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ON LEFT FIXED MAPS OF BE-ALGEBRAS

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ABSTRACT. In this paper, the concept of a left fixed map in an BE-algebra is discussed and some fundamental properties to BE-algebras are discussed.

1. **Introduction.** Y. Imai and K. Iséki introduced two classes of abstract algebras: BCK-algebras and BCI-algebras([3, 4]). It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. In [1, 2], Q. P. Hu and X. Li introduced a wide class of abstracts: BCH-algebras. They have shown that the class of BCI-algebras is a proper subclass of the class of BCH-algebras. In [5], H. S. Kim and Y. H. Kim introduced the notion of a BE-algebra as a dualization of a generation of a BCK-algebras. In this paper, the concept of a left fixed map in an BE-algebra is discussed and some fundamental properties to BE-algebras are discussed.

2. **Preliminaries.** In what follows, let X denote an BE-algebra unless otherwise specified.

By an *BE-algebra* we mean an algebra $(X; *, 1)$ of type (2, 0) with a single binary operation "$*$" that satisfies the following identities: for any $x, y, z \in X$,

(BE1) $x * x = 1$ for all $x \in X$,
(BE2) $x * 1 = 1$ for all $x \in X$,
(BE3) $1 * x = x$ for all $x \in X$,
(BE4) $x * (y * z) = y * (x * z)$ for all $x, y, z \in X$.

We introduce a relation "$\leq$" on X by $x \leq y$ imply $x * y = 1$. An BE-algebra $(X, *, 1)$ is said to be *self-distributive* if $x * (y * z) = (x * y) * (x * z)$ for all $x, y, z \in X$. A non-empty subset S of an BE-algebra X is said to be a *subalgebra* of X if $x * y \in S$ whenever $x, y \in S$.

In an BE-algebra, the following identities are true:

(p1) $x * (y * x) = 1$.
(p2) $x * ((x * y) * y)) = 1$.

Definition 2.1. Let $(X, *, 1)$ be an BE-algebra and F a non-empty subset of X. Then F is said to be a *filter* of X if

(F1) $1 \in F$,
(F2) If $x \in F$ and $x * y \in F$, then $y \in F$.

Example 2.1. Let $X = \{1, a, b, c, d\}$ in which "$*$" is defined by

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$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	c	d
b	1	a	1	c	c
c	1	1	b	1	b
d	1	1	1	1	1

It is easy to know that X is a BE-algebra, and $F_1 = \{1, a\}, F_2 = \{1, b\}, F_3 = \{1, c\}, F_4 = \{1, a, b\}$ are filters of X.

3. **Left fixed maps in BE-Algebras.**

Definition 3.1. A *left fixed map* α of X is defined to be a self map $\alpha : X \to X$ satisfying $\alpha(x * y) = x * \alpha(y)$ for all $x, y \in X$.

Example 3.1. Let $X = \{1, a, b\}$ in which "$*$" is defined by

$*$	1	a	b
1	1	a	b
a	1	1	1
b	1	1	1

Then X is an BE-algebra. It can be easily very that the self map α of X defined by $\alpha(1) = 1, \alpha(a) = 1$ and $\alpha(b) = b$ is a left fixed map.

Example 3.2. Let $X = \{1, a, b, c\}$ in which "$*$" is defined by

$*$	1	a	b	c
1	1	a	b	c
a	1	1	a	a
b	1	1	1	a
c	1	1	a	1

Then X is an BE-algebra. Let $\alpha : X \to X$ be defined by $\alpha(1) = 1, \alpha(a) = 1, \alpha(b) = c$ and $\alpha(c) = c$. Then α is not a left fixed map of X since $\alpha(b * c) \neq b * \alpha(c)$.

Lemma 3.2. *If α is a left fixed map of X, then we have*

(1) $\alpha(1) = 1$,
(2) $x \leq \alpha(x)$ *for all* $x \in X$,
(3) $x \leq y$ *implies* $x \leq \alpha(y)$ *for all* $x, y \in X$.

Proof. (1) For every $x, y \in X$, we have

$$\alpha(1) = \alpha(\alpha(1) * 1) = \alpha(1) * \alpha(1) = 1.$$

(2) For every $x \in X$, we have $1 = \alpha(1) = \alpha(x * x) = x * \alpha(x)$, and so $x \leq \alpha(x)$.
(3) Suppose that $x \leq y$ for every $x, y \in X$. Then

$$1 = \alpha(1) = \alpha(x * y) = x * \alpha(y),$$

and so $x \leq \alpha(y)$. □

Proposition 3.3. *If α and β are left fixed maps of X, then $\alpha \circ \beta$ is a left fixed map of X.*

Proof. Let α and β be left fixed maps of X. Then

$$(\alpha \circ \beta)(x * y) = \alpha(\beta(x * y)) = \alpha(x * \beta(y)) = x * \alpha(\beta(y)) = x * (\alpha \circ \beta)(y)$$

for all $x, y \in X$. □

Theorem 3.4. *Let α be a left fixed map of X. Then α is one-to-one if and only if α is the identity map.*

Proof. Sufficiency is obvious. Suppose that α is one-to-one. For $x \in X$, we have

$$\alpha(\alpha(x) * x) = \alpha(x) * \alpha(x) = 1 = \alpha(1)$$

and so $\alpha(x) * x = 1$, i.e., $\alpha(x) \leq x$. Since $x \leq \alpha(x)$ for all $x \in X$, it follows that $\alpha(x) = x$ so that α is the identity map. □

Let α be a left fixed map of X. Define a set F by

$$F := \{x \mid \alpha(x) = x\}$$

for all $x \in X$.

Proposition 3.5. *Let α be a left fixed map of X. Then F is a subalgebra of X.*

Proof. Clearly, $1 \in F$ and so F is nonempty. Let $x, y \in F$. Then we have $\alpha(x) = x$ and $\alpha(y) = y$, and so

$$\alpha(x * y) = x * \alpha(y) = x * y.$$

This implies $x * y \in F$. Hence F is a subalgebra of X. □

Proposition 3.6. *Let α be a left fixed map of X. If $x \in F$, then we have $(\alpha \circ \alpha)(x) = x$.*

Proof. Let $x \in F$. Then we have

$$(\alpha \circ \alpha)(x) = \alpha(\alpha(x)) = \alpha(x) = x.$$

This completes the proof. □

Let α be a left fixed map of X. Define a $Ker(\alpha)$ by

$$Ker(\alpha) = \{x \mid \alpha(x) = 1\}$$

for all $x \in X$.

Proposition 3.7. *Let α be a left fixed map of X. Then $Ker(\alpha)$ is a subalgebra of X.*

Proof. Clearly, $1 \in Ker(\alpha)$, and so $Ker(\alpha)$ is nonempty. Let $x, y \in Ker(\alpha)$. Then $\alpha(x) = 1$ and $\alpha(y) = 1$. Hence we have

$$\alpha(x * y) = x * \alpha(y) = x * 1 = 1,$$

and so $x * y \in Ker(\alpha)$. Thus $Ker(\alpha)$ is a subalgebra of X. □

An BE-algebra X is said to be *commutative* if for all $x, y \in X$,

$$(y * x) * x = (x * y) * y.$$

Proposition 3.8. *Let X be a commutative BE-algebra. If $x \in Kerd$ and $x \leq y$, then we have $y \in Ker(\alpha)$.*

Proof. Let $x \in Ker(\alpha)$ and $x \leq y$. Then $\alpha(x) = 1$ and $x * y = 1$.

$$\begin{aligned}
\alpha(y) &= \alpha(1 * y) = \alpha((x * y) * y) \\
&= \alpha((y * x) * x) \\
&= (y * x) * \alpha(x) \\
&= (y * x) * 1 \\
&= 1,
\end{aligned}$$

and so $y \in Ker(\alpha)$. This completes the proof. □

Proposition 3.9. *Let α be a left fixed map of X and an endomorphism. Then $Ker(\alpha)$ is a filter of X.*

Proof. Clearly, $1 \in Ker(\alpha)$. Let $x \in Ker(\alpha)$ and $x * y \in Ker(\alpha)$. Then we have $\alpha(x) = \alpha(x * y) = 1$, and so

$$1 = \alpha(x * y) = \alpha(x) * \alpha(y) = 1 * \alpha(y) = \alpha(y).$$

This implies $y \in Ker(\alpha)$. This completes the proof. □

Proposition 3.10. *Let α be a left fixed map of X. If α is one-to-one, then $Ker(\alpha) = 1$.*

Proof. Suppose that α is one-to-one and $x \in Ker(\alpha)$. Then $\alpha(x) = 1 = \alpha(1)$, and thus $x = 1$, i.e., $Ker(\alpha) = \{1\}$. □

Denote by $LF(X)$ the set of all left fixes maps of X. Let $\otimes$ be a binary operation on $LF(X)$ defined by

$$(\alpha \otimes \beta)(x) = \alpha(x) * \beta(x)$$

for all $\alpha, \beta \in LF(X)$ and $x \in X$.

Proposition 3.11. *Let X be an BE-algebra. Then $(LF(X), \otimes)$ is an BE-algebra of X.*

Proof. Let $\alpha \in LF(X)$ and $x \in X$. Then we get $(\alpha \otimes \alpha)(x) = \alpha(x) * \alpha(x) = 1$, which proves (BE1). Similarly, we can prove (BE2), (BE3) and (BE4). □

Let $ILF(X)$ denote the set of all idempotent left fixed maps of X.

Theorem 3.12. *Let X be a self-distributive BE-algebra of X and $\alpha, \beta \in ILF(X)$. Then we have*

(1) $\alpha \otimes \beta \in LF(X)$,
(2) *If $\alpha(\beta(x)) = \beta(\alpha(x))$ for all $x \in X$, then $\alpha \otimes \beta \in ILF(X)$.*

Proof. (1) For every $x, y \in X$, we get

$$\begin{aligned}(\alpha \otimes \beta)(x * y) &= \alpha(x * y) * \beta(x * y) \\ &= (x * \alpha(y)) * (x * \beta(y)) \\ &= x * (\alpha(y) * \beta(y)) \\ &= x * (\alpha \otimes \beta)(y),\end{aligned}$$

and so $\alpha \otimes \beta \in LF(X)$.

(2) Suppose that $\alpha(\beta(x)) = \beta(\alpha(x))$ for all $x \in X$. Then

$$\begin{aligned}(\alpha \otimes \beta)((\alpha \otimes \beta)(x)) &= (\alpha \otimes \beta)(\alpha(x) * \beta(x)) \\ &= \alpha(\alpha(x) * \beta(x)) * \beta(\alpha(x) * \beta(x)) \\ &= (\alpha(x) * \alpha(\beta(x)) * (\alpha(x) * \beta(\beta(x))) \\ &= (\alpha(x) * \alpha(\beta(x)) * (\alpha(x) * (\beta(x)) \\ &= (\alpha(x) * \beta(\alpha(x)) * (\alpha(x) * (\beta(x)) \\ &= (\alpha(x) * \beta(\alpha(x)) * (\alpha(x) * (\beta(x)) \\ &= \beta(\alpha(x) * \alpha(x)) * (\alpha(x) * \beta(x)) \\ &= \beta(1) * (\alpha(x) * \beta(x)) = 1 * (\alpha(x) * \beta(x)) \\ &= \alpha(x) * \beta(x) = (\alpha \otimes \beta)(x),\end{aligned}$$

that is, $\alpha \otimes \beta$ is idempotent. Hence we obtain $(\alpha \otimes \beta) \in ILF(X)$. □

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