COMMENTS ON "ON A CERTAIN REPEATING PROCESSES PROBLEM IN ARITHMETIC"

Paul Cull

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Abstract. We show that the iteration

$$A_{n+1} = \begin{cases} A_n/3 & \text{if} A_n \equiv 0 \pmod 3 \\ [\sum a_i]^2 & \text{if} A_n \not\equiv 0 \pmod 3 \end{cases}$$

converges for every positive integer A_0, and that for $A_0 = 3^i B$, ($B \not\equiv 0 \pmod 3$)
A_n converges to the fixed point 1 when $B \equiv 1$ or $8 \pmod 9$ and,
A_n converges to the cycle $169 \longleftrightarrow 256$ when $B \equiv 2$ or 4 or 5 or $7 \pmod 9$.
Further, this convergence takes $i + O(\log^* B)$ steps.

A recent paper [1] deals with iterates of the function:

$$f(A) = \begin{cases} A/3 & \text{if} A \equiv 0 \pmod 3 \\ [\sum a_i]^2 & \text{if} A \not\equiv 0 \pmod 3 \end{cases}$$

where $A = a_{k-1}10^{k-1} + \ldots + a_1 10^1 + a_0$.
{There is a typo in the abstract where $\sum (a_i)^2$ is written instead of $[\sum a_i]^2$. }
The point of this note is to show that the results of [1] can be obtained more easily and, in fact, a more refined theorem can be proven. A complete description of the discrete dynamical system $x_{t+1} = f(x_t)$ is given by the following theorem.

Theorem 1. *For every positive integer $A = 3^i B$, ($B \not\equiv 0 \pmod 3$)*
$f^{(n)}(A)$ converges to the fixed point 1 when $B \equiv 1$ or $8 \pmod 9$ and,
$f^{(n)}(A)$ converges to the cycle $169 \longleftrightarrow 256$ when $B \equiv 2$ or 4 or 5 or $7 \pmod 9$.
This convergence takes $i + O(\log^* B)$ steps.

A fairly general statement can be made about mappings like $f(A)$, namely:
Let $f(A)$ be a mapping from the positive integers to the positive integers. If $f(A) < A$ for all big enough A, then there are a finite number of finite length cycles of iterates of f, and for each A_0 the sequence $\langle f^{(n)}(A_0) \rangle$ converges to one of these cycles.

This statement can be proved by noting that f must map a finite subset of the integers to itself, and that each A_0 is eventually mapped to this subset.

For our specific f, this convergence takes $i + O(\log^* B)$ steps. The powers of 3 disappear in i steps and the squaring operation can never produce a multiple of 3 from a non-multiple of 3. For $B \not\equiv 0 \pmod 3$, the squaring operation takes a k digit number to at most $(9k)^2$ which has at most $O(\log k)$ digits. So, f maps B to the finite subset in at most $O(\log^* B)$

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steps. (Here, $\log^*$ is the iterated logarithm.) Of course, the mapping on a finite subset can only take a constant number of steps to map any of its members to a cycle.

The following three observations can be easily demonstrated by considering the largest possible digit sums and using induction if necessary:

(a) If A has k digits where $k \geq 5$ then $f(A) < A$. More strongly, if $A \not\equiv 0 \pmod 3$, then $f(A)$ has at most $k-1$ digits.
(b) For $k = 4$, $A \not\equiv 0 \pmod 3$, $f^{(2)}(A)$ has at most 3 digits.
(c) If A has at most 3 dgits, then $f(A)$ has at most 3 digits.

Hence, only numbers with at most 3 digits can form cycles. One could use a computer to search for the behavior of iterates starting from each of these numbers, but the search can be greatly simplified.

A useful observation is that $A \equiv [\sum a_i] \pmod 9$.
{ This is the familar "casting out 9's " used in bookkeeping. }
So, $A^2 \equiv [\sum a_i]^2 \pmod 9$ and one can simply look at iterated squaring $\mod 9$. Considering numbers $\not\equiv 0 \pmod 3$ we find the following dynamics:

$$8 \rightarrow 1 \hookleftarrow \qquad \text{and} \qquad 2 \rightarrow 4 \longleftrightarrow 7 \leftarrow 5,$$

That is, there is a fixed point 1, and a cycle consisting of 4 and 7.
Hence any cycle for f must contain either a number $\equiv 1 \pmod 9$ or a number $\equiv 4 \pmod 9$. To find the cycles of f we need only consider 3 digit numbers whose digit sum is 1 or 4 $\pmod 9$ and this means that the only digit sums we need to look at are 1 and 10 and 19 which are $\equiv 1 \pmod 9$, and 4 and 13 and 22 which are $\equiv 4 \pmod 9$. The first three posibilities quicky map to the fixed point 1. The other three possibilities quicky map to the the period 2 cycle $169 \longleftrightarrow 256$.

Finally, the dynamics $\mod 9$ tell us that the numbers $\equiv 1$ or $8 \pmod 9$ must eventually map to the fixed point 1, and that the numbers $\equiv 2$ or 4 or 5 or $7 \pmod 9$ must eventually map to the cycle $169 \longleftrightarrow 256$.

References

[1] Susumu Oda and Junro Sato. On A Certain Repeating Processes Problem in Arithmetic. *Scientiae Mathematicae Japonicae,* 71(260): 217–222, March 2010.

Computer Science, Oregon State University, Corvallis, OR 97331 USA
E-mail address : pc@cs.orst.edu