq \lambda \leq 1$, then $\hat{g}_\lambda(u, s)$ is monotone decreasing for $u \geq 0, s \geq 1$.

**Proof.** First note that $\lambda\left(\begin{array}{c}t \nabla_s q \end{array}\right) (= p) \geq 1$ since $\lambda \geq \frac{1}{q}$ and $t \nabla_s q \geq q$.

Now for (i), it is easy, since $\Phi(u, p; \Psi(B))$ is monotone decreasing for $u \geq 0$. 
Remark 3.3. For (ii), we can see that $0 \leq \frac{1+\lambda}{\lambda(q-1)} \leq 1$ for $\frac{1}{q} \leq \lambda \leq 1$. By $A \geq B > 0$ and the Löwner-Heinz inequality, we have

$$\hat{g}_\lambda(u,s) = A^{-u\lambda} \left( A^{t\lambda}_s B^q \right)^\lambda \geq B^{-u\lambda} \left( A^{t\lambda}_s B^q \right)^\lambda = \hat{g}_\lambda(u,0,1).$$

For (iii), it suffices to show that $\hat{g}_\lambda(u,ss) \leq \hat{g}_\lambda(u,s)$ for $1 \leq s_1 \leq 2$. Since $A^{t\lambda}_s ss B^q = A^{t\lambda}_s B^q$, we see that

$$A^{t\lambda}_s ss B^q = A^{t\lambda}_s B^q t\lambda \leq A^{t\lambda}_s B^q t\lambda \leq B^q$$

for all $t \geq 1$. Hence, we have

$$\hat{g}_\lambda(u,ss) = A^{-u\lambda} \left( A^{t\lambda}_s ss B^q \right)^\lambda = A^{-u\lambda} B^q \leq 1 \implies \hat{g}_\lambda(u,ss) \leq \hat{g}_\lambda(u,s).$$

The last inequality above is obtained from monotone decrease of $\hat{g}(u,s)$ (with respect $s$).

Remark 3.3. Is $\hat{g}_\lambda(u,s)$ monotone decreasing, if $\lambda \geq 1$? To this question, we give a negative answer from the following numerical computation. Let

$$A = \begin{bmatrix} 2.1 & 1.1 \\ 1.1 & 1.1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad \lambda = 1.3, \quad t = 0.7, \quad q = 2, \quad u = 8.$$

Define $D_{i,j} = \hat{g}_\lambda(8,i) - \hat{g}_\lambda(8,j)$. Then we obtain (discarded less than $10^{-5}$):

$$D_{1,1} = \hat{g}_\lambda(8,1) - \hat{g}_\lambda(8,3) = \begin{bmatrix} 0.3142 & 0.1805 \\ 0.1805 & 0.1389 \end{bmatrix} \quad \text{and} \quad \det D_{1,1} = 0.0110 > 0.$$

$$D_{3,5} = \hat{g}_\lambda(8,3) - \hat{g}_\lambda(8,5) = \begin{bmatrix} 0.1653 & 0.0931 \\ 0.0931 & 0.0709 \end{bmatrix} \quad \text{and} \quad \det D_{3,5} = 0.0030 > 0.$$

$$D_{5,7} = \hat{g}_\lambda(8,5) - \hat{g}_\lambda(8,7) = \begin{bmatrix} -0.0965 & 0.3633 \\ 0.3633 & -0.4328 \end{bmatrix} \quad \text{and} \quad \det D_{5,7} = -0.0902 < 0.$$

$$D_{7,9} = \hat{g}_\lambda(8,7) - \hat{g}_\lambda(8,9) = \begin{bmatrix} -1.0401 & 1.7586 \\ 1.7586 & -2.6428 \end{bmatrix} \quad \text{and} \quad \det D_{7,9} = -0.3438 < 0.$$

**References**


[6] T. FURUTA, $A \geq B \geq 0$ assures $(B'A^p B^r)^{1/q} \geq B^{(p+2r)/q}$ for $r \geq 0, p \geq 0, q \geq 1$ with $(1+2r)q \geq p + 2r$, Proc. Amer. Math. Soc., 101 (1987), 85-88.


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