A THEOREM ON THE SUBJECT OF COOK’S INEQUALITY

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Abstract. We show that the span of an arbitrary simple closed curve $X$ does not exceed the span of any starlike curve contained in the closure of the unbounded component of the complement of $X$.

1. Definitions and auxiliary lemmas

We shall begin by reviewing the definitions introduced by A. Lelek in [6] and [7]. Let $X$ be a connected nonempty metric space. The span $\sigma(X)$ of $X$ is the least upper bound of the set of nonnegative numbers $r$ that satisfy the following condition: there exists a connected space $Y$ and a pair of continuous functions $f, g : Y \to X$ such that $f(Y) = g(Y)$ and $\text{dist}(f(y), g(y)) \geq r$ for every $y \in Y$. To obtain the definition of the semispan $\sigma_0(X)$ of $X$, the equality $f(Y) = g(Y)$ is relaxed to the inclusion of $f(Y) \supset g(Y)$. Requiring that $f$ be onto gives the definitions of surjective span $\sigma_0^*(X)$ and surjective semispan $\sigma_0^*(X)$ of $X$. The last two concepts coincide with the span and semispan, respectively, when $X$ is a simple closed curve.

In general, as was pointed out in [7], $0 \leq \sigma(X) \leq \sigma_0(X) \leq \text{diam}(X)$. Furthermore, it follows from the more general result of A. Lelek [7, Th 2.1, p39] that when $X$ is a continuum then $\sigma_0(X) \leq \varepsilon(X)$, where $\varepsilon(X)$ denotes the infimum of the set of meshes of the chains that cover $X$. A different, direct proof of this inequality can be found in [1]. The span of an arbitrary simple closed curve $X$ that is a boundary of a convex region has been determined in [5]. It has been proven to be equal to its semispan, the infimum of the set of its directional diameters, called the breadth of $X$ in [8], and $\varepsilon(X)$.

A simple closed curve $X$ is starlike if there is a point $Q$ in the bounded component $D$ of $C \setminus X$ such that for each point $P, P \in X$, the line segment $PQ$ is contained in the closure of $D$. For prior work on starlike curves related to span see [2] and [3].

The following versions of the Mountain–Climbing Theorem shall be needed (see the work of J. V. Whittaker in [9]).

Lemma 1.1. Let $0 < a < b, c > 0$. Suppose $f : [a, b] \to [0, c]$ is continuous, increasing, and $f(a) = 0$, $f(b) = c$. Suppose also that $g : [a, b] \to [0, c]$ is continuous, piecewise weakly monotone, and $g(a) = 0$, $g(b) = c$. Then, there exists a continuous mapping $\phi : [a, b] \to [a, b]$ such that $\phi(a) = a$, $\phi(b) = b$ and $f(\phi(t)) = g(t)$ for each $t \in [a, b]$.

Lemma 1.2. Let $0 < a < b, c > 0$. Suppose $f : [a, b] \to [0, c]$ is continuous, decreasing, and $f(a) = c$, $f(b) = 0$. Suppose also that $g : [a, b] \to [0, c]$ is continuous, piecewise weakly monotone, and $g(a) = c$, $g(b) = 0$. Then there exists a continuous mapping $\phi : [a, b] \to [a, b]$ such that $\phi(a) = a$, $\phi(b) = b$ and $f(\phi(t)) = g(t)$ for each $t \in [a, b]$.

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2. The main result

The famous problem of Howard Cook: Do there exist, in the plane, two simple closed curves $X$ and $Y$, such that $X$ is in the bounded component of the complement of $Y$ and the span of $X$ is greater than the span of $Y$? [Problem 173 of “A list of problems known as the Houston Problem Book,” Lecture Notes in Pure and Applied Mathematics, 170, Marcel Dekker, Inc., New York, Basel and Hong Kong, 365–398] has been answered, in the negative, in special cases only. For a survey of related conditions, imposed on either $X$ or $Y$, or both, that guarantee the negative answer, see [4].

Let $h$ be an arbitrary function with values in $C \setminus \{0\}$. In the following theorem, $\text{Arg} h(t)$ denotes the counterclockwise angle between the positive $x$-axis and the ray containing the line segment $0h(t)$ connecting the points 0 and $h(t)$. Notice that $\text{Arg} h(t) \in [0, 2\pi)$.

**Theorem.** Let $X$ be a simple closed curve in the plane $C$. If $Y$ is a starlike curve contained in the closure of the unbounded component of $C \setminus X$ then $\sigma(X) \leq \sigma(Y)$.

**Proof.** Without loss of generality, we shall assume that 0 lies in the bounded component of $C \setminus X$. Let $\varepsilon, \varepsilon > 0$, be an arbitrarily small number. It follows from the definition of span that there exist two continuous functions $G_1, G_2 : [0, 1] \to X$ such that $\sigma([0, 1]) = G_2([0, 1]) = X$ and

$$
\sigma(X) \geq \inf_{t \in [0, 1]} \text{dist}[G_1(t), G_2(t)] > \sigma(X) - \varepsilon/2.
$$

The Weierstrass Approximation Theorem implies the existence of two polynomials $\sim G_1$, $\sim G_2$ such that

$$
\forall t \in [0, 1] \quad |G_i(t) - \sim G_i(t)| < \varepsilon/4, \quad i = 1, 2.
$$

Note that $\text{Arg} \sim G_1$, and $\text{Arg} \sim G_2$ are not continuous. Let $t_1, \ldots, t_m$ be the points of discontinuity of $\text{Arg} \sim G_1$ on $[0, 1]$. Assume, without loss of generality, that $0 < t_1 < \cdots < t_m \leq 1$, and that $\text{Arg} \sim G_1(0) = 0$. Furthermore, if $t_m < 1$ put $t_{m+1} = 1$.

We shall also assume, without loss of generality, that $Y$ is a starlike polygonal line with strictly increasing argument. Let $F : [0, 1] \to Y$ be the mapping that defines $Y$. $F$ is one-to-one on $[0, 1]$, and $F(0) = F(1)$. We can also assume, without loss of generality, that $\text{Arg} F(0) = 0$. Let

$$f(t) = \begin{cases} \text{Arg} F(t), & \text{for } t \in [0, 1) \\ 2\pi, & \text{for } t = 1. \end{cases}$$

Thus, $f$ is increasing and continuous on $[0, 1]$. Let $t_0 = 0$. Note that for each $n \in N \cup \{0\}$, $0 \leq n \leq m$, $\text{Arg} \sim G_1(t_n) = 0$. We shall modify $\text{Arg} \sim G_1$ at some of its points of discontinuity, by changing its value from 0 to $2\pi$, so that on every interval $[t_n, t_{n+1}]$ thus modified portion of $\text{Arg} \sim G_1$ can be continuous, with values in $[0, 2\pi]$, and piecewise weakly monotone.

There are four cases regarding the behavior of $\text{Arg} \sim G_1$ on an arbitrary $[t_n, t_{n+1}]$.

**Case 1.** The restriction of $\text{Arg} \sim G_1$ to $[t_n, t_{n+1}]$ is continuous on $[t_n, t_{n+1}]$ only. See Figure 1.

**Case 2.** The restriction of $\text{Arg} \sim G_1$ to $[t_n, t_{n+1}]$ is continuous on $(t_n, t_{n+1})$ only. See Figure 2.

Notice that, in both case 1 and case 2,

$$\sup_{t \in [t_n, t_{n+1}]} \text{Arg} \sim G_1 = 2\pi \quad \text{and} \quad \inf_{t \in [t_n, t_{n+1}]} \text{Arg} \sim G_1 = 0.$$
Figure 1

Figure 2

Figure 3
Case 3. The restriction of $\text{Arg} \sim G_1$ to $[t_n, t_{n+1}]$ is continuous. See Figure 3.

Note that in case 3 $\sup_{t \in [t_n, t_{n+1}]} \text{Arg} \sim G_1 < 2\pi$

Case 4. The restriction of $\text{Arg} \sim G_1$ to $[t_n, t_{n+1}]$ is continuous on $(t_n, t_{n+1})$ only. See Figure 4.

![Figure 4](image-url)

In case 1, we define $g_1$ as follows.

$$g_1(t) = \begin{cases} 
\text{Arg} \sim G_1(t) & \text{for } t \in [t_n, t_{n+1}) \\
2\pi & \text{for } t = t_{n+1}.
\end{cases}$$

Next, let $h_n$ be an affine mapping from $[t_n, t_{n+1}]$ onto $[0, 1]$ such that $h_n(t_n) = 0$ and $h_n(t_{n+1}) = 1$, and put $f_n(t) = f(h_n(t))$ for all $t \in [t_n, t_{n+1}]$. Since $f_n$ is continuous and increasing on $[t_n, t_{n+1}]$, $g_1$ is continuous and piecewise weakly monotone on $[t_n, t_{n+1}]$, $f_n(t_n) = g_1(t_n) = 0$ and $f_n(t_{n+1}) = g_1(t_{n+1}) = 2\pi$, by virtue of Lemma 1.1 there exists a continuous mapping $\phi_n : [t_n, t_{n+1}] \to [t_n, t_{n+1}]$ such that $\phi_n(t_n) = t_n$, $\phi_n(t_{n+1}) = t_{n+1}$ and $f_n(\phi_n(t)) = g_1(t)$ for all $t \in [t_n, t_{n+1}]$.

In case 2, we define $g_1$ as follows.

$$g_1(t) = \begin{cases} 
2\pi & \text{for } t = t_n \\
\text{Arg} \sim G_1(t) & \text{for } t \in (t_n, t_{n+1}].
\end{cases}$$

With $h_n$ defined as in case 1, put $f_n(t) = f(h_n(t_{n+1} - (t - t_n)))$. Notice that $f_n(t_n) = f(h_n(t_{n+1})) = 2\pi = g_1(t_n)$, and $f_n(t_{n+1}) = f(h_n(t_n)) = 0 = g_1(t_{n+1})$. Since $f_n$ is decreasing and $g_1$ is piecewise weakly monotone, by virtue of Lemma 1.2, there exists a continuous mapping $\phi_n : [t_n, t_{n+1}] \to [t_n, t_{n+1}]$ such that $\phi_n(t_n) = t_n$, $\phi_n(t_{n+1}) = t_{n+1}$ and $f_n(\phi_n(t)) = g_1(t)$ for all $t \in [t_n, t_{n+1}]$.

In case 3, put $g_1(t) = \text{Arg} \sim G_1(t)$ for all $t \in [t_n, t_{n+1}]$ and let $c = \sup_{t \in [t_n, t_{n+1}]} g_1(t)$.

Furthermore, let $t_c$ be such that $g_1(t_c) = c$ and $g_1(t) < c$ for all $t \in [t_n, t_c)$. Next, with $h_n$ defined as in case 1, put $f_n^*(t) = f(h_n(t))$ for all $t \in [t_n, t_{n+1}]$. Since $c < 2\pi$ there exists a number $t_s$, $t_s \in (t_n, t_{n+1})$ such that $f_n^*(t_s) = c$. If $t_s = t_c$, put $f_n^*(t) = f_n^*(t)$ for all...
$t \in [t_n, t_c]$. If not, let $k_n$ be an affine mapping from $[t_n, t_c]$ onto $[t_n, t_s]$ such that $k_n(t_n) = t_n$ and $k_n(t_c) = t_s$ and put $f^n(t) = f_n^\sim(k_n(t))$ for all $t \in [t_n, t_c]$. We define $f_n$ as follows

$$f_n(t) = \begin{cases} f_n^\sim(t), & \text{when } t \in [t_n, t_c] \\ f_n^\sim(t + (t_c - t_n)(t_{n+1} - t)/(t_{n+1} - t_c)), & t \in [t_c, t_{n+1}]. \end{cases}$$

Notice that $f_n(t_c) = c$, $f_n(t_n) = f_n(t_{n+1}) = 0$, $f_n$ is increasing on $[t_n, t_c]$ and decreasing on $[t_c, t_{n+1}]$. By applying Lemma 1.1 on $[t_n, t_c]$ and Lemma 1.2 on $[t_c, t_{n+1}]$ we obtain a continuous mapping $\phi_n : [t_n, t_{n+1}] \to [t_n, t_{n+1}]$ such that $\phi_n(t_n) = t_n$, $\phi_n(t_{n+1}) = t_{n+1}$ and $f_n(\phi_n(t)) = g_1(t)$ for all $t \in [t_n, t_{n+1}]$.

In case 4, we define $g_1$ as follows.

$$g_1(t) = \begin{cases} 2\pi & \text{for } t = t_n \\ \Arg \sim G_1(t) & \text{for } t \in (t_n, t_{n+1}) \\ 2\pi & \text{for } t = t_{n+1}. \end{cases}$$

Let $c = \inf_{t \in [t_n, t_{n+1}]} g_1(t)$. Notice that $c \geq 0$. Let $t_c$ be such that $g_1(t_c) = c$ and $g_1(t) > c$ for all $t \in [t_n, t_c)$. We shall define $f_n$ differently depending on whether $c$ is positive or not.

If $c = 0$ then let $h_{nc}$ be an affine mapping from $[t_n, t_c]$ onto $[0, 1]$ such that $h_{nc}(t_n) = 0$ and $h_{nc}(t_c) = 1$, and put $f_n^\sim(t) = f(h_{nc}(t - t_n))$ for all $t \in [t_n, t_c]$. Notice that $f_n^\sim(t) = f(h_{nc}(t_c)) = f(1) = 2\pi$, $f_n^\sim(t_c) = f(h_{nc}(t_n)) = f(0) = 0$, and $f_n^\sim$ is decreasing. Next, let $h_c$ be an affine mapping from $[t_c, t_{n+1}]$ onto $[0, 1]$ such that $h_c(t_c) = 0$ and $h_c(t_{n+1}) = 1$, and define $f_n$ as follows

$$f_n(t) = \begin{cases} f_n^\sim(t), & \text{when } t \in [t_n, t_c] \\ f(h_c(t)), & t \in [t_c, t_{n+1}]. \end{cases}$$

If $c > 0$ then, with $h_n$ defined as in case 1, put $f_n^\sim(t) = f(h_n(t))$ for all $t \in [t_c, t_{n+1}]$. There exists a number $t_s$, $t_s \in (t_n, t_{n+1})$, such that $f_n^\sim(t_s) = c$. If $t_s = t_c$, put $f_n^\sim(t) = f_n^\sim(t_c)$ for all $t \in [t_n, t_c]$. If not, let $k_n$ be an affine mapping from $[t_c, t_{n+1}]$ onto $[t_s, t_{n+1}]$ such that $k_n(t_c) = t_s$ and $k_n(t_n) = t_{n+1}$ and put $f_n^\sim(t) = f_n^\sim(k_n(t))$ for all $t \in [t_c, t_{n+1}]$. We define $f_n$ as follows

$$f_n(t) = \begin{cases} f_n^\sim(t), & \text{when } t \in [t_n, t_c] \\ f_n^\sim(t + (t_s - t_n)(t_{n+1} - t_s)/(t_{n+1} - t_c)), & t \in [t_c, t_{n+1}]. \end{cases}$$

Both (2.3) and (2.4) give us $f_n$ that is decreasing on $[t_n, t_c]$ and increasing on $[t_c, t_{n+1}]$. Furthermore, $f_n(t_c) = c$ and $f_n(t_n) = f_n(t_{n+1}) = 2\pi$. We apply Lemma 1.2 on $[t_n, t_c]$ and Lemma 1.1 on $[t_c, t_{n+1}]$ to obtain a continuous mapping $\phi_n : [t_n, t_{n+1}] \to [t_n, t_{n+1}]$ such that $\phi_n(t_n) = t_n$, $\phi_n(t_{n+1}) = t_{n+1}$ and $f_n(\phi_n(t)) = g_1(t)$ for all $t \in [t_n, t_{n+1}]$.

In all four cases, $f_n(\phi_n(t)) = g_1(t)$ for all $t \in [t_n, t_{n+1}]$. Furthermore, the principal value of the argument $\Arg g_1(t) = \Arg \sim G_1(t)$ for all $t \in [0, 1]$. We shall now define a mapping $F_1 : [0, 1] \to Y$ in the following manner. For each $n$, $0 \leq n \leq m$, put $F_1(t_n) = F(0)$ and if $t_m = 1$ then also put $F_1(1) = F(0)$. Suppose $t \in (0, 1)$, $t \neq t_n$, $n = 1, \ldots, m$. Then, $t \in (t_n, t_{n+1})$ for some $n$, $0 \leq n \leq m$, and $f_n(\phi_n(t)) \in [0, 2\pi)$. If $f_n(\phi_n(t)) = 0$ then put $F_1(t) = F(0)$, otherwise, for each $s \in (0, 1)$ such that $\Arg F(s) = f_n(\phi_n(t))$. Put $F_1(t) = F(s)$. Note that $F_1([0, 1]) = Y$ and

$$\Arg F_1(t) = \Arg \sim G_1(t) \quad \text{for all } t \in [0, 1].$$

Taking analogous steps with respect to $\sim G_2$, we define an onto mapping $F_2 : [0, 1] \to Y$ such that

$$\Arg F_2(t) = \Arg \sim G_2(t) \quad \text{for all } t \in [0, 1].$$
Since $Y$ is starlike, the equalities (2.5) and (2.6) imply that for all $t \in [0,1]$,
\begin{equation}
|F_1(t) - F_2(t)| \geq |\sim G_1(t) - \sim G_2(t)|.
\end{equation}

Consequently, taking (2.1) and (2.2) into account, it follows that
\begin{equation*}
\sigma(Y) \geq \inf_{t \in [0,1]} |F_1(t) - F_2(t)| \geq \inf_{t \in [0,1]} |\sim G_1(t) - \sim G_2(t)| \geq \inf_{t \in [0,1]} |G_1(t) - G_2(t)| - \varepsilon/2 > \sigma(X) - \varepsilon/2 - \varepsilon/2 = \sigma(X) - \varepsilon.
\end{equation*}

Finally, since $\varepsilon$ was an arbitrary positive number, we conclude that $\sigma(Y) \geq \sigma(X)$.

\begin{thebibliography}{9}
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